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STRONG COULOMB COUPLING IN RELATIVISTIC QUANTUM CONSTRAINT DYNAMICS

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We study, in the framework of relativistic quantum constraint dynamics, the bound state problem of two oppositely charged spin 1/2 particles, with masses m_1 and m_2, in mutual electromagnetic interaction. We search for the critical value of the coupling constant α for which the bound state energy reaches the lower continuum, thus indicating the instability of the heavier particle or of the strongly coupled QED vacuum in the equal mass case. Two different choices of the electromagnetic potential are considered, corresponding to different extensions of the substitution rule into the nonperturbative region of α: (i) the Todorov potential, already introduced in the quasipotential approach and used by Crater and Van Alstine in *Constraint Dynamics*; (ii) a second potential (potential II), characterized by a regular behavior at short distances. For the Todorov potential we find that for $m_2 > m_1$ there is always a critical value α_c of α, depending on m_2/m_1, for which instability occurs. In the equal mass case, instability is reached at $\alpha_c = 1/2$ with a vanishing value of the cutoff radius, generally needed for this potential at short distances. For potential II, on the other hand, we find that instability occurs only for $m_2 > 2.16\, m_1$.

1. Introduction

Strong external Coulomb fields are known to be unstable under spontaneous electron–positron pair creation.[1] It is naturally expected that when the mass of the heavy particle, responsible for the external strong Coulomb field, becomes finite but still large enough, the aforementioned phenomenon should persist. This situation is best described by means of a two-particle bound system, where particle 1 is the electron (with mass m_1) and particle 2 is the positively charged heavy particle (with mass m_2), interacting electromagnetically through a strong Coulomb coupling. When the coupling constant α increases, the bound state mass decreases and, for a critical value α_c of α (~ 1), it reaches the lowest value $(m_2 - m_1)$. This

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is an indication that for $\alpha = \alpha_c$ the positively charged heavy particle may spontaneously emit, without loss of energy, an electron–positron pair, the electron having the binding energy $-2m_1$ and the positron being free with zero kinetic energy. The phenomenon is amplified for $\alpha > \alpha_c$.

The theoretical question then arises as to whether this phenomenon would persist if the mass of the heavy particle became continuously smaller. In particular, in the limit $m_2 = m_1$, one would obtain, for the corresponding critical coupling constant, a zero mass bound state, having the same quantum numbers as a pseudoscalar boson, which might lead to a spontaneous breakdown of chiral symmetry in QED.

Therefore, the study of the bound state spectrum of two oppositely charged particles, interacting electromagnetically through a strong Coulomb coupling, with the search for a massless bound state in the equal mass case, is a key probe of the possible existence of a phase transition in QED.

In this connection, lattice calculations of quenched QED seem to predict a new phase of QED for $\alpha > \alpha_c$ (~ 0.3), with spontaneous chiral symmetry breaking.[2] This conclusion is supported by the solution of the Bethe–Salpeter equation in the ladder approximation.[3,4] These results would imply an instability of the QED vacuum itself accompanied by the occurrence of a zero mass bound state in the two-body equal mass case.

The purpose of this paper is to investigate, in the framework of relativistic quantum constraint dynamics,[5–8] the stability properties, with respect to spontaneous pair creation, of charged particles heavier than the electron as well as of the QED vacuum. This is done by studying the bound state problem of two oppositely charged spin 1/2 particles with masses m_1 and m_2 in mutual electromagnetic interaction and searching for the possibility that the bound state mass reaches the lower continuum.

One of the main advantages of relativistic quantum constraint dynamics is that it provides a three-dimensional manifestly covariant description of the internal motion of the two-body system, once the redundant relative energy variable is eliminated by means of the constraint. These equations correctly reproduce the muonium and positronium spectra[8] up to terms of order α^4. Furthermore, one can establish a correspondence between the Feynman diagrams of the kernel of the Bethe–Salpeter equation and the interaction potential of the constraint theory wave equations.[9] When one of the particles becomes infinitely massive, the two-body wave equations reduce to the one-body Dirac or Klein–Gordon equation in an external Coulomb field.

Having the correct Dirac (or Klein–Gordon) limit, relativistic constraint dynamics necessarily implies that for sufficiently large values of one of the particle masses, the heavier particle becomes unstable with respect to spontaneous pair creation above some critical value α_c of the coupling constant α, as is the case for strong external Coulomb fields[1] interacting with Dirac or Klein–Gordon particles.

Specifically, we wish to answer the following questions:

(1) How do the stability properties depend on the masses m_2 and m_1 ($m_2 \geq m_1$) of the charged particles? In particular, does instability under spontaneous pair creation, valid in the case of large mass asymmetries, also exist for the QED vacuum, i.e. in the equal mass case?
(2) Do the stability properties of a bound state in relativistic constraint dynamics depend on the specific way the interaction is introduced in the formalism?

The second question arises because introduction of the electromagnetic interaction in the nonperturbative regime in relativistic constraint dynamics is not unique. Extension of the minimal substitution rule to higher orders depends on the stage at which the latter is applied. In particular, two different expressions of the interaction potential are found, depending on whether one applies the substitution rule in the final eigenvalue equation of the relative motion, or in the initial individual wave equations of relativistic constraint dynamics. In the first case, one finds the Todorov potential, already introduced in the quasipotential approach[10] and later used by Crater and Van Alstine and their collaborators in relativistic constraint dynamics.[6,8] The main feature of the Todorov potential is that it is dominated by the one-photon-exchange contribution even for large values of the coupling constant and necessitates in this domain the use of a short distance regularization cutoff. In the second case, another potential is found, the main feature of which is to be less dominated by the one-photon-exchange contribution in the large coupling regime, exhibiting a regular behavior at short distances. This potential will henceforth be designated as "potential II."

The Todorov potential in the equal mass case was studied in its nonperturbative regime by two of us.[11] It was found that the strongly coupled positronium remains stable for any value of the coupling constant α, as long as a regularization cutoff is used for the short distance singularity. In the zero cutoff limit, however, one finds zero energy solutions for $\alpha > 1/2$, which can be interpreted as a signal of vacuum instability.

The main results of the present paper can be summarized as follows:

(1) Potential II yields a ground state c.m. energy E such that, in the equal mass case ($m_2 = m_1 = m$), one has $E > 2m/e$, for all values of the coupling constant α, thus leading to a stable vacuum. Only for $m_2 \geq 2.16\ m_1$ do we find that there is a critical value of α above which particle 2 becomes unstable under spontaneous pair creation.
(2) For the Todorov potential, there exists a critical value α_c for any $m_2 > m_1$. As particular examples, we find that

$$\alpha_c \simeq \frac{1}{2} \quad \text{for} \quad m_2 \simeq m_1\,, \qquad \alpha_c \simeq 0.75 \quad \text{for} \quad m_2 \simeq 4m_1\,, \tag{1.1}$$

for $mr_0 \simeq 10^{-3}$, r_0 being a regularization radius needed for this form of interaction.[11] In the equal mass case, instability occurs for a vanishing value of the cutoff radius r_0.

Thus, the Todorov potential in relativistic constraint dynamics points to the onset of a new phase of QED (with a condensate of particle–antiparticle pairs in the ground state) for sufficiently strong coupling constants, in qualitative agreement with the results obtained from lattice calculations of quenched QED[2] and the Bethe–Salpeter equation in the ladder approximation.[3,4]

Although the main aim of this paper is to investigate whether a reasonable relativistic two-body wave equation can reproduce the instability phenomenon (with respect to α) of *quenched* QED, let us, for the sake of completeness, comment on the possible influence of radiative corrections on the above results. In lattice QED, deviations from the quenched theory have been estimated by Göckeler *et al.*:[12] for $\alpha \simeq 0.5$ they are of the order of 20% — a typical order of magnitude — but these authors conclude, for other reasons, that QED, considered as the limit of a cutoff theory formulated on the lattice, is inconsistent for $\alpha > 1/50$.

In the continuum theory, it has been known for a long time[13,14] that when the coupling constant is an ultraviolet stable fixed point of the theory, then the photon propagator behaves at short distances as a free propagator and contributions of vacuum polarization become negligible in the first approximation. In that case the electron bare mass vanishes and its physical mass is entirely of dynamical origin, with a possible spontaneous chiral symmetry breaking.[15] This is precisely the situation that is expected from the occurrence of the critical Coulomb coupling α_c. For this reason it is generally thought, although not yet proved, that α_c, if it exists, must at the same time be an ultraviolet stable fixed point of the theory.[2,4,16] When this circumstance is achieved, vacuum polarization contributions can be neglected for $\alpha \sim \alpha_c$.

The situation is different for the case of potential II, which does not lead to any phase transition. In this case, however, the multiphoton exchange diagrams are assumed to sum up in such a way that the resulting potential is regular at the origin. Radiative corrections have the tendency to increase the effective coupling constant at short distances. This, however, has no *qualitative* effect on the potential, since the latter is regular for *any* α.

A more accurate study in the present framework of the influence of vertex corrections could be done by the introduction of effective anomalous magnetic moments of fermions. The presence of such terms modifies, however, the structure of the wave equations we are considering, by introducing in them new types of interaction. We intend to analyze these effects in a future work.

2. Basic Equations of Relativistic Quantum Constraint Dynamics

The wave equations of relativistic quantum constraint dynamics for two particles of masses m_1 (fermion) and m_2 (antifermion) in mutual interaction are given by[7]

$$(\gamma_1 \cdot p_1 - m_1)\tilde{\Psi} = -(\gamma_2 \cdot p_2 - m_2)\tilde{V}\tilde{\Psi}\,, \tag{2.1}$$

$$(\gamma_2 \cdot p_2 + m_2)\tilde{\Psi} = -(\gamma_1 \cdot p_1 + m_1)\tilde{V}\tilde{\Psi}\,, \tag{2.2}$$

where the indices 1 and 2 refer to particles 1 and 2, respectively; p_1 and p_2 are the four-momentum operators and γ_1 and γ_2 are the Dirac matrices relative to particles 1 and 2. $\tilde{V}$ is a Poincaré-invariant interaction potential. The wave function $\tilde{\Psi}$ is a 4×4 matrix function and the matrices γ_2, acting on the antifermion indices, act on $\tilde{\Psi}$ from the right. The properties of these wave equations are studied in the Appendix.

The potential $\tilde{V}$ is parametrized according to the expression[17]

$$\tilde{V} = \tanh\left[\frac{1}{2}\,C(\gamma_1\cdot\gamma_2)\right], \tag{2.3}$$

where the Feynman gauge has been chosen in the lowest order of the interaction. To this order the new potential C is related to the photon propagator. (The factor 1/2 has been introduced for convenience.)

The requirement that Eqs. (2.1)–(2.3) possess the correct $O(\alpha^4)$ nonrelativistic limit in known vector interactions and the Dirac limit when one of the masses becomes infinite does not, however, uniquely determine C. We shall discuss below two possible choices of C suggested by the substitution rule.

Equations (2.1)–(2.3) can be solved by decomposing the wave function $\tilde{\Psi}$ along 2×2 matrix components and eliminating these with respect to one of them. For the sector of solutions with quantum numbers $j=\ell=s=0$, the reduced wave function φ satisfies in the c.m. frame the eigenvalue equation (see Appendix)

$$\left[\frac{E^2}{4}\,e^{4C} - \frac{1}{2}\,(m_1^2+m_2^2)e^{2C} + \frac{(m_1^2-m_2^2)^2}{4E^2} + \nabla^2 - 4r^2h'^2 + 6h' + 4r^2h''\right]\varphi = 0\,, \tag{2.4}$$

where one has

$$h = \ln\left[1 - \frac{(m_1-m_2)^2}{E^2}\,e^{-2C}\right]^{\frac{1}{2}}, \tag{2.5}$$

E being the c.m. value of the total energy,

$$E^2 = (p_1+p_2)^2 = P^2\,, \qquad P = (p_1+p_2)\,, \tag{2.6}$$

while r is the invariant c.m. distance (or transverse relative coordinate),

$$r^2 = -(x^T)^2\,, \tag{2.7}$$

$$x_\mu^T = x_\mu - \frac{P\cdot x}{P^2}\,P_\mu\,, \qquad x = x_1 - x_2\,, \tag{2.8}$$

and where

$$h' \equiv \frac{\partial h}{\partial(r^2)}\,. \tag{2.9}$$

The expression of potential C can be fixed with recourse to an extension of the minimal substitution rule. The result, however, depends on the stage at which this rule is applied. Two possibilities can be distinguished.

In the first place, one can apply the substitution rule in the final eigenvalue equation (2.4). Actually, this equation, without the last three terms, which are reminiscent of the fermionic nature of the constituents, is similar to the quasi-potential equation obtained by Todorov[10] in the spin 0 case. By redefinitions of mass and energy variables, one can bring Eq. (2.4) (without the last three terms) into a form similar to that of the Klein–Gordon equation of a fictitious particle of mass m_1m_2/E and energy $(E^2 - m_1^2 - m_2^2)/2E$ in the presence of an external electromagnetic field. The electric part of the latter is then identified with the Coulomb potential. With the variables we are using, this procedure amounts to identifying the term $\frac{E^2}{4}e^{4C}$ of Eq. (2.4) with $\left(\frac{E}{2} - V\right)^2$, where V is just the Coulomb potential:

$$V = -\frac{\alpha}{r}\,. \tag{2.10}$$

This identification leads to the following expression of C (henceforth designated as the Todorov potential, or potential I):

$$C = \frac{1}{2}\ln\left(1 - \frac{2V}{E}\right). \tag{2.11}$$

The Todorov potential was extensively used in the literature by Crater and Van Alstine and their collaborators[6,8] in their spectroscopic evaluations of positronium, muonium and quarkonium spectra, in the framework of constraint dynamics.

In the second place, the substitution rule can also be used in the individual equations (2.1) and (2.2). To the lowest order of perturbation theory, the expression of C can be determined from the Bethe–Salpeter kernel[9] (in the Feynman gauge). One finds that

$$C = -\frac{V}{E}\,, \tag{2.12}$$

V being the Coulomb potential (2.10). It is seen in Eqs. (2.1) and (2.2) [more explicitly, in Eqs. (A.4) and (A.5)] that the total energy E undergoes, to the lowest order in V, the modification $E \to E - V$. It is then natural to extend this substitution to higher orders, by demanding that in the expression of C itself E be replaced by $E - V$. This substitution then yields the following expression of C (henceforth designated as potential II):

$$C = -\frac{V}{E - V}\,, \tag{2.13}$$

where V is as defined in Eq. (2.10).

The passage from the lowest order expression (2.12) to the complete expression (2.13) of C can also be understood on the basis of the gauge invariance property of

Eqs. (2.1) and (2.2) combined with a definite behavior of the multiphoton exchange diagrams. We sketch here the main steps leading to the result (2.13). (A more detailed description of this aspect of the problem will be presented elsewhere.)

In the *leading infrared* approximation the effective potential C must be a function of the lowest order potential given on the right hand side of Eq. (2.12), the nth power of it in the series expansion of C corresponding to the leading contribution of the n-photon exchange diagrams. (Furthermore, the perturbation expansion in constraint theory, which is very similar to that of the quasipotential theory, seems to be free of spurious logarithms; this has been checked with two-photon exchange diagrams.[18]) In this approximation the gauge transformation operator of the wave function can explicitly be constructed. It is

$$U(\xi) = \exp\left[\frac{i\xi}{2}\left(fp^T \cdot x^T + x^T \cdot p^T f\right)\right], \qquad f = -\frac{\alpha}{2Er}, \tag{2.14}$$

where ξ is the gauge parameter of the usual covariant photon propagator. [$U(\xi)$ brings the wave function from the ξ gauge representation to the Feynman gauge representation.] This expression is obtained by first analyzing the structure of Eqs. (2.1) and (2.2) in the lowest order of the interaction with the photon propagator taken in an arbitrary covariant gauge. Then the group property of the gauge transformation and the fact that in the leading infrared approximation all operators and potentials must be functions of the lowest order potential, with a one-to-one correspondence between its power and the order of the perturbation expansion, completely determine $U(\xi)$. [The expression for $U(\xi)$ can also be obtained, in its leading order, by starting from the gauge transformation operator of the Bethe–Salpeter wave function[19] and then reducing it through its connection with constraint theory[9] to a three-dimensional form.]

The knowledge of this operator, together with certain assumptions about the structure of the multiphoton exchange diagrams (in the leading infrared approximation), allows one to set up differential equations for the potentials with respect to the gauge parameter ξ. In particular, the factorization assumption of the multiphoton contributions to products, with unknown multiplicative coefficients, generated by the one- and two-photon contributions, leads to the expression (2.13) of C.

Therefore, the shift $E \to E-V$ in the denominator of the lowest order expression (2.12) of the potential, which extends the minimal substitution rule concerning the total c.m. energy factor to higher orders, can also be understood as a consequence, in the leading infrared approximation, of a gauge-invariant summation of multiphoton exchange contributions, provided the above-mentioned factorization rule is assumed.

Comparing potentials (2.11) and (2.13) we notice that they formally coincide up to $O(V^2)$, and hence lead to the same $O(\alpha^4)$ effects in perturbation theory. For both choices of C, (2.11) and (2.13), Eq. (2.4) reproduces the correct $O(\alpha^4)$ muonium and positronium spectra for the ${}^1S_0$ sector and reduces to the Dirac equation for the (properly normalized) ground state radial wave function and its radial excitations when one of the masses becomes infinite.[8]

In the following, we first consider the case of potential II.

3. Potential II

Potential II corresponds to the following choice of the function C in Eq. (2.3):

$$C = -\frac{V}{E - V}, \tag{3.1}$$

where V is the Coulomb potential:

$$V = -\frac{\alpha}{r}. \tag{3.2}$$

We have solved Eq. (2.4) with (3.1) and (3.2). A particular feature of this potential is that it does not need any short distance regularization: no V-dependent term in Eq. (2.4) is singular. Figure 1 shows the variation of the lowest $^1S_0$ eigenvalue E with respect to α in the equal mass case. The quantity E approaches a constant value for large α values and remains positive for any α. This can be understood by noting that Eq. (2.4) defines a Schrödinger-like equation with effective potential V^{eff} given by

$$V^{\text{eff}} = -\frac{E^2}{4}(e^{4C} - 1) + m^2(e^{2C} - 1). \tag{3.3}$$

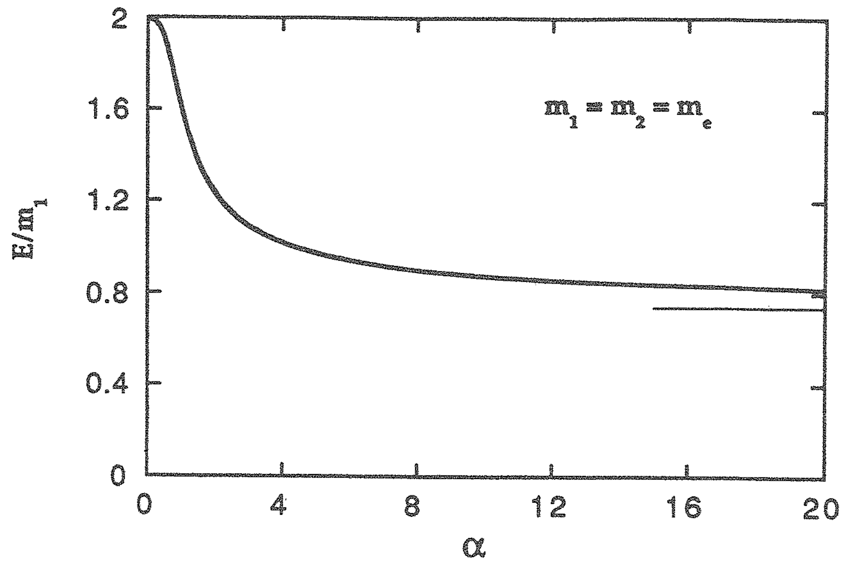

Fig. 1. Lowest $^1S_0$ eigenvalue E, as a function of the coupling constant for potential II between equal mass particles. The horizontal line indicates the bound (3.5b).

A necessary condition for $V^{\rm eff}$ to support a bound state is that $V^{\rm eff}$ be attractive (negative). From Eq. (2.4), one can see that for $|V| \gg E$ (we are interested in the behavior for large values of the coupling constant), one has $C \sim 1$, C', $C'' \sim 0$. The requirement that $V^{\rm eff}$ be negative then gives

$$E > \frac{2m}{\sqrt{1+e^2}}\,. \tag{3.4}$$

In fact, one can obtain a better lower bound by noting that when α tends to ∞, $C \to 1$ [see Eq. (3.1)] and h' and $h'' \to 0$ [see Eq. (2.5)]; there remains no term in Eq. (2.4), except the Laplacian, which depends upon r. Therefore, the lowest energy should correspond to a zero kinetic energy ($\langle \nabla^2 \rangle = 0$). This is possible, since then the effective potential extends over all space with the same value. One then gets

$$\frac{E^2}{4} e^4 - m^2 e^2 > 0\,, \tag{3.5a}$$

or simply

$$E > \frac{2m}{e}\,. \tag{3.5b}$$

Our numerical results are consistent with Eq. (3.5b) and suggest that the lower bound in (3.5b) is the limiting value of the energy for $\alpha \to \infty$.

Figure 2 shows how E varies with α in the unequal mass case for various values of m_2, while m_1 ($m_2 \geq m_1$) is kept fixed and equal to the electron mass. The function which is plotted there (and also in Fig. 3) is the quantity

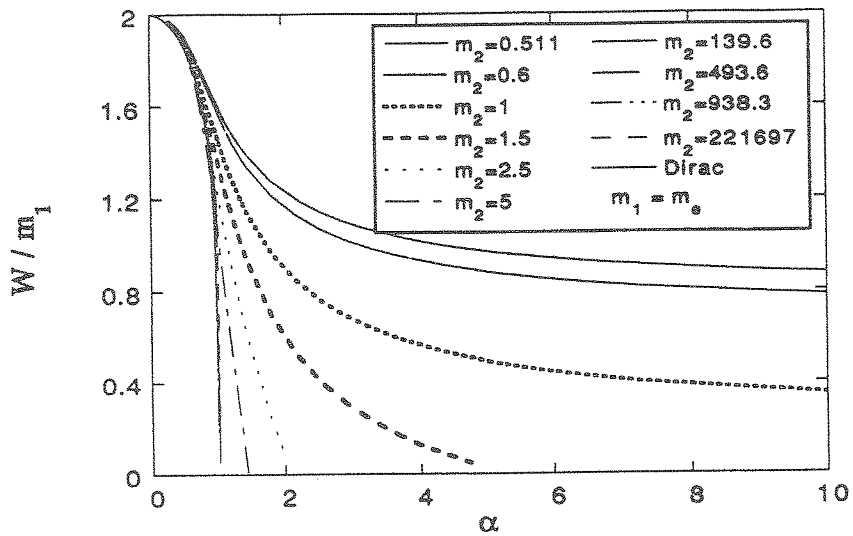

Fig. 2. Same as Fig. 1, but for the unequal mass case. The mass m_1 is taken equal to the electron mass. The ordinate quantity is equal to 2 when $E = m_2 + m_1$ (noninteracting limit) and equal to 0 when $E = m_2 - m_1$. (See text for details.)

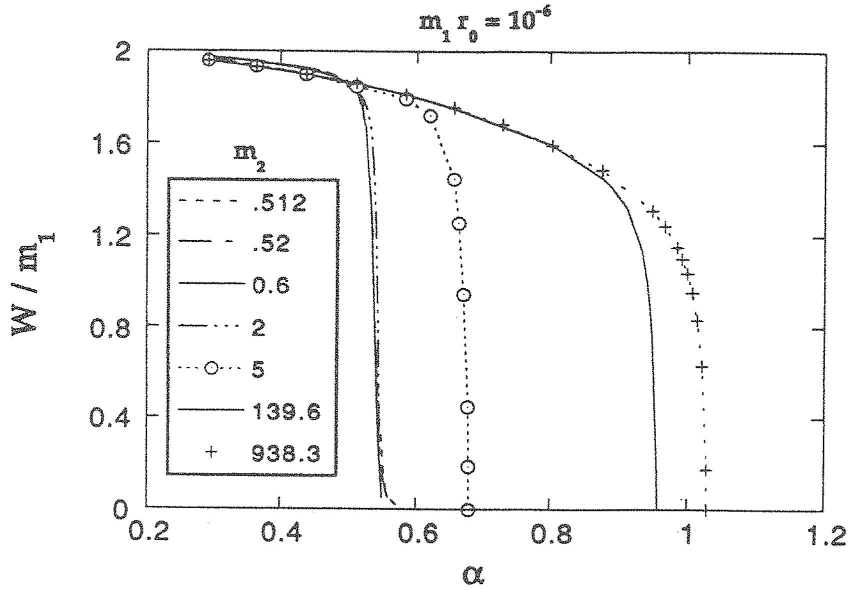

Fig. 3. Same as Fig. 2, but for the Todorov potential. A cutoff radius satisfying $m_1 r_0 = 10^{-6}$ has been chosen.

$$\frac{W}{m_1} = \frac{E - (m_2 - m_1)}{m_1}\,. \tag{3.6}$$

It should be noted that the upper continuum lies above $m_2 + m_1$ and the lower continuum lies below $m_2 - m_1$ [see Eq. (2.4) with $C = 0$]. Therefore the quantity (3.6) is always contained in the interval [0, 2] for a bound state.

One can see from Fig. 2 that, as long as m_2 is greater than a lower bound m_{20} (which will be specified below), there exists a critical value α_c of α, for which $E = m_2 - m_1$. This value of α is critical in the sense that when $\alpha = \alpha_c$, one can have the spontaneous decay

$$\mu^{\pm} \to (e^{\mp}\mu^{\pm}) + e^{\pm}\,, \tag{3.7}$$

where μ is the heavier particle (of mass m_2) and e is the lighter particle (of mass m_1), since the bound system $(e^-\mu^+)$ has energy $m_2 - m_1$. In the equal mass case, a zero energy state for the (e^+e^-) system would imply the instability of the vacuum itself, according to the same argument.

The existence of a lower bound m_{20} of m_2 for the occurrence of α_c can be understood by noting that in the unequal mass case the effective potential V^{eff} from Eq. (2.4) is

$$V^{\text{eff}} = -\frac{E^2}{4}\,e^{4C} + \frac{1}{2}\,(m_1^2 + m_2^2)e^{2C} + \text{terms in } h' \text{ and } h''\,. \tag{3.8}$$

For $|V| \gg E(\alpha \to \infty)$, terms in h' and h'' can be neglected, so that (3.8) yields, at the threshold value $E = m_2 - m_1$, the following condition for V^{eff} to be attractive:

$$m_2 - m_1 > \frac{\sqrt{2(m_1^2 + m_2^2)}}{\sqrt{1+e^2}}, \tag{3.9}$$

which leads to a lower bound m_{20} of m_2. Actually, m_{20} can be determined more accurately, by generalizing, for $m_1 \le m_2 \le m_{20}$, the reasoning following Eq. (3.4). One finds for the energy the bound

$$E > \frac{m_1 + m_2}{e}. \tag{3.10}$$

This bound actually corresponds to the limiting value $\alpha \to \infty$ (see Fig. 2). Therefore E is equal to $m_2 - m_1$ when

$$m_2 = m_{20} = m_1 \frac{e+1}{e-1} = 2.16\ m_1. \tag{3.11}$$

This particular value ($E = m_2 - m_1$) is obtained at $\alpha_c = \infty$ only. For larger values of m_2, α_c is finite and smaller. As m_2 increases, the critical value α_c decreases smoothly down to $\alpha_c = 1$, which is the limiting value for $m_2 \to \infty$, as expected from the static limit.

As noted above, these results disagree with studies of strongly coupled QED on the lattice or with the (ladder) Bethe–Salpeter equation, so that potential II is not appropriate for reproducing a transition to a "condensate" phase.

4. The Todorov Potential

The Todorov potential corresponds to the choice

$$C = \frac{1}{2} \ln\left(1 - \frac{2V}{E}\right), \tag{4.1}$$

with V given by Eq. (2.10). Because of the singularity in $1/r^2$, a cutoff radius is needed to solve Eq. (2.4) with (4.1) for values of α greater than 1/2. We adopt the cutoff

$$\begin{aligned} V(r) &= -\frac{\alpha}{r}, \qquad r > r_0, \\ &= -\frac{\alpha}{r_0}, \qquad r \le r_0. \end{aligned} \tag{4.2}$$

For $m_2 \neq m_1$, Eq. (2.4) leads to a critical value $\alpha = \alpha_c$, depending on m_2/m_1, for any value of m_2 ($> m_1$). Indeed, by definition, α_c is that value of α for which there exists a bound state of energy $E = m_2 - m_1$. In that case, Eq. (2.4) can be written with (4.1) as

$$\left[\frac{(m_2-m_1)^2}{4}\left(1-\frac{2V}{m_2-m_1}\right)^2-\frac{1}{2}(m_1^2+m_2^2)\left(1-\frac{2V}{m_2-m_1}\right)\right.$$
$$\left.+\frac{(m_1+m_2)^2}{4}+\nabla^2-4r^2h'^2+6h'+4r^2h''\right]\varphi=0\,, \tag{4.3}$$

with

$$h=\ln\left[1-\frac{(m_1-m_2)^2}{E(E-2V)}\right]^{\frac{1}{2}}. \tag{4.4}$$

Putting $V=-\frac{\alpha}{r}$ in Eq. (4.3), one sees that this equation contains a singular interaction term ($\sim \alpha^2/r^2$). It is well known that such a singular term implies an instability of the bound state equation for $\alpha > 1/2$.[1,20] The instability is not removed when a cutoff of the type (4.2) is introduced.[20]

However, Eq. (4.3) also contains the term $V(m_1^2+m_2^2)/(m_2-m_1)$, which prevents us from concluding at once that one also has $\alpha_c = 1/2$ for $m_2 = m_1$. The equal mass case was studied in detail in Ref. 11. As long as r_0 is different from 0, there is no critical value of α for $m_2 = m_1$ and E remains positive. On the other hand, one observes a critical value whenever $m_2 = m_1+\epsilon$, with ϵ arbitrarily small and positive. The function representing the lowest eigenvalue E versus α needs to be only slightly distorted when m_2 is slightly increased from $m_2 = m_1$ to have a crossing point with the line $E = m_2 - m_1$ (compare Fig. 3 with Fig. 1 of Ref. 11), and thus to present a critical point. One sees from Fig. 3 that $\alpha_c \approx 79/137$ for $m_2 = 0.6$ MeV. There is, however, a continuity of the energy eigenvalue as $m_2 \to m_1$. As is suggested in Fig. 3, for $m_2 = 0.512$ MeV the energy curve bends up when m_2 further decreases; the bending pushes the crossing point with the horizontal axis to larger and larger values of α. At the limit $m_2 = m_1$, one recovers a curve with no crossing point as in Ref. 11.

This is also illustrated in Fig. 4, which gives α_c as a function of m_2/m_1 for different values of m_1r_0. One can infer from dimensionless quantities that

$$\frac{W}{m_1}=f\left(\frac{m_2}{m_1},\,m_1r_0,\alpha\right). \tag{4.5}$$

[W is defined in Eq. (3.6).] For m_1r_0 small but different from 0, the critical value α_c tends to infinity as $m_2 \to m_1$, in agreement with Ref. 11 (Fig. 1). It displays a minimum for some value of m_2 and then increases when $m_2 \to m_1$. However, it is clear from Fig. 4 that the minimum value of α_c tends to 1/2 when the cutoff radius is made smaller and smaller. Similarly, the value of α_c at very large m_2 will tend to unity, in agreement with the Dirac limit. Our numerical results do not allow us to state with certainty what the function $\alpha_c(m_2/m_1)$ for $r_0 = 0$ is. In all likelihood, it starts at $\alpha_c = 1/2$ for $m_2 = m_1$ and goes continuously and smoothly to $\alpha_c = 1$ for $m_2 = \infty$.

It is interesting to note the completely different behavior of the function $\alpha_c(m_2/m_1)$ for potential II (Sec. 3). For this case, one has a continuous decrease of α_c when m_2 increases from the lower bound m_{20} [Eq. (3.11)], up to infinity (see Fig. 4).

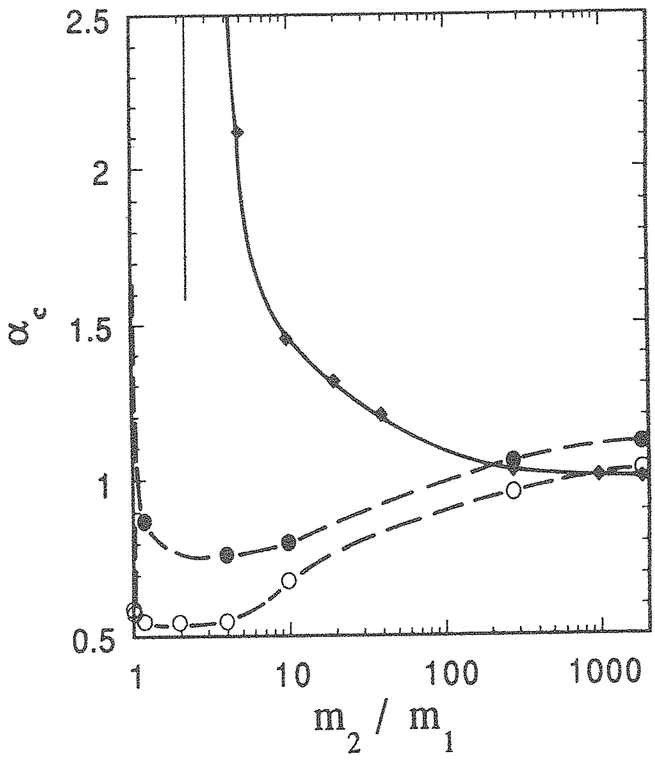

Fig. 4. Critical value α_c as a function of the ratio m_2/m_1. The dashed curves correspond to the Todorov potential, for two values of the cutoff radius: $m_1 r_0 = 10^{-3}$ for the upper curve and $m_1 r_0 = 10^{-6}$ for the lower curve. The full curve corresponds to potential II; the vertical line represents the lower bound m_{20} of m_2 [Eq. (3.11)].

5. Conclusion

We found, in the framework of relativistic quantum constraint dynamics, that the instability, due to spontaneous pair creation, in electromagnetically bound systems, depends crucially on the way electromagnetic interaction is extended to the strong coupling regime. The two potentials we considered coincide up to $O(\alpha^4)$ effects, but drastically differ in the nonperturbative region of the coupling constant α. The Todorov potential continues to be dominated for large α by the one-photon-exchange contribution and hence displays short distance singularities that are typical of the relativistic Coulomb potential. In potential II, the multiphoton exchange contributions add up in such a way that they regularize the potential at the origin.

The implications of these two potentials go in two different directions. The Todorov potential leads to instability for all values of the ratio m_2/m_1 and with $1/2 \lesssim \alpha_c \lesssim 1$, the upper bound being reached for $m_2/m_1 = \infty$ and the lower bound for $m_2 \simeq m_1$; in the equal mass case instability occurs only for a vanishing value of the cutoff radius r_0. Qualitatively, these results agree with those obtained from lattice calculations of quenched QED and from the Bethe–Salpeter equation in the ladder approximation.

Potential II, on the other hand, does not lead to an instability of the QED vacuum, although it predicts an instability of the bound system for $2.16 < m_2/m_1 \leq \infty$, with $1 \leq \alpha_c < \infty$, the lower bound of α_c corresponding to $m_2/m_1 = \infty$ and the upper bound to $m_2/m_1 = 2.16$.

In order to better understand, at the quantum field theory level, the properties of the above potentials, the knowledge of the subsets of Feynman diagrams that generate them would be of great interest. Furthermore, the introduction of effective anomalous magnetic moments of fermions would allow one to have an estimate of the influence of vertex corrections on the previous results. Also, a more detailed (analytic) study of the zero cutoff radius limit of the Todorov potential might clarify its connection with spontaneous chiral symmetry breaking.

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Appendix

In this appendix we show how the wave equations (2.1) and (2.2) can be reduced, for the states ${}^1S_0$, to the final eigenvalue equation (2.4). Equations (2.1) and (2.2) imply the constraints[7]

$$(p_1^2 - p_2^2)\tilde{\Psi} = (m_1^2 - m_2^2)\tilde{\Psi}\,, \tag{A.1}$$

$$[p_1^2 - p_2^2, \tilde{V}]\tilde{\Psi} = 0\,. \tag{A.2}$$

Equation (A.1) allows the elimination of the relative energy variable, while Eq. (A.2) means that $\tilde{V}$ depends on the relative coordinates through the transverse component x^T [Eq. (2.8)].

Equations (2.1) and (2.2) can also be obtained from a covariant three-dimensional reduction of the Bethe–Salpeter equation, and the potential $\tilde{V}$ can be computed in perturbation theory from the Bethe–Salpeter kernel.[9]

Upon bringing the operators on the right hand side of Eqs. (2.1) and (2.2) to the right of $\tilde{V}$ and using again the wave equations, one can transform Eqs. (2.1) and (2.2) into two Dirac type equations, where each particle appears as being placed in the external field created by the other particle.[6] Crater and Van Alstine[17] observed that if $\tilde{V}$ is chosen in the hyperbolic form, as in Eq. (2.3), for the vector interactions, then in the Dirac type equations the resulting effective potential will also be pure vector. Therefore, the parametrization (2.3) of the potential $\tilde{V}$ results in a simple correspondence between the "external field" interpretation and the quantum field theory interpretation. However, it should be noted that the parametrization (2.3) does not only contain effects of one-particle-exchange diagrams; rather, it corresponds to a particular summation of subsets of ladder and crossed ladder higher order diagrams. (The explicit rules of this summation have not yet been derived in the literature.)

In order to solve Eqs. (2.1) and (2.2) with the potential (2.3), one first uses the wave function transformation

$$\tilde{\Psi} = \cosh\left(\frac{1}{2}C\gamma_1 \cdot \gamma_2\right)\Psi . \tag{A.3}$$

Then, bringing the momentum operators to the right of C, one sees that Eqs. (2.1) and (2.2) become, in terms of the internal motion wave function $\psi(x^T)$,

$$\begin{aligned}\Bigg[\left(\frac{P_L}{2}e^C + \frac{m_1^2 - m_2^2}{2P_L}e^{-C}\right)\gamma_{1L} + e^{-C}\gamma_1^T \cdot p^T \\ - i\dot{C}e^{-C}\gamma_2^T \cdot x^T\gamma_1 \cdot \gamma_2 - m_1\Bigg]\psi = 0 ,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\Bigg[\left(\frac{P_L}{2}e^C - \frac{m_1^2 - m_2^2}{2P_L}e^{-C}\right)\gamma_{2L} - e^{-C}\gamma_2^T \cdot p^T \\ + i\dot{C}e^{-C}\gamma_1^T \cdot x^T\gamma_1 \cdot \gamma_2 + m_2\Bigg]\psi = 0 ,\end{aligned} \tag{A.5}$$

where the transverse and longitudinal components are defined as follows:

$$\begin{aligned} &q_\mu^T = q_\mu - (q \cdot \hat{P})\hat{P}_\mu , \qquad q_L = q \cdot \hat{P} , \qquad \hat{P}_\mu = \frac{P_\mu}{(P^2)^{1/2}} , \\ &P_L = (P^2)^{1/2} , \qquad p = \frac{1}{2}(p_1 - p_2) , \end{aligned} \tag{A.6}$$

and

$$\dot{C} = \frac{\partial C}{\partial x^{T2}} . \tag{A.7}$$

Equations (A.4) and (A.5) can be solved with respect to one of the 2×2 components of ψ. Upon decomposing ψ on the basis of the γ_L and γ_5 matrices as

$$\begin{aligned}\psi = \frac{1}{2}(1 + \gamma_L)\psi_{+-} + \frac{1}{2}(1 - \gamma_L)\psi_{-+} + \frac{1}{2}(1 + \gamma_L)\gamma_5\psi_{++} \\ + \frac{1}{2}(1 - \gamma_L)\gamma_5\psi_{--} ,\end{aligned} \tag{A.8}$$

we see that Eqs. (A.4) and (A.5) yield eight coupled compatible equations for the above components. The component ψ_{++} is the dominant one in the nonrelativistic limit and determines the quantum numbers.

In general, in the unequal mass case, ψ_{++} is not an eigenfunction of the total spin and orbital angular momentum operators W_S^2 and W_L^2. However, the ground state of the mass spectrum and its radial excitations are eigenfunctions with quantum numbers $s = 0$ and $\ell = 0$, and the corresponding equations simplify. Since in this work we are interested in the ground state energy alone, we shall present the final eigenvalue equation and the relationships between the components in this case only.

One obtains, in the c.m. frame, the following relationships among the components:

$$(\psi_{+-} - \psi_{-+}) = \frac{m_1 - m_2}{E}\, e^{-C}(\psi_{+-} + \psi_{-+})\,, \tag{A.9}$$

$$(\psi_{++} - \psi_{--}) = \frac{m_1 + m_2}{E}\, e^{-C}(\psi_{++} + \psi_{--})\,, \tag{A.10}$$

$$(\psi_{+-} + \psi_{-+}) = \frac{2}{E}\, e^{-2h}(\mathbf{s}_1 - \mathbf{s}_2)\cdot \mathbf{p}[e^{-2C}(\psi_{++} + \psi_{--})]\,, \tag{A.11}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are the spin operators of particles 1 and 2, respectively, $E = P_L$ [cf. Eq. (2.6)] and h is as defined in Eq. (2.5).

Upon defining the wave function φ by the relation

$$(\psi_{++} + \psi_{--}) = e^{2C+h}\varphi\,, \tag{A.12}$$

one also ends up with the eigenvalue equation (2.4):

$$\left\{e^{2C}\left[\frac{E^2}{4}\, e^{2C} - \frac{1}{2}\,(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4E^2}\, e^{-2C}\right] + \nabla^2 - 4r^2h'^2 + 6h' + 4r^2h''\right\}\varphi = 0\,. \tag{A.13}$$

The term e^{2C}, which factorizes the brackets in the above expression, represents the contribution of the spacelike part of the interaction (2.3), all other C-dependent terms coming from its timelike part.

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