How to count leaves in the trees from Tetris?

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Trees in Tetris
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A polyomino is a connected union of unit squares, called cells.
Solomon W. Golomb (1932 - 2016)
Combinatorial Problems

- Monomino: 🟢
Combinatorial Problems

- Monomino:

- Domino:
Combinatorial Problems

Monomino:

Domino:

Tromino:
Combinatorial Problems

- **Monomino:**
- **Domino:**
- **Tromino:**
- **Tetromino:**

Unknown for $n \geq 57$
How many polyominoes with $n$ cells?
Combinatorial Problems

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How many polyominoes with $n$ cells?

*Unknown for $n \geq 57$*
Can we pave the plane with a finite set of polyominoes?

Examples:

- Polyominoes:
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Examples:
- Polyominoes: Yes, we can!
Can we pave the plane with a finite set of polyominoes?

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- Polyominoes: 
  ![Polyomino Example](image)
  Yes, we can!

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  ![Polyomino Example](image)
Can we pave the plane with a finite set of polyominoes?

Examples:

- Polyominoes:
  ![Yes, we can!]
  Yes, we can!

- Polyominoes:
  ![No, we can’t!]
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Can we pave the plane with a finite set of polyominoes?

Examples:

- Polyominoes:
  ![Polyominoes](image)
  Yes, we can!

- Polyominoes:
  ![Polyominoes](image)
  No, we can’t!

In general, the problem is **undecidable**.
Can we pave a finite area with a finite set of polyominoes?
Example:
- Area: $45 \times 36$ rectangle
- Polyominoes: all pentominoes
Combinatorial Problems

Can we pave a finite area with a finite set of polyominoes?

http://www.knowltonmosaics.com
Combinatorial Problems

Can we pave a finite area with a finite set of polyominoes?
Example:

- **Area:** a checkerboard

- **Polyominoes:** a domino
Can we pave a finite area with a finite set of polyominoes?

Example:

- Area: a checkerboard
- Polyominoes: a domino

Yes, we can!
Can we pave a finite area with a finite set of polyominoes?
Example:

- Area: a checkerboard without two opposite corners
- Polyominoes: a domino
Can we pave a finite area with a finite set of polyominoes?

Example:

- **Area:** a checkerboard *without two opposite corners*

- **Polyominoes:** a domino

No, we can’t!
Can we pave a finite area with a finite set of polyominoes?
Example:

- Area: a checkerboard without two opposite corners

- Polyominoes: a domino

No, we can’t!

In general, the problem is **NP-complete**.
Behind each polyomino hides a graph

Vertices are the cells. Two vertices are adjacent if their corresponding cells have a common side. Such graph can contain cycles.
Behind each polyomino hides a graph

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Behind each polyomino hides a graph

- Vertices are the cells.
- Two vertices are adjacent if their corresponding cells have a common side.
- Such graph can contain cycles.
If there are no cycles, we call it a tree-like polyomino.
If there are no cycles, we call it a *tree-like polyomino*.

This tree-like polyomino has 9 cells and 4 leaves.
Tree-like Polyominoes

- If there are no cycles, we call it a tree-like polyomino.
- This tree-like polyomino has 9 cells and 4 leaves.
- For a given $n$, what is the maximal number of leaves realized by a tree-like polyomino with $n$ cells?
Maximal number of leaves?

Theorem (Blondin Massé, de Carufel, Goupil, Samson 2017)

For $n \geq 1$, the maximal number of leaves realized by a tree-like polyomino with $n$ cells is given by

$$
\ell(n) = \begin{cases} 
0 & \text{if } n = 1 \\
2 & \text{if } n = 2 \\
(n - 1) & \text{if } n = 3, 4, 5 \\
\ell(n - 4) + 2 & \text{if } n \geq 6
\end{cases}
$$
Trees in Minecraft

polyominoes $\Rightarrow$ polycubes

The maximal number of leaves realized by a tree-like polycube satisfies a linear recurrence.

Theorem (Blondin Massèe, de Carufel, Goupil, Samson 2017)
Trees in Minecraft

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The maximal number of leaves realized by a tree-like polycube satisfies a linear recurrence.

Theorem (Blondin Massé, de Carufel, Goupil, Samson 2017)
Tree-like polyominoes are induced subtrees of $\mathbb{Z}^2$. 
- Tree-like polyominoes are induced subtrees of $\mathbb{Z}^2$.
- Tree-like polycubes are induced subtrees of $\mathbb{Z}^3$. 
Tree-like polyominoes are induced subtrees of $\mathbb{Z}^2$.
Tree-like polycubes are induced subtrees of $\mathbb{Z}^3$.

Can we study this problem in other graphs?
Definition

For a graph $G = (V, E)$ and $n \geq 2$

- $T_n = \text{set of induced subtrees with } n \text{ vertices}$
- $L_G(n) = \max \{ \# \text{ leaves in } T | T \in T_n \}$
- Leafed sequence of $G$: $L_G(n)_{n \in \{2, \ldots, |V|\}}$

\[
\begin{array}{c|cccccccc}
  n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  L_G(n) & 2 & & & & & & \\
\end{array}
\]
Definition

For a graph $G = (V, E)$ and $n \geq 2$

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\[
\begin{array}{ccccccc}
\begin{array}{c}
1 \\
2 \\
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& \begin{array}{c}
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\end{array}

\begin{array}{c|ccccccc}
\hline
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
L_G(n) & 2 & 2 & 3 & \hline
\end{array}
\end{array}
\]
For a graph $G = (V, E)$ and $n \geq 2$

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```
 1 2 3
/  \
4 5 6
|  |
8 7
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\begin{center}
\begin{tabular}{c|cccccccc}
$n$ & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
$\quad L_G(n)$ & 2 & 2 & 3 & 4 & 4 & 5 & -\infty
\end{tabular}
\end{center}
Particular cases

\[ L_{K_p}(n) = \begin{cases} 
2 & \text{if } n = 2 \text{ et } p \geq 2 \\
-\infty & \text{if } 3 \leq n \leq p
\end{cases} \]
Particular cases

\[ L_{K_p}(n) = \begin{cases} 2 & \text{if } n = 2 \text{ et } p \geq 2 \\ -\infty & \text{if } 3 \leq n \leq p \end{cases} \]

\[ L_{C_p}(n) = \begin{cases} 2 & \text{if } 2 \leq n < p \\ -\infty & \text{if } n = p \end{cases} \]
Particular cases

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2 & \text{if } n = 2 \text{ et } p \geq 2 \\
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\end{cases} \]

\[ L_{C_p}(n) = \begin{cases} 
2 & \text{if } 2 \leq n < p \\
-\infty & \text{if } n = p
\end{cases} \]

\[ L_{R_p}(n) = \begin{cases} 
2 & \text{if } 2 = n \text{ ou } \left\lfloor \frac{p}{2} \right\rfloor + 1 < n < p \\
 n - 1 & \text{if } 3 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor + 1 \\
-\infty & \text{if } p \leq n \leq p + 1
\end{cases} \]
Particular cases

\[ \begin{align*}
L_{K_p}(n) &= \begin{cases} 
2 & \text{if } n = 2 \text{ et } p \geq 2 \\
-\infty & \text{if } 3 \leq n \leq p
\end{cases} \\
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-\infty & \text{if } n = p
\end{cases} \\
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\text{ord} & \text{if } 3 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor + 1 \\
-\infty & \text{if } p \leq n \leq p + 1
\end{cases} \\
L_{K_{p,q}}(n) &= \begin{cases} 
2 & \text{if } n = 2 \\
\text{ord} & \text{if } 3 \leq n \leq \max(p, q) + 1 \\
-\infty & \text{if } \max(p, q) + 1 < n \leq p + q
\end{cases}
\end{align*} \]
Particular cases: hypercubes

Only partial results:

\[ L_{H_3}(n) = \begin{cases} 
2 & \text{if } 2 \leq n \leq 3 \\
3 & \text{if } n = 4 \\
-\infty & \text{if } 5 \leq n \leq 8 
\end{cases} \]

\[ L_{H_4}(n) = \begin{cases} 
2 & \text{if } n \in \{2, 3\} \\
3 & \text{if } n \in \{4, 6, 8\} \\
4 & \text{if } n \in \{5, 7, 9\} \\
-\infty & \text{if } 10 \leq n \leq 16 
\end{cases} \]
Particular cases: hypercubes

Only partial results:

\[
L_{H3}(n) = \begin{cases} 
2 & \text{if } 2 \leq n \leq 3 \\
3 & \text{if } n = 4 \\
-\infty & \text{if } 5 \leq n \leq 8 
\end{cases}
\]

\[
L_{H4}(n) = \begin{cases} 
2 & \text{if } n \in \{2, 3\} \\
3 & \text{if } n \in \{4, 6, 8\} \\
4 & \text{if } n \in \{5, 7, 9\} \\
-\infty & \text{if } 10 \leq n \leq 16 
\end{cases}
\]

Observations:

- The leafed sequence can increase by at most 1 each step.
- The leafed sequence is not always non-decreasing.
- The leafed sequence \(L_G(n)_{n\in\{2,\ldots,|V|\}}\) is non-decreasing iff \(G\) is a tree.
Complexity

**Problem LIS**

- **Instance:** a graph $G$ and two integers $n, k \geq 1$
- **Question:** Is there an induced subtree of $G$ with $n$ vertices and $k$ leaves?
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**Theorem** (Blondin Massé, de Carufel, Goupil, V. 2017)

The LIS problem is NP-complete.
Complexity

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- Instance: a graph $G$ and two integers $n, k \geq 1$
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**Theorem** (Blondin Massé, de Carufel, Goupil, V. 2017)

The LIS problem is NP-complete.

Reduction to the problem:

- Instance: a graph $G$ and an integer $n \geq 1$
- Question: Is there an induced subtree of $G$ with more than $n$ vertices?

which is NP-complete.  [Erdös, Saks, Sós 1986]
LIS is NP-complete

- Polynomial transformation $f : (G, n) \mapsto (H, 2(n + 1), n + 1)$
LIS is NP-complete

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If $G$ has an induced subtree with $> n$ vertices
LIS is NP-complete

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If $G$ has an induced subtree with $> n$ vertices

$\Rightarrow \exists T$ induced subtree with $n + 1$ vertices \{v_1, \ldots, v_{n+1}\}
LIS is NP-complete

- Polynomial transformation $f : (G, n) \mapsto (H, 2(n + 1), n + 1)$

- If $G$ has an induced subtree with $> n$ vertices
  $\Rightarrow \exists T$ induced subtree with $n + 1$ vertices $\{v_1, \ldots, v_{n+1}\}$
  $\Rightarrow H$ has a subtree induced by $\{v_1, \ldots, v_{n+1}\} \cup \{v'_1, \ldots, v'_{n+1}\}$
LIS is NP-complete

- Polynomial transformation $f : (G, n) \mapsto (H, 2(n + 1), n + 1)$

- If $G$ has an induced subtree with $> n$ vertices
  \[
  \Rightarrow \exists T \text{ induced subtree with } n + 1 \text{ vertices } \{v_1, \ldots, v_{n+1}\}
  \]
  \[
  \Rightarrow H \text{ has a subtree induced by } \{v_1, \ldots, v_{n+1}\} \cup \{v'_1, \ldots, v'_{n+1}\}
  \]
  with $2(n + 1)$ vertices and $n + 1$ leaves
LIS is NP-complete

- Polynomial transformation \( f : (G, n) \mapsto (H, 2(n + 1), n + 1) \)

- If \( H \) has an induced subtree \( T \) with \( 2(n + 1) \) vertices and \( n + 1 \) leaves
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- Polynomial transformation $f : (G, n) \mapsto (H, 2(n + 1), n + 1)$

If $H$ has an induced subtree $T$ with $2(n + 1)$ vertices and $n + 1$ leaves

$\Rightarrow T$ has $\geq n + 1$ vertices in $V$
LIS is NP-complete

- Polynomial transformation $f : (G, n) \mapsto (H, 2(n + 1), n + 1)$

If $H$ has an induced subtree $T$ with $2(n + 1)$ vertices and $n + 1$ leaves

$\Rightarrow T$ has $\geq n + 1$ vertices in $V$

$\Rightarrow G$ has an induced subtree with $\geq n$ vertices
What about trees?

**Theorem** (Blondin Massé, de Carufel, Goupil, V. 2017)

For a tree $T$ with $m$ vertices, the leafed sequence $L_T(n)$ is computed in polynomial time and space.

Algorithm based on the dynamic programming paradigm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm

\[ \hat{T}_u = ? \]
Idea of the algorithm

For a directed tree $\hat{T}_u$ rooted in $u$

- $f(\hat{T}_u) = \# \{ x \in \hat{T}_u \mid \deg^+(x) = 0 \}$
- $L_{\hat{T}_u}(n) = \max \{ f(\hat{T}_u') : \hat{T}_u' \subseteq \hat{T}_u, |\hat{T}_u'| = n \}$

Generalization to a directed forest $\hat{F}$ with $k$ connected components $\hat{F}_i$:

$$L_{\hat{F}}(n) = \max \left\{ \sum_{i=1}^{k} L_{\hat{F}_i}(\lambda(i)) \mid \lambda \in C(n, k) \right\}$$

- If $\hat{F}$ is the forest of the subtrees rooted in the children of $u$,

$$L_{\hat{T}_u}(n) = \begin{cases} n & \text{if } n = 0, 1 \\ L_{\hat{F}}(n - 1) & \text{otherwise} \end{cases}$$
Idea of the algorithm

$L_{\widehat{T_u}} = ?$
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm

Assuming $L_{\hat{F}_i}$ are known,

- Naive computation of $L_{\hat{F}}$ via

$$L_{\hat{F}}(n) = \max \left\{ \sum_{i=1}^{k} L_{\hat{F}_i}(\lambda(i)) \mid \lambda \in C(n, k) \right\}$$

is not polynomial
Idea of the algorithm

Assuming $L_{\hat{F}_i}$ are known,

- Naive computation of $L_{\hat{F}}$ via

$$L_{\hat{F}}(n) = \max \left\{ \sum_{i=1}^{k} L_{\hat{F}_i}(\lambda(i)) \left| \begin{array}{c} \lambda \in C(n, k) \end{array} \right. \right\}$$

is not polynomial

- Computation of $L_{\hat{F}}$ via

$$L_{\hat{F}}(n) = \max\{L_{F_{-F_1}}(i) + L_{F_1}(n-i) \mid 0 \leq i \leq n\}$$

in time $\Theta(k|F|^2)$
Idea of the algorithm
Idea of the algorithm

012 01 01 01

01123

011234556

0112

01

01123

0112

01 01

01 01

u v
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm
Idea of the algorithm

- Maximal number of leaves in the subtrees with $n$ vertices, containing the edge $\{u, v\}$:

$$L_{\{u, v\}}(n) = \max \left\{ L_{\tilde{T}_u}(i) + L_{\tilde{T}_v}(n - i) \middle| 1 \leq i \leq n - 1 \right\}$$

$$L_{\{u, v\}} = 012234456677889$$
Idea of the algorithm

- Maximal number of leaves in the subtrees with $n$ vertices, containing the edge $\{u, v\}$ : 

$$L_{\{u,v\}}(n) = \max \left\{ L_{\hat{T}_u}(i) + L_{\hat{T}_v}(n - i) \mid 1 \leq i \leq n - 1 \right\}$$

$$L_{\{u,v\}} = 012234456677889$$

- $L_{T}(n) = \max \{ L_{\{u,v\}}(n) \mid \{u, v\} \in E \}$
Could we improve the time and space complexity?

- We can not hope to obtain a procedure computing $L_T(n)$ which deletes leaves successively.
- Counter-example:

\[ L_T(7) = 5 \quad \text{et} \quad L_T(9) = 6 \]
Each tree has a non-decreasing leaf sequence $L = (L_i)_{i \geq 2}$.

For any graph, $L_{i+1} - L_i \leq 1$.

The difference word $\Delta L := (L_{i+1} - L_i)_{i \geq 3}$ is a binary word.
From graph theory to combinatorics on words

- Each tree has a non-decreasing leaf sequence $L = (L_i)_{i \geq 2}$.
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The difference word $ΔL := (L_{i+1} - L_i)_{i \geq 3}$ is a binary word.

Leafed sequence: $L = 223445567789$
Difference word: $ΔL = 011010110111$
From graph theory to combinatorics on words

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Leafed sequence: $L = 223445567789$

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The language of a class of graphs is the set of all their difference words.
Each tree has a non-decreasing leaf sequence $L = (L_i)_{i \geq 2}$.

For any graph, $L_{i+1} - L_i \leq 1$.

The **difference word** $\Delta L := (L_{i+1} - L_i)_{i \geq 3}$ is a binary word.

Leafed sequence: $L = 223445567789$

Difference word: $\Delta L = 1101011011$

The **language** of a class of graphs is the set of all their difference words.

Language of Trees =???
Caterpillars

\[ L = 223455667778 \text{ and } \Delta L = 1110101001 \]

What is the language of caterpillars?

Language of caterpillars $\subseteq$ Language of trees

Enumerate caterpillars
Get their associated difference words $\epsilon, 0, 1, 00, 10, 11, 0000, 1000, 1001, 1010, 1100, 1101, \ldots$

Count them by length

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Check OEIS
It is A194850!
Caterpillars

$L = 223455667778$ and $\Delta L = 1110101001$

What is the language of caterpillars?
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Language of caterpillars $\subsetneq$ Language of trees
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29/31
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Prefix normal word

A prefix normal word $w_1 \cdots w_n$ is a binary word such that
\[ \forall \ell \leq n, i \leq n - \ell, \]
\[ |w_1 \cdots w_\ell|_1 \geq |w_i \cdots w_{i+\ell}|_1. \]

$L = 223455667778$ and $\Delta L = 1110101001$
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Conjecture:

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Language of Caterpillars $= \{\text{prefix normal words}\}$

**Proposition** (Blondin Massé, de Carufel, Goupil, Lapointe, V.)

Language of Caterpillars $\supseteq \{\text{prefix normal words}\}$
Determine when an integer sequence corresponds to a tree or a graph.

Find an application in Chemistry or Biology.

Extend the results obtained in $\mathbb{Z}_2$ to the infinite triangular grid (polyiamonds) and to the infinite hexagonal grid (polyhexes).
Perspectives

- Determine when an integer sequence corresponds to a tree or to a graph
Perspectives

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