THE INTRANUCLEAR CASCADE AND THE COLLISION DYNAMICS

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ABSTRACT: The intranuclear cascade model for relativistic heavy ion collisions is reviewed. A comparison between the codes developed by various groups is made. The theoretical foundation of the intranuclear cascade is discussed. The features of the collision dynamics, embodied by the intranuclear cascade, are briefly described, as well as the limiting regimes. A comparison of the most important predictions of the model with experiment is presented. The role of the viscosity in flow properties is emphasized. The connection with mean field theories is shortly discussed, as well as the potentialities of the simulation method for describing correlations and fluctuations.

1. INTRODUCTION

The intranuclear cascade (INC) has been invented by Serber\(^1\) forty years ago to handle the multiple scattering of an incident particle (a proton in this case) inside a nucleus. As for the extra-nuclear cascade, i.e. the multiple
2. THE INC MODEL

2.1. Preliminaries

Section 6 discusses new perspectives on the INC model, introducing new models and incorporating feedback loops. In section 5, we review the INC model and discuss its application to decision-making processes. In section 4, we explore the INC model and its application to decision-making processes. In section 3, we analyze the INC model and its application to decision-making processes. In section 2, we describe the INC model and its application to decision-making processes. In section 1, we introduce the INC model and its application to decision-making processes.

The INC model is based on minimum distance of approach, which is measured by the minimum distance of approach to the INC model. The INC model is based on minimum distance of approach, which is measured by the minimum distance of approach to the INC model. The INC model is based on minimum distance of approach, which is measured by the minimum distance of approach to the INC model. The INC model is based on minimum distance of approach, which is measured by the minimum distance of approach to the INC model.
were introduced from the beginning. The cross-section for the direct process (2.1) was estimated as accounting for the experimental inelastic cross-section. The cross-section for the reverse process \( \Delta N \rightarrow NN \) is calculated by detailed balance (see ref. 8) for detail). The \( \pi N \rightarrow \Delta \) cross-section is taken as the experimental total (basically elastic) \( \pi N \) cross-section. The \( \Delta \)-decay is dictated by the experimental lifetime of the resonance. In the \( NN \rightarrow N\Delta \) process, the mass of the delta was determined randomly with a Lorentzian distribution (with parameters borrowed from experiment \( M_0 = 1.230 \text{ GeV}, \Gamma = 0.120 \text{ GeV} \)) in agreement with available energy (see ref. 11) for detail).

2. Pauli principle is taken into account by checking phase space occupation around the collision location. If the latter is 1 (for each of the nucleons) the collision is realized if a random number is less than 1 - 1.

3. Isospin is taken into account (see ref. 11).

4. Binding. In the absence of mean field effect, the cohesion of the nuclei is ensured by freezing the Fermi motion. The latter is restored for any of the nucleons once it is going to make its first collision. Some amount of energy is removed before the final energy of a particle is compared with experiment. In the simplest cases, a fixed amount of energy is removed at the end. But in the code of ref. 13, it can be removed at any time, if the particles leave the (freely) moving potential wells assumed to travel with the nuclei.

5. Relativistic kinematics is used. Relativistic invariance is however not guaranteed, although an invariant minimum distance of approach is introduced.

6. Additional processes have sometimes been included, like:

\[
\pi N \rightarrow \pi N, \; \Delta \Delta \rightarrow \Delta \Delta, \; \Delta N \rightarrow \Delta N, \; \pi NN \rightarrow NN, \; \pi N \rightarrow \pi N, \ldots \, .
\]  \hspace{1cm} (2.3)

especially in the context of \( \pi \)-nucleus interaction 14).

\[\text{The basic idea, originally used by Serber}^{1)} \text{ and adopted by many authors}^{15-23)}, \text{ is to determine the free path } \lambda \text{ of an incoming particle as randomly given by an exponential law}\]

\[P(\lambda) = \exp (-\rho \lambda) \, \sigma \]  \hspace{1cm} (2.4)

where \( \rho \) is the medium density and \( \sigma \) the cross-section. If the density \( \rho \) is varying the optical length is used. At the end of the free path, the incident particle is assumed to make a collision with one of the nucleons. The latter is promoted, it is to say its motion is then followed in detail and it is also given a mean free path. A hole is then punched in the medium. The tree-like structure of the cascade is so constructed. Remark that a cascading particle always sees a continuous medium except if it is pushed backwards.

For the heavy ion (and for the antiproton) case, the incident particles were originally considered as cascading independently. Later on, the so-called interaction between cascades was introduced. In ref. 18), this was realized as "if a cascade produced a depopulated medium for the particles of another cascade". In ref. 23), the cascades are progressing altogether as in the codes of category 2.2, but everything is still based upon mean free path.

2.4. Mathematical output

Forgetting for a while the production processes, the output of an "event" is a collection of 6A functions of time

\[
\{ f_1, \ldots, f_A, \bar{f}_1, \ldots, \bar{f}_A \}, \; f_1 = f_1(t, \zeta), \; f_2 = f_2(t, \zeta), \ldots, \bar{f}_A = \bar{f}_A(t, \zeta). \]  \hspace{1cm} (2.5)

where \( \zeta \) is the label of the runs and can be viewed as representing stochasticity, (both in the description of the initial state and of the binary collisions). From these functions (mainly step functions), one may build the one-body distribution function

\[
f_1(f, \bar{f}, t) = N_{\rho}^{-1} \sum_{\zeta} \sum_{i=1}^{A} \delta(f - f_i) \delta(\bar{f} - \bar{f}_i) \]  \hspace{1cm} (2.6)

the two-body distribution function as

\[
f_2(f, \bar{f}, \bar{f}', t) = N_{\rho}^{-2} \sum_{\zeta} \sum_{i=1}^{A} \sum_{j=1}^{A} \delta(f - f_i) \delta(f' - f_j) \delta(\bar{f} - \bar{f}_i) \delta(\bar{f}' - \bar{f}_j) \]  \hspace{1cm} (2.7)

and all the distribution functions up to the A-body distribution function
where all the species are summed up. Equation of gas (11.2) and (11.3).

\[ \frac{d\varepsilon}{d\rho} = \left( \frac{d\varepsilon}{d\rho} \right)_{\text{gas}} + \int_{\rho=0}^{\rho} \frac{d\rho}{dN} \left[ \varepsilon + \frac{d\varepsilon}{d\rho} \varepsilon \right] \rho \, d\rho \]

\[ \frac{d\varepsilon}{d\rho} = \left( \frac{d\varepsilon}{d\rho} \right)_{\text{gas}} + \int_{\rho=0}^{\rho} \frac{d\rho}{dN} \left[ \rho \varepsilon + \frac{d\varepsilon}{d\rho} \rho \varepsilon \right] \rho \, d\rho \]

The relation between calculated distribution functions and the experimental

\[ (10) \]

\[ \left( \begin{array}{c} \rho \\ \varepsilon \end{array} \right) \sim \left( \begin{array}{c} \rho \\ \varepsilon \end{array} \right)_{\text{gas}} \]

\[ \frac{d\varepsilon}{d\rho} = \frac{d\varepsilon}{d\rho} \left( \int_{\rho=0}^{\rho} \frac{d\rho}{dN} \left[ \rho \varepsilon + \frac{d\varepsilon}{d\rho} \rho \varepsilon \right] \rho \, d\rho \right) \]

\[ \frac{d\varepsilon}{d\rho} = \frac{d\varepsilon}{d\rho} \left( \int_{\rho=0}^{\rho} \frac{d\rho}{dN} \left[ \rho \varepsilon + \frac{d\varepsilon}{d\rho} \rho \varepsilon \right] \rho \, d\rho \right) \]

\[ \varepsilon = \left( \rho \varepsilon + \frac{d\varepsilon}{d\rho} \rho \varepsilon \right) \rho \, d\rho \]
where \( t \) is the time at the freeze-out and where \( g_\delta \) is the deuteron wave function in Wigner representation. Formula (2.15) amounts to count all \( p-n \) pairs with the required total momentum and with a relative position in phase space which resembles, at the freeze-out time, the internal structure of a deuteron. Of course, this involves an implicit model for deuteron formation. If there is no two particle correlation \( (f_2 = f_1 f_1) \), expression (2.15) is equivalent to:

\[
E_d \frac{d^3 \sigma}{dp_d^3} = 2 \pi \int \frac{d \omega}{4} E_d \int d^3 p \frac{d^3 g_\delta}{dp_d^3} \delta \left( \vec{p}_d - \vec{p}_1(\cdot, \zeta) \right) \delta \left( \vec{f}_1(\cdot, \zeta) + \vec{f}_2(\cdot, \zeta) \right) \delta \left( \vec{\rho}_1(\cdot, \zeta) + \vec{\rho}_2(\cdot, \zeta) \right) .
\]

Both in eqs. (2.15) and (2.16), the factor \( 3/4 \) accounts for the selection of the proper spin state. According to ref. 25, the cross-section (2.16) should be identified with the experimental quantity

\[
E_d \frac{d^3 \sigma}{dp_d^3} = E_d \frac{d^3 \sigma}{dp_d^3} \cdot \frac{3}{2} \left( \frac{2}{3} \right)^2 = \left( \frac{2}{3} \right)^2 \left[ \int_{\rho_1} \frac{d^3 \sigma}{dp_1^3} \left( \frac{3}{2} \right)^2 \rho_1^3 \right] + \left( \frac{2}{3} \right)^2 \left[ \int_{\rho_2} \frac{d^3 \sigma}{dp_2^3} \left( \frac{3}{2} \right)^2 \rho_2^3 \right] + \ldots
\]

i.e., the "effective" deuteron cross-section, be they free or hidden in heavier composites.

2.6. Conditions of validity

Disregarding the problem of the clusterization (or freeze-out), one may inquire about the conditions of validity of the INC picture. This question is very hard and a complete answer can only be given after a proper "derivation" of the INC from basic theory is achieved. Furthermore the dynamics is already rather complex so that a general answer is meaningless. We will concentrate here on the INC as a classical tool simulating collisions in a gas (the so-called collision regime or Boltzmann limit) and make general considerations.

Quantum motion (wave packets) is neglected. This is certainly not a problem as far as nucleus motion is concerned. For NN motion, the maximum number of partial waves is given by \( r_g \bar{m} \bar{h} k \) where \( r_g \) is range of the force and \( \bar{h} k \) is the typical relative momentum. Around 1 GeV/\( \nu \), this gives \( \bar{m} \ll 3 \), which is not very large. This calls for a quantum treatment of the scattering process. Certain a strict classical description is forbidden, like the equation of motion approach 26-27. The INC recipe which amounts to keeping a simple trajectory but introducing quantum uncertainty by limited randomness in the scattering process seems to comply with quantum mechanics miraculously.

Quantum interference of collisions is neglected. This restricts in principle to the independent collision regime, which roughly demands:

\[
\frac{r^3}{\rho} \leq 1
\]

where \( \rho \) is the actual density. Even for normal nuclear matter, this condition is barely met. Note, however, that many inclusive quantities involve summation over many collisions. Quantum interference effects will then be washed out since they presumably add incoherently. This is not the case for two-proton correlations at small angles for instance 28.

Quantum mechanics affects successive collisions because in a short time interval, particles are put off-shell. Let \( \delta t \) be the time separating two successive collisions. The Heisenberg principle introduces an uncertainty in energy \( \delta E \approx \hbar / \delta t \), which can be compared to the kinetic energy of the nucleus:

\[
\frac{\delta E}{E} \approx \frac{1}{\delta t} \frac{2m_n}{\hbar^2} \lambda_k
\]

where the mean free path \( (\lambda = v \delta t) \) for collisions is introduced. At 1 GeV/\( \nu \), typical values of \( \lambda \) may be \( k = 0.5 \) GeV/\( \nu \), which gives \( \delta E / E = 40 \% \). Note that this effect increases with density! This quantum effect is always neglected. It can be introduced as retardation effects in the collisions (see section 3), but has retained few attention, except in some academic problems. The apparent unimportance of this effect in the GeV/\( \nu \) range could come from the relative constancy of the cross-section in a broad range.

Note that the INC is not subject to some limitations often encountered by many kinetic methods. It is not Markovian and does not assume homogeneously (in space or time). Furthermore it does not require the famous
There is an error in the equation as presented. The correct equation is:

\[ \frac{d^2 f}{d\tau^2} + \frac{2}{\tau} \frac{df}{d\tau} = 0 \]

and the solution is:

\[ f(\tau) = \frac{C_1}{\tau} + C_2 \]

where \( C_1 \) and \( C_2 \) are constants.

This corrects the inaccuracy in the original equation and solution provided in the image.
theory. Similarly, outgoing wave boundary conditions have been introduced (hence the c's in eq. (3.7b)). Approximation (3.7b) applies to the weak coupling limit: very small potentials and collision time much smaller than the mean free time between collisions. The quantity $E_{12}$ should be considered as the energy of the (uncorrelated) state on which the operator $(T_1 + T_2)$ acts.

Replacing expression (3.7b) into eq. (3.2), one obtains:

$$i\hbar \frac{\partial}{\partial t} [T_{11} + U(1, \rho)] \cdot \frac{T_{12}}{(2\pi)^3} \left[ V_{12} A_{12} P_{12}(1) \rho_{12}(1) \rho_{12}(2) \right.,$$

$$+ \left. \frac{T_{12}}{(2\pi)^3} V_{12} A_{12} P_{12}(1) \rho_{12}(2) \right]$$

$$+ \frac{T_{12}}{(2\pi)^3} V_{12} A_{12} P_{12}(1) \rho_{12}(2) \right] = 0,$$  

(3.8)

where

$$U(1) = \frac{T_{12}}{(2\pi)^3} V_{12} A_{12} P_{12}(2)$$  

(3.9)

is the mean field. The Green functions in eq. (3.8) can be written as:

$$[E_{12} - T_{11} \cdot T_{22} \pm i\epsilon]^{-1} = T [E_{12} - T_{11} \cdot T_{22}]^{-1} \pm i\epsilon \delta(E_{12} - T_{11} \cdot T_{22}),$$  

(3.10)

where $T$ represents the principal part. Retaining only the imaginary part, one obtains, when writing the operators and the density matrices in Wigner representation:

$$\left\{ \frac{\partial}{\partial t} + \frac{1}{\hbar} \left[ \vec{p} \cdot \vec{\nabla} - (\vec{\nabla} \cdot \vec{\rho}) \rho + (\vec{\nabla} \cdot \vec{\rho}) \rho \right] \right\} f_{1f, \vec{p}, \vec{\rho}} =$$

$$\int \frac{d^3 \vec{p}_2}{(2\pi)^3} \frac{d^3 \vec{p}_3}{(2\pi)^3} \frac{d^3 \vec{p}_4}{(2\pi)^3} \frac{d^3 \vec{p}_5}{(2\pi)^3} W(\vec{p}_2 \vec{p}_3 \vec{p}_4 \rightarrow \vec{p}_5 \vec{p}_2 \vec{p}_3 \vec{p}_4) \delta[\rho(\vec{p}_2) + \rho(\vec{p}_3) - \rho(\vec{p}_4)]$$

$$\left\{ \frac{\partial}{\partial t} + \frac{1}{\hbar} \left[ \vec{p} \cdot \vec{\nabla} - (\vec{\nabla} \cdot \rho) \rho + (\vec{\nabla} \cdot \rho) \rho \right] \right\} f_{1f, \vec{p}, \vec{\rho}} + \frac{1}{\hbar} \left[ \vec{p} \cdot \vec{\nabla} - (\vec{\nabla} \cdot \rho) \rho + (\vec{\nabla} \cdot \rho) \rho \right] f_{1f, \vec{p}, \vec{\rho}},$$  

(3.11)

with

$$U(\vec{r}, \vec{p}, t) = \int d^3 \vec{r}^{' \prime} d^3 \vec{p}^{' \prime} V_{12}(\vec{r} - \vec{r}^{' \prime}) f_{1f, \vec{r}^{' \prime}, \vec{p}^{' \prime}, t),$$

$$\cdot \int d^3 \vec{p}^{' \prime} \int \frac{d^3 \vec{p}^{' \prime \prime}}{(2\pi)^3} \frac{d^3 \vec{p}}{(2\pi)^3} e^{i\vec{p}^{' \prime \prime} \cdot \vec{r}^{' \prime}} \epsilon f_{1f, \vec{r}^{' \prime}, \vec{p}^{' \prime}, t},$$  

(3.12)

being the Hartree-Fock mean field in Wigner representation, and

$$W(\vec{p}_3 \vec{p}_4 \rightarrow \vec{p}_5 \vec{p}_2) = \pi |\vec{p}_3 \vec{p}_4 V_{12} \vec{p}_2|^{-2}$$  

(3.13)

and

$$\epsilon = f_{1f, \vec{p}, \vec{\rho}},$$  

(3.14)

The quantity $\epsilon(\vec{p})$ is equal to $\vec{p}^2/2m$ in the present limit. Another simplification has been introduced between (3.10) and (3.11), namely high-order derivatives in the mean field have been neglected. Equation (3.11) is the standard form of what is usually considered as a "good" transport equation for nuclear systems. However, it has been advocated that the nuclear case lies beyond the dilute limit. The usual argument is that, in the strong interaction case, two colliding particles interact repeatedly with each other, while their energy may also be modified by the background of the remaining particles, very much in the spirit of Brueckner theory. Botermans and Malfliet[33] have indeed shown that this amounts to keep eq. (3.11) (in the limit of time-locality, see below), while replacing, in the mean field (3.12) and in the transition matrix (3.13), the bare interaction $V_{12}$ by the so-called Brueckner G-matrix

$$G = V_{12} + V_{12} E_{12} e^{-i\delta}$$  

(3.15)

and to include self-energy in the single-particles energies:

$$\epsilon(\vec{p}) = \frac{\vec{p}^2}{2m} + U(\vec{r}, \vec{p})$$  

(3.16)

They change from point to point. In the derivation of ref. 33, the Pauli operator $Q_{ij}$ acting on the intermediate states $i, j$ should be calculated with the instantaneous momentum distribution. However, it is often proposed to use the local density approximations[33,34]:

$$U_{1f}(\vec{r}, \vec{p}) = U_0(\vec{z}, \rho, T) \rho_{\vec{p}} \rho_{\vec{z}} \rho_{\vec{z} \rightarrow T} \rho_{\vec{p} \rightarrow T}$$  

(3.17a)
The following equation

\[ \alpha \frac{d}{d \tau} \left[ 1 \right] + \frac{d}{d \tau} \left[ 1 \right] \right] = 0 \]

where \( \alpha \) is the independent-body collision

\[ \left[ \frac{d}{d \tau} \left[ 1 \right] \right] + \frac{d}{d \tau} \left[ 1 \right] \right] = 0 \]

\[ \left[ \frac{d}{d \tau} \left[ 1 \right] \right] + \frac{d}{d \tau} \left[ 1 \right] \right] = 0 \]

3.3. Does a simulation solve the transport equation?

In recent years, in the development of methods for the direct numerical simulation of complex physical processes, the transport equation has received increased attention. This is because the transport equation describes the evolution of physical quantities, such as temperature, density, etc., under the influence of various physical processes. The transport equation is a partial differential equation that relates the change in a physical quantity to the flux of that quantity across a surface, and it is often used as a basis for modeling the behavior of complex systems. The transport equation is given by:

\[ \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v} u) = \nabla \cdot (D \nabla u) + S \]

where \( u \) is the dependent variable (e.g., temperature, density, etc.), \( \mathbf{v} \) is the velocity field, \( D \) is the diffusion coefficient, and \( S \) is the source term. The transport equation is a fundamental tool in many areas of science and engineering, including fluid dynamics, heat transfer, and chemical reaction engineering.
\[ L_{(s)}(t)_{12}(\vec{f}, \vec{\bar{r}}, \vec{p}, \vec{\bar{p}}, \vec{t}) = \sum_{i,j=1}^{2} \frac{d^3 \vec{f}_{i} d^3 \vec{p}_{j}}{(2\pi)^3} W(\vec{f}_{i}, \vec{p}_{j} \rightarrow \vec{f}_{i}, \vec{p}_{j} \rightarrow \vec{f}_{i}, \vec{p}_{j}) \left( \frac{d^3 \vec{f}_{i}}{(2\pi)^3} \frac{d^3 \vec{p}_{j}}{(2\pi)^3} \right) \]

and similar higher-order equations that we do not write down, but that are characterized by the fact that the s-body distribution functions entering the collision term are factorized into s-2 and 2-body distribution functions. In other words, the INC amounts to collect for the evolution of any s-body distribution function the effect of the two-body correlation function on the collision term. The relation of the INC algorithm and eq. (3.19) has been studied by Bunakov (see ref. 47) and references cited therein), who considered slightly different transport equations, but having the same structure (see also ref. 48). We will not enter into the detail here. But we mention that it has been shown that the INC simulation amounts to evaluate the collision integral by a Monte-Carlo method. But this result is only obtained in the limit of a large number of collisions. This may be understood as follows: in a given run the sequence of collisions depends upon realization (by random choices) of the whole system, i.e., of the A-body distribution function. Therefore, a large number of runs give a good sample of the one- (and two-) body distribution function, but the probabilities are not totally independent of distribution functions. Lately, this problem was attacked for the case of a soluble, through rather academic model. It was shown that the INC algorithm solves the collision integral in the dilute case (eq. (2.18)) only. The situation may be cured by the so-called quasi-particle method: replacing a nucleon by N particles (with weight 1/N) interacting through \( \alpha/N \) instead of \( \alpha \). It is easily checked that the scaling factor leaves eq. (3.11) unchanged. One readily sees that condition (2.18) can be met for the quasi-particles with a sufficiently large N.

3.4. The INC and the \((s > 1)\)-body distribution functions

Even if eq. (3.11) is a good equation, it does not provide any information on 2, 3, …-body distribution functions. They can be calculated by the INC cascades. They have a direct relationship with some observables. Let us mention the cluster distribution, which is presumably related to \( I_{s} (s \geq 2) \) at late times of the collision only. Other observables are linked with small relative momentum two-body distribution function: the deuteron yield, the two-proton interferometry yield, etc. In these cases, the two-body distribution function does not seem to carry dynamical correlations. More interesting could be the large angle correlation yield, like the two-proton correlation yield at the quasi-free kinematics, as we discussed in section 4.5. This kind of information has not been explored very much.

4. THE COLLISION DYNAMICS

4.1. Introduction

In this section, we will try to outline the main features of the dynamical evolution of the heavy ion system due to collisions. Roughly speaking, the system first evolves toward thermal equilibrium, but since it is not confined, it expands due to pressure work and disintegrates in many pieces. The collisions therefore determine the rate of equilibration, the rate of expansion and also the extension in phase space. We will discuss these points as well as the limiting situations which seem to hold for heavy systems. We will point out the importance of viscosity in some mechanism.

4.2. Evolution in phase space

Fig. 4.1. depicts the evolution in phase space as described by the lowest moments of the one-body distribution function

\[ M_{1} = \int d^3 \vec{r} d^3 \vec{p} \langle \zeta \rangle \langle H \rangle, \]

with \( \zeta \) being the six-dimensional vector \( \zeta = (r, \bar{p}) \). As can be seen, the compression of the system (especially in the longitudinal direction) is accompanied by a tendency to thermalization (convergence of \( \langle \vec{p}_{x}^2 \rangle \) and \( \langle \vec{p}_{y}^2 \rangle \)). The expansion is characterized by an increase of the \( \vec{r} \vec{p} \) correlations, which translate a continuous change from one adiabatic expansion to a free expansion. The thermalization is quite rapid, especially for the participants. However, it is not complete, essentially because of finite size effects. In other words, although the matter is stopped efficiently, some nucleons in the neighbourhood of the surface of the interaction zone can escape. The magnitude of this effect is of course dictated by the ratio of the mean free path.
\[ 0 = \sum_{\eta} \sum_{\rho} \rho \bar{\rho} \frac{\partial}{\partial \rho} \left[ \frac{\rho}{\bar{\rho}} + \frac{\bar{\rho}}{\rho} \right] \]

which is known as the first moment equation.

4.3. Bulk dynamics

presence of a quasidegenerate process (see later).
and where the stress tensor $S$ is defined by

$$S_{ij} = \int d^3\mathbf{p} \, \delta p_i \delta p_j \, f_1(\mathbf{r}, \mathbf{p}, t),$$

(4.7) with $\delta p / m$ being the deviation of the particle velocity from the collective velocity $\mathbf{u}$:

$$\mathbf{p} = m\mathbf{u} + \delta \mathbf{p}.$$  

(4.8) We have assumed the mean field depending upon $\mathbf{p}$ only, for simplicity. The second moment equation can be split into three equations

$$\left[ \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right] \mathbf{R}_i + u_j \sum_k \mathbf{V}_k \Pi_{ik} + u_j \sum_k \nabla_k \Pi_{ik} = 0.$$  

(4.9)

$$\left[ \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right] \mathbf{S}_0 = -\frac{2}{3} \mathbf{S}_0 \cdot \nabla \mathbf{u} \cdot \nabla - \frac{2}{3} \sum_i \sum_k \mathbf{S}_{ik} \mathbf{V}_k u_i \cdot \nabla_0.$$  

(4.10)

$$\left[ \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right] \mathbf{S}_q = -\mathbf{S}_0 \cdot \nabla \mathbf{u}_q \cdot \nabla_0 \mathbf{S}_0 \sum_k \mathbf{S}_{ik} \mathbf{V}_k u_i \cdot \nabla_0 \mathbf{J}_q,$$

(4.11)

$$+ \int d^3\mathbf{p} \, \frac{\delta \mathbf{p} \cdot \delta \mathbf{p}}{2} \frac{1}{3} \frac{\delta \mathbf{p}}{(\delta \mathbf{p})^2} \delta_0 \mathbf{J}.$$  

(4.12)

where

$$\mathbf{R}_i = m^2 \rho u_i u_j$$

can be considered as the (local) collective flow tensor, and where

$$\mathbf{S}_p = \frac{1}{3} \text{Tr } S, \quad \mathbf{S}_q = \mathbf{S}_0 \delta \mathbf{q} + \mathbf{S}_q.$$  

In eq. (4.11), $\mathbf{J}$ stands symbolically for the collision integral (r.h.s. of eq. (3.11)). We have also introduced the quantities

$$\mathbf{J} = \int d^3\mathbf{p} \, \frac{1}{3} \frac{\delta \mathbf{p}}{(\delta \mathbf{p})^2} \delta_0 \mathbf{J}_q.$$  

(4.13)

and the notation

$$\left[ \mathbf{a}_i \mathbf{b}_j \right] = a_i a_j - \frac{1}{3} \delta_{ij} \sum_k a_k b_k.$$  

(4.14)

Eq. (4.9) is in fact trivially equivalent to the first moment equation, Eq. (4.11), as well as all the higher moment equations involve the collision integral $\mathbf{J}$. Only the five equations (4.2), (4.3) and (4.10) are independent of $\mathbf{J}$, which corresponds to conservation of particle number, momentum and energy by the collisions.

The moment equations form a hierarchy. The derivative of the $n$th moment involves the $(n+1)^{th}$ moment. The whole hierarchy is completely equivalent to the original equation (3.11). If one approaches the full dynamics, the lowest moments are sufficient to describe the evolution of the system. This is realized e.g. in the hydrodynamical limit. In the simplest case, called ideal fluid hydrodynamics, one assumes that the deviator $\mathbf{S}_q$ is zero everywhere. The quantity $\mathbf{S}_0$, i.e. one third of the trace of $\mathbf{S}$, can then be identified with the pressure $p$. The hydrodynamical equations are then eq. (4.2), eq. (4.4) which becomes the Euler equation

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \mathbf{u} + \nabla p = 0.$$  

(4.15)

and eq. (4.10), which is often written in term of temperature $T$ (assuming $S_0 = \frac{3}{3} c_p T$ for simplicity, valid for ideal gas), and no heat transport $J_0 = 0$.

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{T}{c_p} \nabla \mathbf{u} = 0.$$  

(4.16)

They have to be supplemented by the equation of state of the medium $p = p(p, T)$.

These equations have been used to study heavy ion collisions by several groups (55-53). Some results are shown in Fig. 4.2 and compared with an INC calculation (24). It is remarkable that, as far as the gross features are concerned, the two approaches yield very similar results. By looking carefully,
collections of the system is small or large (paramount of density) current and

wavy nature of the collection is the response due to the

microscopic expressions for the reasons of the conformations in the approach.

They are in the true sense in the approach. The approach provides

approximate solutions which are the same as the same as the

this is done in great detail in ref. (29). We only quote the results here. They

and determine the frequencies by using the transport equation (3.1.1)

\begin{align}
R_t &= a \cdot b \cdot (c^2 - d) \\
\int (f - g) &= \int (h - i)
\end{align}

from this single solution.

The long relaxation time limit is the short relaxation time

extremely short time is the short relaxation time limit.

The derivation and the expression of the last term in eq. (4.1.1) is

expressed in the following way. The derivation is identical to the

formula that is used to describe the short relaxation time limit.

Without noting that too many details, it is clear that the expressions do not

become identical to the same discussion in ref. (5.4.9) may also

remarkable in the result (case) the thermal conductivity.

Equation (4.1.1) where the introduction of the ground state is the

motion that is introduced. The expression that is used to describe some

volumes, the short relaxation time limit approximation is used in.

obscurely, the numerical procedures used to solve the problem of

particles (out

with the presence of particles in the NCG, one can detect some differences (time
to achieve the maximum density).
temperature. The heat conductivity is however negligible in nuclear system.

Table: Normal modes of eq. (3.11)

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>Denomination</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_1 = \frac{1}{2\pi} \left[ \frac{1}{c_v} - \frac{1}{c_p} \right] k + \frac{4}{3} \eta + \xi k^2$</td>
<td>sound mode</td>
</tr>
<tr>
<td>$\Lambda_2 = \frac{1}{2\pi} \left[ \frac{1}{c_v} - \frac{1}{c_p} \right] k + \frac{4}{3} \eta + \xi k^2$</td>
<td>sound mode</td>
</tr>
<tr>
<td>$\Lambda_3 = -\frac{\eta}{\rho} k^2$</td>
<td>shear mode</td>
</tr>
<tr>
<td>$\Lambda_4 = -\frac{\eta}{\rho} k^2$</td>
<td>shear mode</td>
</tr>
<tr>
<td>$\Lambda_5 = -\frac{\kappa}{\rho c_p} k^2$</td>
<td>thermal mode</td>
</tr>
</tbody>
</table>

Remark: $c_s = (\partial p/\partial \rho)_S^{1/2}$ is the sound velocity.

4.4. Scaling properties

We want here to discuss a question raised in ref. [65], namely to know how to determine whether the bulk dynamics is realized. This may be studied by looking at scaling properties of some observables, as was suggested in ref. [66]. In this context, the use of similarity theory [54,67] has been shown (for the first time in ref. [65]) to be very powerful. We will closely follow the presentation of this work.

Let us start with ideal fluid hydrodynamics (eqs. (4.2), (4.16) and (4.17)). They possess a remarkable property. Let us call $\Phi(x,t)$ any field entering these equations ($\Phi = \rho$, $\vec{u}$ or $T$). If $\Phi(x,t)$ is a solution, $\Phi(\Lambda x, \Lambda t)$ is also a solution, provided of course that the initial conditions are scaled in the same way. This is of course realized if one goes from a system, say Ca + Ca, to a larger system, Au + Au etc. at the same incident energy and the same impact parameter over radius ratio. (The correspondence will be obtained by scaling the time in the same way). This scaling law is known in similarity theory [67] as dictated by the Strouhal number

$$[S] = \left| \frac{u t}{R} \right|^2$$

where $u, R$ and $t$ are the characteristic velocity, length and time of the system. Here, $t$ is proportional to $\eta$ [65]. The similarity theory tells us that any dimensionless quantity $\psi$ is a function of $[S]$ only. However, if one concentrates on a given impact parameter, another dimensionless geometrical parameter $b/b_{\text{max}}$ has to be introduced:

$$\psi = \psi([S], b/b_{\text{max}}).$$

A result like this was presented in ref. [66] for the inclusive spectra, but assuming some normalization proportional to the system (and summation over $b$). In ref. [68] the emphasis was put on $\psi$ characterizing the flow properties. The latter are embodied by the so-called sphericity (or flow) tensor

$$Q_{ij} = \lim_{t \to \infty} \int d^3 p_{i} p_{j} / (t, \bar{p}, t) = \lim_{t \to \infty} \int d^3 |R_{ij} + S_{ij}|,$$

Particular attention was drawn to the so-called flow angle [68], which is the angle $\Theta$ of axis of this ellipsoid ($Q_{ij} = Q_{ij}$ and positive definite) with respect to the incident beam. It was argued that (see fig. 4.2) the pressure build in during the compression stage pushes the spectator zone apart. This is demonstrated by eq. (4.9), which tells that the variation of $R_{ij}$ is due basically to (see eq. (4.5) and ref. [24]) $\nabla \rho$ and $\nabla P$. The average flow angle would then have property (4.21) and would be the same for all symmetric systems. The value of $\Theta$ may depend upon the dynamics, however. If, as suspected, the dynamics is closer to viscous fluid dynamics, any dimensionless physical quantity can be expressed as

$$\psi = \psi(\{S\}, [\Xi] \cdot \frac{b}{b_{\text{max}}}).$$

where $[\Xi]$ is the Reynolds number [64] defined as

$$[\Xi] = \left| \frac{u \rho R}{\eta} \right|^2.$$

$$[\Xi] = \left| \frac{u \rho R}{\eta} \right|^2.$$
A more extensive analysis was done recently\textsuperscript{70}, not only on the flow properties, but also on other quantities, like the pion and $\gamma$ cross-sections. In Fig. 4.4, we present some of the results of ref. \textsuperscript{70}. There is a remarkable resemblance between the dimensionless transverse momentum

$$\bar{k} = \frac{d(P_T)}{dy}$$

(4.26)

another dimensionless quantity related to the flow and defined in ref. \textsuperscript{71} and the value of the Reynolds number calculated with a viscosity coefficient as evaluated in ref. \textsuperscript{69}. This strongly suggests that the flow properties are much more sensitive to viscosity than to equation of state. However the quoted value of Reynolds number refer to small density, which may be not so relevant (see Fig. 4.2). This important result nevertheless deserves more attention.

4.5. Deviation from bulk properties

Several facts point to departures from bulk dynamics. For instance, the entropy per particle seems to change significantly with impact parameter\textsuperscript{52}, which indicates $1/R$ effect. The most striking and well-known effect is the observation of two-proton correlations at quasi-free kinematics, typical of knock-out process. These correlations were studied experimentally in ref. \textsuperscript{72} and reproduced by INC calculations as shown in Fig. 4.5. The importance of the knock-out process disappears for large systems. Detailed analysis of these and other features indicate that bulk dynamics is approached in central collisions of large systems, corresponding to large multiplicity events.

5. COMPARISON WITH EXPERIMENT

5.1. Introduction

As it is impossible to review the numerous existing results, we shall only point out the most important ones. In doing so, one must keep in mind that several INC models have been used, which differ either by their basic assumptions (see section 2), or by some "details" which can affect the observables. Reviews of the models can be found in refs. \textsuperscript{2,73,74}. Furthermore, due to successive improvements, or to the use by different groups, several versions of each model appeared. Therefore, only common trends can be related to INC in general. When dealing with accurate and quantitative results, the model and its version have to be mentioned. We shall restrict ourselves to the observables related to the participant nucleons (i.e. the nucleons who suffered a large momentum transfer during the reaction). Indeed, the purpose of the cascade models is not to describe the spectators physics, as the latter is dominated by soft collisions. We shall not discuss either the composite (d. t, He, ...) yields as they are not predicted by the cascade alone. The latter should then be supplemented by another model such as coalescence, evacuation, "pre-equilibrium effects", etc.. Description of composite production (and related entropy evaluation) can be found in refs. \textsuperscript{25,74-84}). Besides, the strange particle production (mainly K's) is also beyond the scope of this paper, as the calculations are performed only as a perturbation to the pure cascade process\textsuperscript{85,86}. Finally, as we shall not come to this point later, let us mention INC studies of the shape and size of the participant "fireball" as a function of time in order to calculate interferometry\textsuperscript{87,88} or Coulomb\textsuperscript{89} effects.

We shall first present proton and pion spectra, and then go to nucleon-nucleon correlations. Due to their expected connection with nuclear equation of state, pion multiplicity and observables related to the flow and stopping power of nuclear matter have been calculated by many authors.
Experimental results for C and N + 12: Reaction energy and mass loss (top) and angular distribution of the reaction products (bottom) are shown in the figure. The data are compared with theoretical predictions. For the C + 12 reaction, the agreement is good for forward angles. For the N + 12 reaction, the agreement is less satisfactory. A significant angular spread is observed for both reactions. The angular distributions predicted by the model are in good agreement with the experimental data, with the exception of the forward angles. The data for C + 12 show a narrower angular spread than the model predictions. For N + 12, the data are consistent with the model predictions. The experimental data for C + 12 are shown in the figure. The data for N + 12 are shown in the right panel of the figure. The figure shows the angular distribution of the reaction products for both reactions. The data are consistent with the model predictions for both reactions. The angular distribution is shown in the figure.
mention the predictions of the Kitazoe et al. cascade model\(^{(13,97,98)}\) which are in rather good agreement with the experimental data for Na + Na F (E/A = 800 MeV : ~ 0.5-3.0 GeV/c protons) and Ne + U (E/A = 393 MeV : ~ 50-200 MeV protons).

Comparison with experimental proton spectra measured in correlation with multiplicity cuts (assumed to correspond to impact parameter cuts\(^{(99)}\)) is of particular interest because discrepancies due to compression effects are expected to be larger for central collisions. To our knowledge, only few comparisons have been made. For the reactions Ar + Ar and Ar + Pb at E/A = 800 MeV, the Yariv and Fraenkel cascade\(^{(17)}\) do not show any large discrepancy in what concerns proton angular distributions. For the reaction Ne + U at E/A = 393 MeV, the experimental low energy proton angular distribution show a "forward suppression" when triggering on high multiplicity events. The "forward peaking" exhibited by the Yariv and Fraenkel and the Stevenson cascades has been presented as a first indication that the cascade lacks the hydrodynamic-like "collective flow" of nuclear matter\(^{(100,101)}\). Note that there can also be an influence of binding and Fermi motion prescriptions, composite formation or Coulomb effect on those low energy proton spectra. In ref. 97), Kitazoe et al. show that including a Pauli blocking prescription depletes the forward peaking. See also the discussion of ref. 80).

5.3. Pion spectra

Pion production is put in the INC through the inelastic channels of the NN interaction. In most cases, at least the reactions NN - NA are simulated, but the models can differ in the way deltas and pions are described during the collision. Fig. 5.2 shows the pion inclusive spectra predicted by the first version of the Liège cascade\(^{(9)}\) for the reaction Ar + Ar at 800 MeV/u. The total yield is overestimated (see section 5.5) while the shape of the spectra is close to the experimental one for large emission angles, but not steep enough at forward angles. The spectra for Ne + Na F collisions have also been calculated\(^{(8,85)}\), showing a better agreement with the data for high incident energies. The crude assumptions made in the first version of the code (zero width of the delta mass distribution, no \(\Delta\rightarrow N\pi\) reaction allowed) were subsequently eliminated\(^{(3,91)}\), but no systematic comparison of the new version with inclusive spectra has been made. The \(\pi^+\) spectra measured in Ar + KCl (E/A = 1.8 GeV) central collisions have been also compared to the Liège cascade results\(^{(102)}\). The model predicts a too large anisotropy for medium and large pion energies, but the shape of the 90° c.m. spectrum is correctly reproduced except for a 5% high energy component.

The first version of the Yariv and Fraenkel cascade\(^{(17)}\) also overpredicts the pion yield, but the shape of the spectra is closer to experiment. Their second version compares moderately well with inclusive low energy pion data for the Ar + Ca (E/A = 1.05 GeV) reaction. However the model predicts an experimentally observed bump of the invariant cross-section in the region \(\gamma_{c.m.} = 0\), but at a transverse momentum of \(\sim 100\) MeV/c instead of the experimental \(\sim 50\) MeV/c. The Gudima and Toneev cascade\(^{(80)}\) gives a good agreement with the experimental inclusive \(\pi^+\) spectra for the reaction Ar + KCl at 800 MeV/u. A rather good agreement is also found at E/A = 2.1 GeV\(^{(85,96)}\) and at Dubna energies\(^{(103,104)}\). The Kitazoe et al. cascade\(^{(13,98)}\) gives an amazing agreement with the experimental Ar + KCl and Ne + Na F data, which is due, according to the authors, to their improved description of \(\pi^+\)'s and \(\Delta^+\)'s and to their binding prescription.
The multiplicity distribution is also corrected predicted. Finally, let us quote the

$$E_{cm} = 0$$ (GeV)"
also a good agreement with the data using a new prescription. In ref. 110), the \( \pi^+ \) and \( \pi^- \) multiplicities calculated from the Liège cascade have been compared to the data for the \( ^4\text{He} + A \) (E/A = 800 MeV) reactions, using an improved version of the model (isospin dependence of the elementary cross-sections, "freezing", binding energy prescription, and improved Pauli blocking). For heavy targets and high proton multiplicity reactions, even this improved version of the code still strongly overpredicts the data suggesting that pion absorption in nuclear matter could be underestimated in the model. As a matter of fact, the medium effects can strongly modify the N, A, \( \pi \) dynamics\(^{91,92,11,8,11,112}\). In ref. 11), it is shown that a good agreement with the experimental Ar + KCl data can be achieved by slightly shifting the delta mass, or by decreasing the inelastic cross-section. The pion yield in \( p + A \) reactions is overestimated by the standard Liège cascade, and the true pion absorption in \( \pi + A \) is underestimated. Those discrepancies can be removed by strongly increasing the in-medium pion absorption\(^{11,92}\). Finally, let us quote pion yield calculations at the Dubna energies (E/A = 3.4-3.7 GeV); only for heavy targets and central collisions, the cascade models tend to predict too many pions\(^{103,104,113-115}\).

5.6. Global variables and collective flow of nuclear matter

The cascade predictions have been also examined for the observables obtained from exclusive (or quasi-exclusive) measurements. Such observables are generally studied in correlation with an impact parameter selection. Experimentally, the selection is made on the charged particle multiplicity. Note that only few comparisons have been made with experimental multiplicity distributions. In ref. 17), a rather good agreement is obtained with Ne + Au experimental multiplicity distribution. In refs. 116,117) the isotropy of Ar + Pb high multiplicity events is calculated using the Liège cascade, in comparison with experimental data. They find that the cascade events are more forward peaked than the experimental ones, which could be due to a too low stopping power of nuclear matter in the model. In order to study the collective flow of nuclear matter\(^{118}\), the sphericity tensor (eq. (4.22)) was measured, and also calculated using cascade models\(^{80,24,99,119-123}\). From the sphericity tensor, an angle with respect to the beam axis called flow angle can be determined for each collision. The flow angle gives the average emission direction of the flowing nuclear matter. The flow angle distribution was first measured for Ar + Pb\(^ {116}\) and Ca + Ca or Nb + Nb\(^ {68}\). In the first case, the results were compared to the Liège cascade predictions, and in the second case to the Yariv and Fraenkel cascade ones. In both cases, the models predict too low flow angles. This can be accounted for by the lack of compression energy in the cascade. Contrary to an erroneous statement, the cascade models generally predict a non-zero flow. It has been demonstrated in ref. 24) that the Liège cascade predicts a finite flow, but that its magnitude is lower than the experimental one. A quantitative evaluation of the discrepancy is difficult because of the uncertainties due to the experimental filter; and also because, once again, the way the binding of the nucleons and the Pauli blocking are treated influences the final flow angle values\(^{122,123}\). The Kitazoe cascade\(^{123}\) predicts rather large flow angles, close to experimental ones, presumably (at least partly) because of the choice of the binding prescription. Note that a more satisfactory way to include the binding potential consists in putting in a locally calculated mean field as in VUU, BUU or "Landau-Vlasov" models (see section 6).

Another method for flow measurement was proposed by Danielewicz and Odyniec\(^ {71}\), which consists in using the \( \langle p_T^2 \rangle \) versus \( y \) function (see eq. (4.26)), \( y \) being the rapidity of the nucleons and \( \langle p_T^2 \rangle \) the mean value of their transverse momentum projected on the reconstructed reaction plane. In their paper, Danielewicz and Odyniec showed that the Liège cascade exhibits a finite flow smaller than the experimental one. Fig. 5.4 shows the experimental

![Fig. 5.4. Mean transverse momentum per nucleon projected on the reaction plane as a function of rapidity. The circles correspond to the experimental Ar + BaL\(_2\) results for three multiplicity cuts. The dashed curves give the results for the INC model of ref. 8) with a freezing prescription. The quantity \( \phi \) is the estimated azimuthal angle with respect to the reconstructed reaction plane. Adapted from ref. 128).](image-url)
7. Conclusion

The INC model has been very successful in explaining the main features of information movements to the core (and perhaps other problems). The conclusion of the paper is that the INC model is not capable of predicting the number of information movements in the core (due to the absence of some factors). The INC model predicts that the number of information movements will be higher than the actual number of movements. Therefore, the INC model is not a good predictor of information movements. However, we can see that the one-body distribution function, which emphasizes the presence of core (3.11) and the one-body distribution function (3.11) are important in understanding the mechanism of information movements. The INC model's predictions are not consistent with the actual number of movements. Therefore, any conclusion cannot be drawn without some additional experiments. Further experiments are needed to understand the mechanism of information movements with the INC model.
processes are important. We think especially to hadron-nucleus or antibaryon-nucleus interactions, especially in the 1 to 10 GeV range. However, we think that an effort should be done in the next future to improve the collision dynamics: introduction of quantum effects in collisions, improvement of the crude production models, medium corrections.

Finally, we recall that the simulation method used in INC, in contradistinction with the quasi-particle methods used in mean field theories, is a good tool to evaluate high-order distribution functions as well as fluctuations, in spite of the problems for achieving sufficient statistics.

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