Strongly coupled positronium and relativistic quantum constraint dynamics

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We study the ground state energy of strongly coupled "positronium" as a function of the coupling constant α in the framework of relativistic quantum constraint dynamics as formulated by Crater and Van Alstine. We use a regularized Coulomb interaction $A(r)$ defined by $A(r)=-\alpha/r$ $(r>r_0)$, $A(r)=-\alpha/r_0$ $(r<r_0)$, where r_0 is an arbitrarily small cut-off radius. We find that the center-of-mass energy W of the system remains positive for any α-value, indicating no instability with respect to spontaneous pair creation, in sharp contrast with corresponding results from the Klein–Gordon or Dirac equation. We do, however, find a drastic change in the scaling properties of W as α passes through the critical value $\alpha_c \sim \frac{1}{2}$. While for $\alpha<\alpha_c$, $mr_0 \ll 1$, W scales with the mass m of the particle, W scales with $m^2 r_0$ for $\alpha>\alpha_c$. The limit $r_0 \to 0$ is briefly discussed.

Crater and Van Alstine (CV) recently derived fully covariant relativistic equations for two spin one-half particles in electromagnetic interaction [1]. The CV equations are a generalization of Todorov's quasipotential approach to the relativistic two-body problem [2] and have a number of remarkable properties.

(1) The CV equations correctly reduce to the Dirac equation (with a Coulomb interaction) when one of the particle masses becomes infinite. It follows that they sum *exactly* all generalized ladder photon exchange graphs in this limit [2].

(2) Relativistic corrections of order $(\alpha)^4$ to the ground state energy of (physical) positronium are correctly given by the CV equations [1]. Note that the CV equations yield the same perturbative results as the Breit equation, which partly include the exchange of transverse photons so that the CV equations, when solved in the nonperturbative regime, include at least partly the effect of nonperturbative transverse photon exchanges as well. Let us note that these properties are *not* shared by the *ladder* approximation to the BS equation [2,3].

(3) It has been shown [4], at least for spinless particles, that the BS equation can be written as a set of equations which have the same structure as the equations derived from relativistic quantum constraint dynamics, thus giving the latter a field-theoretic foundation.

The purpose of this work is to study strongly coupled positronium (SCP) in the framework of the CV equations. Specifically, we wish to study whether SCP becomes unstable with respect to spontaneous particle–antiparticle pair creation at some critical coupling strength. The interest of such a problem is (at least) twofold. On the one hand, one can hope that strongly coupled QED (as modelled by the CV equations) may give us some insight into short distance QCD [5]. On the other hand, recent heavy-ion experiments at Darmstadt [6] have led to the conjecture that a new phase of QED might occur in sufficiently strong Coulomb fields [7]. It is conceivable that in such a phase the electromagnetic coupling between elementary fermions would be "strong", i.e. much greater than 1/137.

Our main result is that the total center-of-mass energy W of SCP remains positive for any finite value of α, so that no instability with respect to spontaneous pair creation occurs, in sharp contrast with corre-

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sponding results from the Dirac [8] or Klein-Gordon [9,10] equation. However, there are two distinct phases in the energy spectrum according to whether α is greater or smaller than a critical value $\alpha_c = \frac{1}{2}$. For $\alpha < \alpha_c$, the energy scale is the mass m of the particle. For $\alpha > \alpha_c$, m is proportional to $m^2 r_0$, where mr_0 is an arbitrarily small but nonzero parameter. As will be briefly discussed, r_0 may be interpreted as associated with a nonperturbative modification of the Coulomb interaction at short distance, although, from a strict mathematical viewpoint, r_0 is an arbitrary scale in the energy spectrum for $\alpha > \frac{1}{2}$.

In order to see this in detail, we follow the CV formalism [1] and start from the coupled Dirac equations describing strongly coupled positronium:

$$\gamma_1 \cdot [(p_1 - A_1) + m]\psi = 0, \tag{1}$$

$$\gamma_2 \cdot [(p_2 - A_2) + m]\psi = 0. \tag{2}$$

In eqs. (1), (2), we take [5] $A_1 = A_2 = (0, A(r))$, where $A(r)$ describes the electromagnetic interaction of the particles of mass m. In the CM system, eqs. (1), (2) reduce to the very simple equation for the $^1S_0$ state [5]

$$[p^2 + m_W^2 - (\varepsilon_W - A)^2]\varphi = 0, \tag{3}$$

where one has

$$\varepsilon_W = \frac{W^2 - 2m^2}{2W}, \tag{4}$$

$$m_W = \frac{m^2}{W}, \tag{5}$$

W being the CM energy of SCP.

We have studied eq. (3) with $A(r)$ given by

$$A(r) = -\frac{\alpha}{r}, \quad r > r_0,$$

$$A(r) = -\frac{\alpha}{r_0}, \quad r < r_0. \tag{6}$$

In eq. (6), α is the strength of the Coulomb interaction in SCP, while r_0 is a (so far) arbitrary cut-off radius. Solutions to eq. (3) can be written down immediately as eq. (3) is formally identical with the S-wave radial Klein-Gordon equation [9]. One gets

$$\varphi(r) = \frac{A}{r} W_{k,\mu}(\rho) \frac{\sin \tilde{K} r_0}{W_{k,\mu}(\rho)|_{r=r_0}} \quad (r > r_0), \tag{7}$$

where $W_{k,\mu}(\rho)$ is Whittaker's function,

$$\varphi(r) = \frac{A}{r} \sin \tilde{K} r \quad (r < r_0) \tag{8}$$

with

$$\mu = (\tfrac{1}{4} - \alpha^2)^{1/2}, \tag{9}$$

$$k = \frac{\varepsilon_W \alpha}{(m_W^2 - \varepsilon_W^2)^{1/2}}, \tag{10}$$

$$\rho = 2Kr, \tag{11}$$

$$K = (m_W^2 - \varepsilon_W^2)^{1/2}, \tag{12}$$

$$\tilde{K} = \left[m_W^2 - \left(\varepsilon_W + \frac{\alpha}{r_0} \right)^2 \right]^{1/2}. \tag{13}$$

Bound state solutions are obtained by matching $r\varphi(r)$ and its derivative at $r = r_0$:

$$\tilde{K} \operatorname{cotan} \tilde{K} r_0 = \left. \frac{(\mathrm{d}/\mathrm{d}r)\, W_{k,\mu}(\rho)}{W_{k,\mu}(\rho)} \right|_{r=r_0}. \tag{14}$$

Results are displayed in fig. 1.

One can see that the energy of the $^1S_0$ state remains positive for all α-values, indicating *no instability* of the two-body system with respect to spontaneous pair creation. This should actually be not too surprising, as eq. (3) contains a term $2\varepsilon_W A$ which becomes infinitely repulsive as W becomes arbitrarily small, as shown by eq. (4). We can see that the existence of strongly attractive interactions in eq. (3) must not be necessarily interpreted as a signal of instability with respect to spontaneous pair creation, in contrast with the corresponding situation for Klein-Gordon particles.

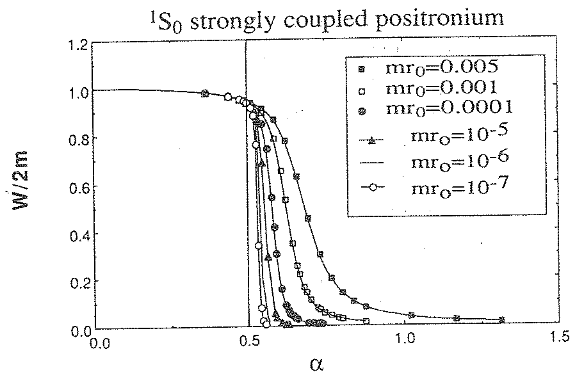

Fig. 1. Energy (W) of the $^1S_0$ state of strongly coupled positronium as a function of α for different values of mr_0.

The curves in fig. 1 do, however, show a striking change in the energy scale as α goes from the region $\alpha < \alpha_{cr}$ to the region $\alpha > \alpha_{cr}$, where $\alpha_{cr} \simeq \frac{1}{2}$ for $mr_0 = 10^{-7}$. As long as α is less than α_{cr}, the mass m of the particle provides the energy scale. For $\alpha > \alpha_{cr}$, there is a sharp variation of W with α. However, for some value of $\alpha = \alpha_0 > \alpha_c$ (but still very close to α_{cr}), W now scales with $m^2 r_0$ and becomes arbitrarily small for any $\alpha > \alpha_0$.

One can illustrate this behavior by means of analytic expressions for W which approximate the exact solution in different α-regions.

(1) For $\alpha < \frac{1}{2}$, $r_0 = 0$, eq. (3) can be solved exactly with interaction (6). The ground state energy is given by [11]

$$W_0^2 = 2m^2 \left[1 + \left(1 + \frac{\alpha^2}{(\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha^2})^2} \right)^{-1/2} \right]. \tag{15}$$

Eq. (15) shows that $W_0 \sim 2m$ for $0 < \alpha < \frac{1}{2}$.

(2) For α slightly larger than $\frac{1}{2}$ [$\lambda \equiv (\alpha^2 - \frac{1}{4})^{1/2}$ real and small], $W/2m \ll 1$, $mr_0 \ll 1$, one can show from (14) that the ground state energy W_0 is given by

$$W_0 = 2\alpha m^2 r_0 \exp\left(\frac{\pi}{\lambda} - 2\gamma \right), \tag{16}$$

where γ is Euler's constant: $\gamma = 0.577\ldots$. Eq. (16) can be derived exactly in the same way as approximate formulas for the spectrum of Dirac particles in strong Coulomb fields have been obtained in the literature [12]. We shall not repeat this derivation here.

(3) For $\alpha \gg 1$, $W/2m \ll 1$, one finds from (14), using the formula [13]

$$W_{-K',\mu/2}(Z) \sim 2^{-1/2} \left(\frac{Z}{K'} \right)^{1/4} \times \exp\left[-K' \log\left(\frac{K'}{e} \right) - 2\sqrt{ZK'} \right], \tag{17}$$

$$W_0 \cong \frac{2m^2 r_0}{\alpha}. \tag{18}$$

A numerical study shows (18) to be already an excellent approximation for $\alpha > 1$. Fig. 2 shows that $W/2m^2 r_0 \alpha$ is a bounded function of (α) as mr_0 becomes arbitrarily small. It shows that our formula (16) provides an upper bound for W_0 for *all* α.

We have so far considered mr_0 to be an arbitrary small but nonzero parameter, so that we can interpret

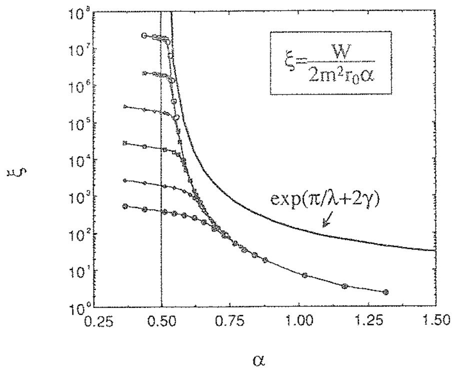

Fig. 2. Values of the ratio ξ, eq. (16), as a function of α for the $^1S_0$ state of strongly coupled positronium and for several values of mr_0 (10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} and 5×10^{-3}, respectively from top to bottom). The full line indicates the limiting value of ξ, given by eq. (16).

r_0 as describing a nonperturbative modification of the Coulomb interaction at small distance. Support for this viewpoint can be found in the literature [14]. We now wish to show that this interpretation is actually not compulsory. Even if we used a pure point Coulomb interaction in eq. (3) a new arbitrary scale would arise in the energy spectrum for $\alpha > \frac{1}{2}$. In order to see this in some detail, consider eq. (3) for $\alpha > \frac{1}{2}$, $W/2m \ll 1$ ($\varepsilon_W^2 - m_W^2 \cong 0$).

Writing

$$x = \beta r^{1/2} \quad (\beta \text{ an arbitrary constant}), \tag{19}$$

$$\varphi = x^{-1/2} \psi(x), \tag{20}$$

one finds that eq. (3) reduces to the following eigenvalue problem:

$$\left(\frac{d^2}{dx^2} + \frac{4\alpha^2 - \frac{3}{4}}{x^2} \right) \varphi = \mu^2 \varphi, \tag{21}$$

where

$$\mu^2 = \frac{8m^2 \alpha}{W \beta^2}. \tag{22}$$

As is well known [15,16], the eigenvalue spectrum corresponding to (21) for $\alpha > \frac{1}{2}$ is given by

$$\mu^2 = C(\alpha) \exp\left(-\frac{n\pi}{2\lambda} \right), \tag{23}$$

where $C(\alpha)$ is an arbitrary function of α and n is any integer. Using (22), (23), one then finds

$$W=\frac{8m^2\alpha}{\beta^2}\frac{1}{C(\alpha)}\exp\left(-\frac{n\pi}{2\lambda}\right). \tag{24}$$

Comparison with (16) shows that provided we take $n=1$ (for the ground state energy), $\beta^2=1/r_0$ and $C(\alpha)=\exp(-2\gamma-\pi/2\lambda)$, (24) will coincide with (16). Obviously μ^2 must be finite in order for (21) to define a self-adjoint operator, so that r_0 *must be nonzero*. Thus the occurrence of an arbitrary scale in the energy spectrum for $\alpha>\frac{1}{2}$ can be regarded as a mathematical consequence of the self-adjointness of the eigenvalue problem (23). It is, however, *equivalent* to introduce a modification of the Coulomb potential at short distance.

To conclude, we found that strongly coupled "positronium" as described by the CV equations is stable with respect to the spontaneous creation of particle-antiparticle pairs at any finite value of the coupling constant. The new feature, however, is that for any $\alpha>\frac{1}{2}$, the ground state energy becomes arbitrarily small (but nonzero). As already emphasized, these results are in sharp contrast with corresponding results from the Dirac equation, even though the CV equations reduce to the Dirac equation when the mass of one of the particles becomes infinite. Thus the stability properties of a two-body system with equal masses may be quite different from the unequal mass case, at least if one trusts the CV equations in the nonperturbative regime.

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