Topological invariants, diametral dimension, and one related question

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PhD Seminars Mathematics - UGent

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Introduction

Diametral dimension

An open question about diametral dimension(s)

Functional analysis

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- Important to *compare structures* of tvs: linear maps, continuous maps, etc.
- **Isomorphism**: a *bijective* map between 2 tvs, which is *linear* and *continuous* and has a *continuous inverse*.

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- dimension in vector spaces;
- being Hausdorff or not in topological spaces;
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- ... and the **diametral dimension** for tvs!

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The n-th *Kolmogorov's diameter* of V with respect to U is

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Some properties

- $0 \leq \delta_{n+1}(V, U) \leq \delta_n(V, U) \leq \mu$;
- if $V_0 \subseteq V \subseteq U \subseteq U_0$, then $\delta_n(V_0, U_0) \leq \delta_n(V, U)$;
- if $T : E \to F$ is linear, $\delta_n(T(V), T(U)) \leq \delta_n(V, U)$.

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Theorem

The diametral dimension is a **topological invariant** (if $E \cong F$, then $\Delta(E) = \Delta(F)$).

Examples

Some reminders

- A *seminorm* on E is a map $p : E \to [0, \infty)$ s.t.
  1. $\forall x, y \in E$, $p(x + y) \leq p(x) + p(y)$;
  2. $\forall x \in E$, $\forall \lambda \in \mathbb{C}$, $p(\lambda x) = |\lambda| p(x)$.

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Examples: holomorphic functions

With the topology defined by the seminorms “suprema on compact sets”, we have

$$\Delta(H(\mathbb{D})) = \bigcap_{k\in\mathbb{N}} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi_n e^{-n/k} \to 0\right\}$$
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If E is a tvs/lcs and $\mathcal{U}$ a 0-neighbourhood basis,

$$\Delta_b(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall B \text{ bounded}, \ \forall U \in \mathcal{U}, \xi_n \delta_n(B, U) \to 0 \right\}.$$

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Question

$\Delta(E) = \Delta_b(E)$ for lcs ???

A partial answer

Notation

$$\Delta^{\infty}(E) := \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U},\ \exists V \in \mathcal{U}, C > 0 \text{ s.t. } |\xi_n|\delta_n(V,U) \leq C\right\}.$$

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Theorem (2016, L.D., L. Frerick, J. Wengenroth)
If E is a Schwartz metrizable lcs s.t. $\Delta(E) = \Delta^\infty(E)$, then $\Delta(E) = \Delta_b(E)$.

- E is **Schwartz** if $\forall U \in \mathcal{U}, \exists V \in \mathcal{U}$ s.t. $\forall \varepsilon > 0$, $\exists F$ a finite part of E s.t. $V \subseteq \varepsilon U + F$.
- A lcs E is **metrizable** iff it is Hausdorff and its topology can be defined by a countable family of seminorms.

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Warning!

There exist Schwartz **non-metrizable** lcs E with $\Delta(E) \subsetneq \Delta_b(E)$ (2017, F. Bastin, L.D.).

One last concept

Look at this:

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$\rightsquigarrow$ Prominent bounded sets (2013, T. Terzioglu)

A bounded B set of E is *prominent* if $\forall U \in \mathcal{U}$, $\exists V \in \mathcal{U}, C > 0$ s.t. $\delta_n(V, U) \leq C\delta_n(B, V)\ \forall n \in \mathbb{N}_0$.

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If E has a prominent set, then $\Delta(E)=\Delta_b(E)$.

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Spaces with prominent sets (2016, L.D., L.F., J. W.)

Metrizable lcs with **property** $(\overline{\Omega})$: if $\mathcal{U} = (U_k)_{k\in\mathbb{N}}$,

$$\forall m, \exists k, \forall j, \exists C > 0 : U_k \subseteq rU_j + \frac{C}{r} U_m \ \forall r > 0.$$

Applications of this theory?

- Multifractal analysis: study of signals and "regularity".

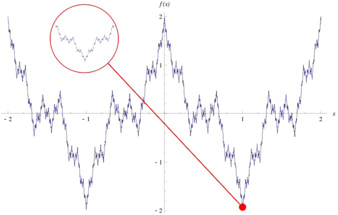

Figure: Weirstrass Function

(https://upload.wikimedia.org/wikipedia/commons/thumb/6/60/WeierstrassFunction.svg/795px-WeierstrassFunction.svg.png)

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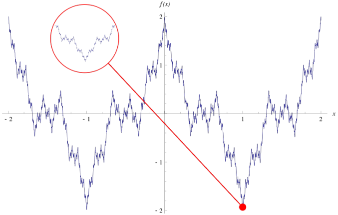

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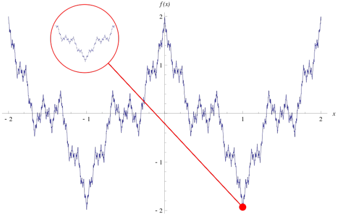

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More details? See you in Han-sur-Lesse on June 8th-9th! ☺

Thank you for your attention!

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