MODAL IDENTIFICATION AND DETERMINATION OF EFFECTIVE MASS USING ENVIRONMENTAL VIBRATION TESTING ON AN ELECTRO-DYNAMIC SHAKER

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ABSTRACT

An experimental determination of effective modal masses of a mechanical structure is proposed using environmental vibration tests on an electro-dynamic shaker. The method is based on the identification of electro-mechanical parameters of the electro-dynamic model of the shaker. An experimental example is presented to illustrate the proposed procedure.

NOMENCLATURE

${}_rA_{jk}$,	$({}_rA'_{jk})$ modal residues
${}_rA^*_{jk}$,	$({}_rA'^*_{jk})$ complex conjugate of modal residues
B	magnetic flux density
c_s	damping coefficient associated to the shaker
$[\mathbf{C}]$	symmetric and non negative damping matrix
$\{\mathbf{e}_x\}$	unit amplitude vector in x-direction
$F(\omega)$	Fourier transform of force
$\lceil H(\omega) \rfloor$	diagonal transfer matrix
$H_{rr}(\omega)$	diagonal element of transfer matrix
$H_{jk}(\omega)$	individual FRF element
$H_V(\omega)$	transfer function
$i(t)$	current intensity
$I(\omega)$	Fourier transform of current
$\lceil \mathbf{I} \rfloor$	identity matrix
k_s	stiffness associated to the shaker
k_r	modal stiffness of r^{th} mode
$\lceil k_r \rfloor$	modal stiffness matrix
$[\mathbf{K}]$	symmetric and non negative stiffness matrix
ℓ	coil length
L	circuit electrical inductance
m	number of included modes in the direction of the excitation
m_t	mass of the test object
m_s	moving mass of the shaker table
m_r	modal mass of r^{th} mode
$\widetilde{m}_r$	effective modal mass of r^{th} mode
$\lceil m_r \rfloor$	modal mass matrix
m_2	mass associated to the support
m_2^*	total structural mass
$[\mathbf{M}]$	symmetric and non negative mass matrix
$[\widetilde{\mathbf{M}}]$	effective modal mass matrix
$[\mathbf{M}]^*_{22}$	condensed mass matrix
$[\mathbf{M}_D]$	dynamic mass matrix
$[\mathbf{M}_D(\omega)]$	Fourier transform of dynamic mass matrix
n_1	number of unrestrained DOFs
n_2	number of restrained DOFs
$\{\mathbf{p}\}$	vector of generalized forces
$\{\mathbf{q}\}$	vector of generalized displacements
r	current mode number
$\{\mathbf{r}_2\}$	vector of reaction forces
R	circuit electrical resistance
$\{\mathbf{R}_2(\omega)\}$	Fourier transform of reaction forces
$[\mathbf{S}]$	matrix of constrained modes matrix
$\{\mathbf{u}\}$	vector of rigid body displacements
v	voltage
$V(\omega)$	Fourier transform of voltage
x	displacement
$X(\omega)$	Fourier transform of displacement
$Z(\omega)$	Fourier transform of relative displacement
$\{\mathbf{y}_1\}$	vector of displacements
$[\mathbf{\Gamma}]$	modal participation matrix
ζ_r	viscous damping ratio of r^{th} mode
$\lceil \zeta_r \rfloor$	viscous damping ratio matrix
$\{\boldsymbol{\eta}\}$	vector of modal coordinates
$\{\boldsymbol{\eta}(\omega)\}$	Fourier transform of $\{\boldsymbol{\eta}\}$
λ_r, (λ_r^*)	eigenvalue (complex conjugate) of r^{th} mode
$\phi(t)$	acceleration law
$[\mathbf{\Psi}]$	mode shape matrix
$[\mathbf{\Psi}]_{11}$	mode shape matrix of the restrained structure
ω_r	natural frequency of r^{th} mode
ω	frequency

1. INTRODUCTION

The effective modal mass concept is very helpful for the dynamic analysis of structures submitted to base excitation. It has been first used in aerospace engineering to characterize the dynamic behavior of satellites or flight equipments. As vibration testing is now widely used in other areas such as mechanical, electrical and nuclear industries, the effective mass concept is more and more applied.

Environmental vibration testing on an electro-dynamic shaker is used to perform qualification and acceptance tests of equipments. It also allows to identify the modal parameters of the structure (eigenvalues and mode shapes) using some adaptation of classical modal identification methods. The advantage of environmental vibration testing compared to other excitation techniques is the possibility to exactly reproduce the vibration levels encountered in service.

Effective modal masses can be computed easily from the structural model of the equipment using for example the finite element method, but are rather difficult to be deduced from vibration tests on an electro-dynamic shaker. The experimental determination of the effective masses requires to measure the reaction forces between the test object and the work-table of the shaker and can be very difficult in practice. Once the electro-mechanical model of the shaker has been clearly identified, the reaction forces can be determined if the drive voltage fed to the shaker and the control acceleration of the work-table are known. Thus, the dynamic mass and the effective modal masses of the structure can be calculated.

2. STRUCTURES EXCITED THROUGH GLOBAL SUPPORT MOTION

2.1. Dynamic equilibrium equations

The dynamic response of a structure submitted to environmental vibration testing (figure 1) may be split into a global acceleration induced by the shaker table and associated to the rigid body motion and a vibration of the structure clamped onto the supports. The general equations of motion of a damped structure are of the form :

$$[\mathbf{M}]\ \{\ddot{\mathbf{q}}\} + [\mathbf{C}]\ \{\dot{\mathbf{q}}\} + [\mathbf{K}]\ \{\mathbf{q}\} = \{\mathbf{p}\} \tag{1}$$

The response of the system shown in figure 1 to global motion of the shaker table can be written by partitioning the degrees of freedom in two sets :

- the n_1 displacements $\{\mathbf{q}_1\}$ remaining completely free;
- the n_2 displacements $\{\mathbf{q}_2\}$ imposed at the supports;

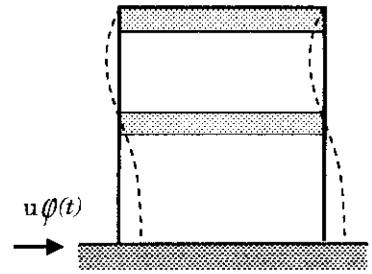

Fig. 1: Environmental vibration testing

In the following, we will consider the conservative system associated to the actual damped system. In terms of absolute displacements, the equations of motion take the form :

$$\begin{bmatrix} [\mathbf{M}]_{11} & [\mathbf{M}]_{12} \\ [\mathbf{M}]_{21} & [\mathbf{M}]_{22} \end{bmatrix} \begin{Bmatrix} \{\ddot{\mathbf{q}}_1\} \\ \{\ddot{\mathbf{q}}_2\} \end{Bmatrix} + \begin{bmatrix} [\mathbf{K}]_{11} & [\mathbf{K}]_{12} \\ [\mathbf{K}]_{21} & [\mathbf{K}]_{22} \end{bmatrix} \begin{Bmatrix} \{\mathbf{q}_1\} \\ \{\mathbf{q}_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{\mathbf{r}_2\} \end{Bmatrix} \tag{2}$$

The first set of equations extracted from (2)

$$[\mathbf{M}]_{11}\ \{\ddot{\mathbf{q}}_1\} + [\mathbf{K}]_{11}\ \{\mathbf{q}_1\} = -\,[\mathbf{K}]_{12}\ \{\mathbf{q}_2\} - [\mathbf{M}]_{12}\ \{\ddot{\mathbf{q}}_2\} \tag{3}$$

allows to compute the response of the unrestrained degrees of freedom, while the second one

$$\{\mathbf{r}_2\} = [\mathbf{K}]_{21}\ \{\mathbf{q}_1\} + [\mathbf{M}]_{21}\ \{\ddot{\mathbf{q}}_1\} + [\mathbf{K}]_{22}\ \{\mathbf{q}_2\} + [\mathbf{M}]_{22}\ \{\ddot{\mathbf{q}}_2\} \tag{4}$$

gives the reactions $\{\mathbf{r}_2\}$ between the structure and the shaker table. The solution of equation (3) can be decomposed in the form :

$$\{\ddot{\mathbf{q}}\} = \begin{Bmatrix} \{\ddot{\mathbf{q}}_1\} \\ \{\ddot{\mathbf{q}}_2\} \end{Bmatrix} = \begin{Bmatrix} \{\ddot{\mathbf{y}}_1\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} \{\mathbf{u}_1\} \\ \{\mathbf{u}_2\} \end{Bmatrix} \phi(t) \tag{5}$$

In this equation,

- $\{\mathbf{y}_1\}$ represents the sole dynamic part of the response, i.e. arising from the vibration of the structure on its support,
- $\{\mathbf{u}\}^T$ is the rigid body displacement mode of the system $\{\mathbf{u}\}^T = [\ \{\mathbf{u}_1\}\ \{\mathbf{u}_2\}\ ]$,
- $\phi(t)$ is the acceleration law of the shaker table.

The rigid body displacement mode satisfies the static equilibrium equation

$$[\mathbf{K}]\ \{\mathbf{u}\} = \{0\} \tag{6}$$

so that the vector of the support displacements $\{\mathbf{u}_2\}$ can be related to the vector of unrestrained displacements $\{\mathbf{u}_1\}$ by

$$\{\mathbf{u}_1\} = [\mathbf{S}]\ \{\mathbf{u}_2\} \quad \text{with} \quad [\mathbf{S}] = -[\mathbf{K}]_{11}^{-1}\ [\mathbf{K}]_{12} \tag{7}$$

where $[\mathbf{S}]$ is defined as the static condensation matrix at the shaker table. Through substitution of the solution decomposition (5) into (3), the equation governing the motion of the unrestrained degrees of freedom takes the form :

$$\begin{gathered}[\mathbf{M}]_{11}\ \{\ddot{\mathbf{y}}_1\} + [\mathbf{K}]_{11}\ \{\mathbf{y}_1\} = \\ -\left([\mathbf{M}]_{11}\ \{\mathbf{u}_1\} + [\mathbf{M}]_{12}\ \{\mathbf{u}_2\}\right)\ \phi(t)\end{gathered} \tag{8}$$

The expanded mode shape matrix $[\mathbf{\Psi}]$ can be defined as

$$[\mathbf{\Psi}] = \begin{bmatrix} [\mathbf{\Psi}]_{11} & [\mathbf{S}] \\ [0] & [\,^{\backprime}\mathbf{I}_{\backslash}] \end{bmatrix} \tag{9}$$

where $[\mathbf{\Psi}]_{11}$ is the mode shape matrix obtained from the solution of the eigenproblem related to the system fixed on its support, i.e.

$$[\mathbf{K}]_{11}\ \{\mathbf{q}_1\} = \omega^2\ [\mathbf{M}]_{11}\ \{\mathbf{q}_1\} \tag{10}$$

and the eigenvector matrix $[\ [\mathbf{S}]\ [\,^{\backprime}\mathbf{I}_{\backslash}]\ ]$ is related to the rigid body mode of the structure.

2.2. The effective modal mass concept [1-5]

The solution of equation (2) can be developed in terms of eigenmodes as follows

$$\{\mathbf{q}\} = [\mathbf{\Psi}]\ \{\boldsymbol{\eta}\} \tag{11}$$

so that the equations of motion become

$$[\mathbf{M}]\ [\mathbf{\Psi}]\ \{\ddot{\boldsymbol{\eta}}\} + [\mathbf{K}]\ [\mathbf{\Psi}]\ \{\boldsymbol{\eta}\} = \{\mathbf{p}\} \tag{12}$$

Premultiplying equation (12) by the transpose of the expanded mode shape matrix $[\mathbf{\Psi}]^T$ and partitioning, one obtains the following normal equation

$$\begin{gathered}\begin{bmatrix} [\,^{\backprime} m_r \,_{\backslash}] & [\mathbf{\Gamma}] \\ [\mathbf{\Gamma}]^T & [\mathbf{M}]_{22}^* \end{bmatrix} \begin{Bmatrix} \{\ddot{\boldsymbol{\eta}}_1\} \\ \{\ddot{\boldsymbol{\eta}}_2\} \end{Bmatrix} + \\ \begin{bmatrix} [\,^{\backprime} k_r \,_{\backslash}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{\eta}_1\} \\ \{\boldsymbol{\eta}_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{\mathbf{r}_2\} \end{Bmatrix}\end{gathered} \tag{13}$$

in which

$[\,^{\backprime} m_r \,_{\backslash}] = [\mathbf{\Psi}]_{11}^T\ [\mathbf{M}]_{11}\ [\mathbf{\Psi}]_{11}$ is the modal mass matrix of the restrained system.

$[\,^{\backprime} k_r \,_{\backslash}] = [\mathbf{\Psi}]_{11}^T\ [\mathbf{K}]_{11}\ [\mathbf{\Psi}]_{11}$ is the modal stiffness matrix of the structure fixed at its support.

$$[\mathbf{\Gamma}] = -[\mathbf{\Psi}]_{11}^T\ \left([\mathbf{M}]_{11}\ [\mathbf{S}] + [\mathbf{M}]_{12}\right) \tag{14}$$

is the modal participation matrix of dimension $n_1 \times n_2$.

$$[\mathbf{M}]_{22}^* = \left[\ [\mathbf{S}]^T\ [\,^{\backprime}\mathbf{I}_{\backslash}]\ \right] \begin{bmatrix} [\mathbf{M}]_{11} & [\mathbf{M}]_{12} \\ [\mathbf{M}]_{21} & [\mathbf{M}]_{22} \end{bmatrix} \begin{bmatrix} [\mathbf{S}] \\ [\,^{\backprime}\mathbf{I}_{\backslash}] \end{bmatrix} \tag{15}$$

is the matrix resulting from the static condensation of the mass at support level.

With the assumption of a lightly damped structure, equation (13) can be generalized to

$$\begin{gathered}\begin{bmatrix} [\,^{\backprime} m_r \,_{\backslash}] & [\mathbf{\Gamma}] \\ [\mathbf{\Gamma}]^T & [\mathbf{M}]_{22}^* \end{bmatrix} \begin{Bmatrix} \{\ddot{\boldsymbol{\eta}}_1\} \\ \{\ddot{\boldsymbol{\eta}}_2\} \end{Bmatrix} + \\ \begin{bmatrix} 2\,[\,^{\backprime} \zeta_r\ m_r\ \omega_r \,_{\backslash}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{\boldsymbol{\eta}}_1\} \\ \{\dot{\boldsymbol{\eta}}_2\} \end{Bmatrix} + \\ \begin{bmatrix} [\,^{\backprime} m_r\ \omega_r^2 \,_{\backslash}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{\eta}_1\} \\ \{\boldsymbol{\eta}_2\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{\mathbf{r}_2\} \end{Bmatrix}\end{gathered} \tag{16}$$

From equation (16), the equation governing the reaction forces between the structure and the shaker table can be written in the frequency domain as

$$\{\mathbf{R}_2(\omega)\} = -\omega^2\ \left([\mathbf{\Gamma}]^T\ \{\boldsymbol{\eta}_1(\omega)\} + [\mathbf{M}]_{22}^*\ \{\boldsymbol{\eta}_2(\omega)\}\right) \tag{17}$$

and the motion equation of the structure clamped on its support can be written in the form

$$\{\boldsymbol{\eta}_1(\omega)\} = -[\,^{\backprime}\ \mathbf{H}(\omega) \,_{\backslash}]\ [\mathbf{\Gamma}]\ \{\boldsymbol{\eta}_2(\omega)\} \tag{18}$$

where $[\,^{\backprime}\ \mathbf{H}(\omega) \,_{\backslash}]$ is a diagonal transfer matrix in which the diagonal element of r^{th} mode is defined by

$$H_{rr}(\omega) = \frac{\omega^2}{\omega^2 - \omega_r^2 - 2\ i\ \zeta_r\ \omega\ \omega_r} \tag{19}$$

Eliminating $\{\boldsymbol{\eta}_1\}$ from equations (18) and (17), it follows

$$\begin{gathered}\{\mathbf{R}_2(\omega)\} = \\ -\omega^2\ \left([\mathbf{M}]_{22}^* - [\mathbf{\Gamma}]^T\ [\mathbf{\Gamma}]\ [\,^{\backprime} m_r \,_{\backslash}]^{-1}\ [\,^{\backprime}\ \mathbf{H}(\omega) \,_{\backslash}]\right)\ \{\boldsymbol{\eta}_2(\omega)\}\end{gathered} \tag{20}$$

The matrix defined by

$$[\widetilde{\mathbf{M}}] = [\mathbf{\Gamma}]^T\ [\mathbf{\Gamma}]\ [\,^{\backprime} m_r \,_{\backslash}]^{-1} \tag{21}$$

is the effective modal mass matrix.

Owing to equation (14), one obtains

$$[\mathbf{\Gamma}]^T\,[\mathbf{\Gamma}]\,[\,\grave{}\, m_r \,\lrcorner]^{-1} = [\mathbf{M}]^*_{22} - [\mathbf{M}]_{22} + [\mathbf{M}]^T_{12}\,[\mathbf{M}]^{-1}_{11}\,[\mathbf{M}]_{12} \tag{22}$$

In many cases the term $[\mathbf{M}]_{12}$ may be neglected (it is strictly zero if the mass matrix is diagonal) and thus the term $[\mathbf{M}]^T_{12}\,[\mathbf{M}]^{-1}_{11}\,[\mathbf{M}]_{12}$ is of second order and is no longer involved into equation (22), i.e.

$$[\mathbf{\Gamma}]^T\,[\mathbf{\Gamma}]\,[\,\grave{}\, m_r \,\lrcorner]^{-1} \simeq [\mathbf{M}]^*_{22} - [\mathbf{M}]_{22} \tag{23}$$

The concept of effective modal masses is very useful to select the system eigenmodes which have dominant participation to the support motion response.

2.3. The dynamic mass concept

The dynamic mass matrix is defined as the transfer function between the reaction force applied by the table on the structure and the table acceleration, i.e.

$$\{\mathbf{r}_2\} = [\mathbf{M}_D]\,\{\ddot{\mathbf{q}}_2\} \tag{24}$$

or, in the frequency domain

$$\{\mathbf{R}_2(\omega)\} = -\omega^2\,[\mathbf{M}_D(\omega)]\,\{\boldsymbol{\eta}_2(\omega)\} \tag{25}$$

From equations (20) and (25), it can be deduced that the dynamic mass matrix is related to the effective modal mass matrix through the relation

$$[\mathbf{M}_D(\omega)] = [\mathbf{M}]^*_{22} - [\widetilde{\mathbf{M}}]\,[\,\grave{}\,\mathbf{H}(\omega)\lrcorner] \tag{26}$$

3. ENVIRONMENTAL TESTING

3.1. Description of the test facility

The main vibration test facility installed at the University of Liège is a LING ELECTRONICS shaker of 7350 daN peak sine force. The shaker is coupled to a KIMBALL slip table mounted on horizontal hydrostatic bearing line. It allows random, sine and shock excitations in the frequency range of 5 Hz to 3000 Hz. The data acquisition is done using a HP 9000 workstation and a DAC-ADC HP3565-S with up to 24 channels. The control and modal identification software used is CADA-X from LMS International.

3.2. Electro-dynamic model of the shaker

The electro-dynamic shaker can be modeled as the lumped-parameter electro-mechanical system shown in figure 2. It consists of a work-table constrained to move in the x-direction in a magnetic field. The table, where the test object is mounted, is supported by an equivalent spring and damper system. The exciting signal is provided to the drive coil by the inductive electrical circuit shown in figure 2. Applying Kirchhoff's voltage law to the drive coil electrical loop, we obtain :

$$v = R\,i + L\,\frac{di}{dt} + B\ell\,\dot{x} \tag{27}$$

The mechanical equation of motion of the system is :

$$(m_s + m_t)\,\ddot{x} + c_s\,\dot{x} + k_s\,x = B\ell\,i \tag{28}$$

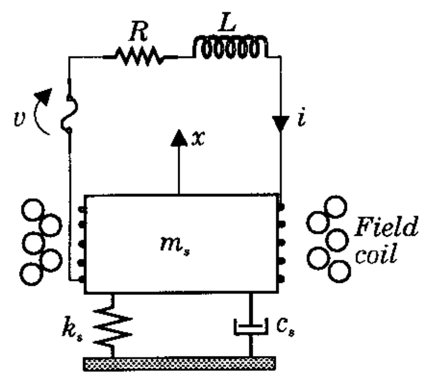

Fig. 2: Electro-dynamic model of the shaker

In the frequency domain, equations (27) and (28) become respectively

$$V(\omega) = R\,I(\omega) + i\,\omega\,L\,I(\omega) + i\,\omega\,B\ell\,X(\omega) \tag{29}$$

$$-(m_s + m_t)\,\omega^2\,X(\omega) + i\,\omega\,c_s\,X(\omega) + k_s\,X(\omega) = B\ell\,I(\omega) \tag{30}$$

The transfer function between the acceleration at the shaker table and the voltage is defined as

$$H_V(\omega) = \frac{\omega^2\,X(\omega)}{V(\omega)} \tag{31}$$

The acceleration at the table and the voltage were measured simultaneously in the whole frequency range during a sine-sweep vibration test without any payload. The identification of the electro-mechanical parameters of the shaker from the measured transfer function (31) gave the following results :

$$\begin{array}{lcll} m_s & = & 161 & \text{kg} \\ k_s & = & 125\,488 & \text{N/m} \\ c_s & = & 10\,925 & \text{N}\times\text{ s/m} \\ B\ell & = & 38.19 & \text{Wb}\times\text{ m /m}^2 \\ L & = & 8.115\;10^{-6} & \text{H} \\ R & = & 0.028 & \Omega \end{array} \tag{32}$$

4. MODAL IDENTIFICATION USING ENVIRONMENTAL VIBRATION

For the purpose of modal identification of parameters, the frequency response function (FRF) of the structure at coordinate j due to excitation at coordinate k is usually written in the general form :

$$\begin{aligned} H_{jk}(\omega) &= \frac{X(\omega)}{F(\omega)} \\ &= \sum_{r=1}^{m} \frac{{}_rA_{jk}}{i\,\omega - \lambda_r} + \frac{{}_rA^*_{jk}}{i\,\omega - \lambda^*_r} \end{aligned} \tag{33}$$

Thus identification can be performed using classical methods such as the least squares complex exponential method (L.S.C.E.), ... [6].

When environmental vibration testing is being performed on a shaker table, the base of the structure is submitted to a rigid body displacement of amplitude x in the direction of the excitation $\{\mathbf{e}_x\}$, i.e.

$$\{\mathbf{q}_2\} = \{\boldsymbol{\eta}_2\} = x\,\{\mathbf{e}_x\} \tag{34}$$

In order to perform identification with measurement data collected during an environmental vibration test, the FRF has to be written in the following form :

$$\begin{aligned} H'_{jk}(\omega) &= \frac{Z(\omega)}{X(\omega)} \\ &= \sum_{r=1}^{m} \frac{{}_rA'_{jk}}{i\,\omega - \lambda_r} + \frac{{}_rA'^*_{jk}}{i\,\omega - \lambda^*_r} \end{aligned} \tag{35}$$

where $Z(\omega)$ represents the relative displacement of the test structure.

5. THE EXPERIMENTAL DETERMINATION OF THE DYNAMIC MASS [5]

From equations (20) and (23), the reaction forces can be written in the form

$$\{\mathbf{R}_2(\omega)\} = -\omega^2\left([\mathbf{M}]_{22} + [\widetilde{\mathbf{M}}] - [\widetilde{\mathbf{M}}]\,[\,^{\backslash}\,\mathbf{H}(\omega)\,_{\backslash}\,]\right)\{\boldsymbol{\eta}_2(\omega)\} \tag{36}$$

By projection of equation (26) in the direction of the excitation (ox), it follows

$$\begin{aligned} M_{D,x}(\omega) &= \{\mathbf{e}_x\}^T\,[\mathbf{M}_D(\omega)]\,\{\mathbf{e}_x\} \\ &= m_2^* - \sum_{r=1}^{m} \widetilde{m}_r\left(\frac{\omega^2}{\omega^2 - \omega_r^2 - 2\,i\,\zeta_r\,\omega\,\omega_r}\right) \\ &= m_2 + \sum_{r=1}^{m} \widetilde{m}_r\left(1 - \frac{\omega^2}{\omega^2 - \omega_r^2 - 2\,i\,\zeta_r\,\omega\,\omega_r}\right) \end{aligned} \tag{37}$$

where

$$m_2^* = \{\mathbf{e}_x\}^T\,[\mathbf{M}]^*_{22}\,\{\mathbf{e}_x\} \tag{38}$$

is the total structural mass associated to the rigid body displacement,

$$m_2 = \{\mathbf{e}_x\}^T\,[\mathbf{M}]_{22}\,\{\mathbf{e}_x\} \tag{39}$$

is the mass associated to the support.

It is worthwhile noticing that

$$\begin{aligned} m_2^* - m_2 &= \{\mathbf{e}_x\}^T\,[\widetilde{\mathbf{M}}]\,\{\mathbf{e}_x\} \\ &= \{\mathbf{e}_x\}^T\,[\boldsymbol{\Gamma}]^T\,[\boldsymbol{\Gamma}][\,^{\backslash}\,m_r\,_{\backslash}\,]^{-1}\,\{\mathbf{e}_x\} \\ &= \sum_{r=1}^{m} \frac{(\{\boldsymbol{\Gamma}\}_r\{\mathbf{e}_x\})^2}{m_r} = \sum_{r=1}^{m} \widetilde{m}_r \end{aligned} \tag{40}$$

In this equation, each term of the sum represents the effective mass of the corresponding mode. It is important to note that the effective modal mass concept is related to a given direction of motion. In equation (40), the summation is expanded to the number of modes defined in the direction of the excitation. For purpose of modal parameter identification, equation (37) may also be written in the form

$$M_{D,x}(\omega) = m_2 + \sum_{r=1}^{m} \frac{{}_rA'_{jk}}{i\,\omega - \lambda_r} + \frac{{}_rA'^*_{jk}}{i\,\omega - \lambda^*_r} \tag{41}$$

with

$$\begin{aligned} \lambda_r &= -\zeta_r\,\omega_r + i\,\omega_r\,\sqrt{1-\zeta_r^2} \\ {}_rA'_{jk} &= \widetilde{m}_r\,\omega_r\left(\zeta_r + i\,\frac{2\,\zeta_r^2 - 1}{2\,\sqrt{1-\zeta_r^2}}\right) \end{aligned} \tag{42}$$

and the effective modal mass of r^{th} mode is related to the modal parameters by the relation

$$\widetilde{m}_r = \frac{2\,|{}_rA'_{jk}|\,\sqrt{1-\zeta_r^2}}{\omega_r} \tag{43}$$

Refering to figure 3, the reaction force applied from the table on the structure in the direction of the excitation is also given by

$$r_{2,x} = B\ell\,i - (m_s + m_t)\,\ddot{x} - c_s\,\dot{x} - k_s\,x \tag{44}$$

or, in the frequency domain,

$$\begin{aligned} R_{2,x}(\omega) &= B\ell\,I(\omega) + \omega^2\,(m_s + m_t)\,X(\omega) \\ &\quad - i\,\omega\,c_s\,X(\omega) - k_s\,X(\omega) \end{aligned} \tag{45}$$

Using equations (25) and (29), it follows

$$\begin{aligned} R_{2,x}(\omega) &= -\omega^2\,M_{D,x}(\omega)\,X(\omega) \\ &= B\ell\left(\frac{V(\omega) - i\,\omega\,B\ell\,X(\omega)}{R + i\,\omega\,L}\right) + \\ &\quad \omega^2\,(m_s + m_t)\,X(\omega) - i\,\omega\,c_s\,X(\omega) - k_s\,X(\omega) \end{aligned} \tag{46}$$

Equation (46) allows to determine the frequency evolution of the dynamic mass by measuring simultaneously the voltage and the table acceleration. Thus the effective modal masses for each identified mode in the direction of the excitation can be found from equation (43) after performing a modal parameter extraction on the basis of equation (41).

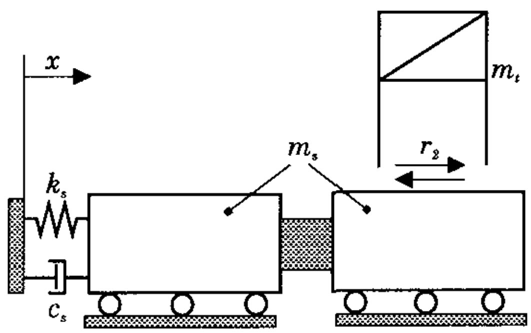

Fig. 3: Environmental vibration testing

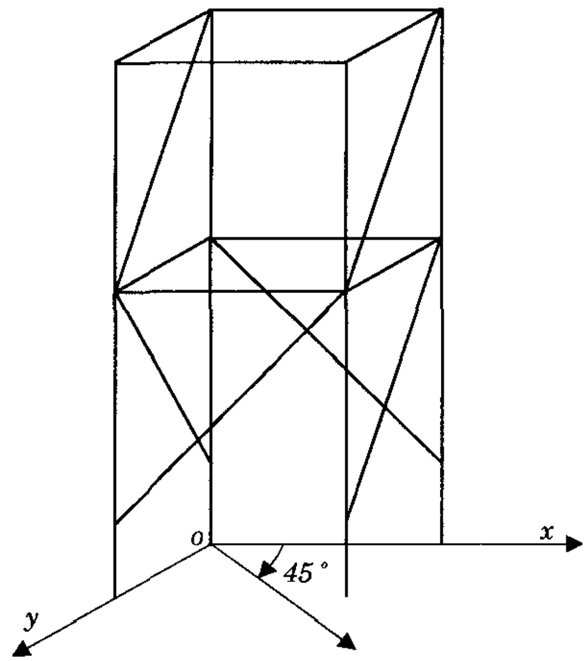

Fig. 4: Three-dimensional beam-like structure

6. PRACTICAL APPLICATION EXAMPLE

The procedure described in the previous section was applied to the three-dimensional beam-like structure shown in figure 4. The structure was tested using a sine-sweep vibration in the frequency range from 5 Hz to 500 Hz. The experimental dynamic masses deduced from equation (46) using the measured voltage and table acceleration are plotted in figure 5 for each test direction. The experimental and theoretical frequencies of the structure are listed in table 1 along with the corresponding damping coefficients. Tables 2 to 4 give the effective modal masses associated to each identified mode in the direction of the excitation (ox, oy and at an angle of 45° respectively). The experimental effective modal masses were found to be in good agreement with the theoretical effective modal masses obtained by a theoretical computation using the finite element program Samcef [7].

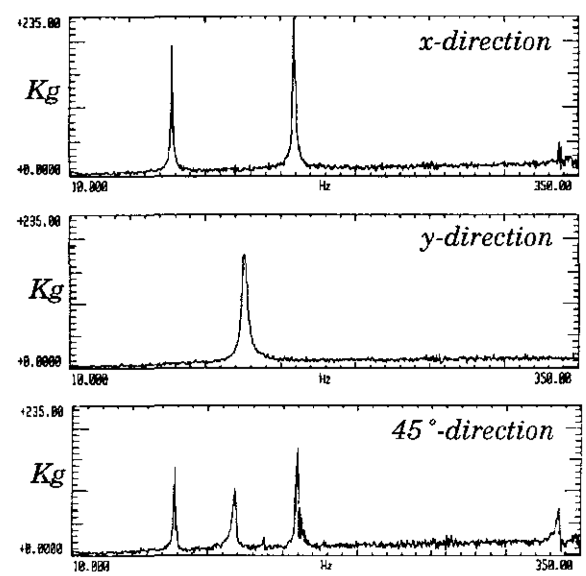

Fig. 5: Experimental dynamic mass

Table 1 : identified modal parameters

Mode n°	Theoretical frequencies (Hz)	Measured frequencies (Hz)	Damping coefficients (%)
1	74.1	77.5	0.45
2	128.8	77.5	0.45
3	140.0	139.2	0.61
4	159.9	159.0	0.27
5	182.2	186.3	0.28
6	195.5	209.4	0.23
7	227.6	244.5	0.59
8	240.2	258.7	0.79
9	277.3	296.6	0.23
10	306.4	332.0	0.22

Table 2 : Effective modal masses in x-direction

Mode n°	Theoretical (f.e.m.) mass (kg)	Experimental mass (kg)
1	1,54 (42.6 %)	1,65 (46.1 %)
4	1.63 (46.0 %)	1.46 (40.8 %)
10	0.15 (4.2 %)	0.15 (4.2 %)
Total	3.32 (92.8 %)	3.26 (91.1 %)

Table 3 : Effective modal masses in y-direction

Mode n°	Theoretical (f.e.m.) mass (kg)	Experimental mass (kg)
2	3.32 (92.8 %)	3.33 (93.1 %)

Table 4 : Effective modal masses in 45°-direction

Mode n°	Theoretical (f.e.m.) mass (kg)	Experimental mass (kg)
1	0.77 (21.5 %)	0.79 (22.1 %)
2	1.62 (45.3 %)	1.54 (43.0 %)
4	0.86 (24.0 %)	0.81 (24.6 %)
10	0.07 (2.0 %)	0.08 (2.2 %)
Total	3.32 (92.8 %)	3.22 (91.9 %)

7. CONCLUSION

An experimental determination of the effective modal masses of a structure submitted to a sine-sweep vibration on an electro-dynamic shaker was presented. The procedure is based on the identification of the electro-dynamic parameters of the shaker and the measurement of the voltage fed to the drive coil of the exciter. As the proposed method *does not require sophisticated instrumentation*, it appears to be very helpful either to verify theoretical results or to be used for mechanical structures for which finite element modelisation is not available or cannot be economically justified.

8. REFERENCES

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