Gelfand duality for compact pospaces

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Let X be a compact Hausdorff space. It is well known that X can be characterized by its ring of real continuous functions, by its set of regular open subsets or more simply by its set of open subsets. These characterizations lead to dualities between the category **KHaus**, of compact Hausdorff space and respectively the categories **C*-alg** (or equivalently **ubal**), of commutative $C^\star$-algebras, **DeV** of de Vries algebras and **KRFrm** of compact regular frames. We thus get a square of dualities. (see [1], [2] and [6]).

Later, G.Bezhanishvili and J.Harding extended in [1] a part square to dualities between the categories **StKSp** of stably compact spaces, **RPrFrm** of regular proximity frames and **StKFrm** of stably compact frames.

We thus get the square of dualities extended this way.

$$\begin{tikzcd}
\mathbf{StKSp} \arrow[rrr, dash] \arrow[dr, dash] & & & \mathbf{RPrFrm} \arrow[dl, dash] \arrow[ddd, dash] \\
& \mathbf{KHaus} \arrow[r, dash] \arrow[d, dash] & \mathbf{DeV} \arrow[d, dash] & \\
& \mathbf{C^\star\text{-}alg} \arrow[r, dash] & \mathbf{KRFrm} \arrow[dr, dash] & \\
& & & \mathbf{StKFrm}
\end{tikzcd}$$

Our aim is to complete the outside triangle, looking for a category generalizing the $C^\star$-algebras.

Using the equivalences between **StKSp** and the category **KPSp** of compact po-spaces (see [4]), an essential fact, due to G.Hansoul in [5] leads us to consider a category of ordered semi-ring. Indeed, we can see that the Nachbin-Stone-Cech compactification of a completely regular ordered po-space X can be realized through its semi-ring of increasing, continuous and real, positive functions, denoted $I(X, \mathbb{R}^+)$.

Following the definitions of G.Bezhanishvili, P.Morandi and B.Olberding in [2], we define the bounded Archimedean ℓ-semi-algebras this way.

Definition 1.

1. An ℓ-semi-ring is an algebra $(A, +, ., 0, 1, \leq)$ with the following axioms :
   - (a) $(A, +, 0)$ and $(A, ., 1)$ are commutative monoids.
   - (b) $(A, +, .)$ is distributive.
   - (c) $a \leq b \Leftrightarrow a + c \leq b + c$.
   - (d) $a \geq 0$ and $a \leq b \Rightarrow a.c \leq b.c$
   - (e) $(A, \leq)$ is a lattice.
2. An ℓ-semi-ring A is bounded if for all $a \in A$, there is $n \in \mathbb{N}$ such that $a \leq n.1$.
3. An ℓ-semi-ring A is Archimedean if for all $a, b, c, d \in A$, whenever $n.a + b \leq n.c + d$, then $a \leq c$.

4. An ℓ-semi-ring A is an ℓ-semi-algebra if it is an $\mathbb{R}^+$-algebra such that for all $a, b \in A$ and $r \in \mathbb{R}^+$, $r.a \leq r.b$.

5. $(a \vee b) + c = (a + c) \vee (b + c)$ and $(a \wedge b) + c = (a + c) \wedge (b + c)$.

We now denote **sbal** the category of bounded Archimedean ℓ-semi-algebras, and defining the morphisms in the natural way.

In order to get the missing duality, we define the $\sim$-relation on $A \times A$, with A an sbal, such as

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c,$$

allowing us to construct the functor $\cdot^b : \mathbf{sbal} \longrightarrow \mathbf{bal}$ which sends A to $A \times A/\sim$. In particular, this functor enable us to easily transfer structures from rings to semi-rings.

With all these tools, we propose the following functors between **KPSp** and **sbal** : the first functor, denoted I, sends a compact po-space X to the set $I(X, \mathbb{R}^+)$ and a continuous increasing function $f : X \longrightarrow Y$ between compact po-spaces to

$$f_\star : I(Y, \mathbb{R}^+) \longrightarrow I(X, \mathbb{R}^+) : g \longmapsto g \circ f.$$

On the other side, the second functor, denoted χ, maps a sbal A to its set of ℓ-congruences, denoted X_A and a morphism $\alpha : A \longrightarrow B$ between sbals to

$$\alpha^\star : X_B \longrightarrow X_A$$

such that, if $\theta \in X_B$, $(a, b) \in \alpha^\star(\theta)$ if and only if $(\alpha(a), \alpha(b)) \in \theta$.

Definition 2. An ℓ-semi-ring A admits difference with constants if for all $a \in A$ and $r \in \mathbb{R}^+$ $a \leq r.1$ implies there is $b \in A$ such that $a = b + r.1$. It is uniformly complete if it is complete for the norm $||a|| = \inf\{\lambda \in \mathbb{R} : a \leq \lambda.1\}$. We then denote **usbal** the full subcategory of **sbal** whose objects are the uniformly complete bounded Archimedean ℓ-semi-algebras with difference with constants.

Theorem 3. *The functors χ and I establish a dual equivalence between* **usbal** *and* **KPSp**

References

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