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OPTICAL-MODEL DESCRIPTION OF THE FRAGMENTATION OF A SINGLE-PARTICLE STATE

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Abstract: Several possible definitions of the $l = 0$ neutron strength function are examined. Their energy dependence is investigated and particular attention is paid to threshold effects when the single-particle state lies in the vicinity of the elastic threshold. Among other results, it is shown that, in actual physical situations, the *S*-matrix strength function never exhibits a cusp at threshold.

1. Introduction

The nuclear single-particle excitation, which is a model state built up by transferring a nucleon to some nucleus, is mixed into many of the actual physical states of the nucleus in a way which can be found from the strength function. Roughly speaking, this quantity is proportional to the overlap of the single-particle state with the nuclear physical states contained in an energy interval of unit length. The question then arises to know the energy variation of the latter quantity. This problem has first been theoretically investigated in the framework of the *R*-matrix theory by Lane, Thomas and Wigner [1]) and in the framework of Kapur-Peierls theory by Brown [2]). In particular, these authors showed that the energy dependence of the strength function should exhibit a giant resonance, i.e. a bump of Lorentzian shape located near the single-particle excitation energy. In the spirit of the *R*-matrix (or Kapur-Peierls) theory, this is only true for the reduced strength function and for a given boundary condition at the nuclear surface. In other words, the *R*-matrix result is only valid for the mixing of a given single-particle state which satisfies given boundary conditions at the surface, into many complicated states which satisfy the *same* boundary conditions. Actually, the physical states correspond to boundary conditions that are different from the *R*-matrix boundary conditions, and that, moreover, can vary with the energy of the state. The form of the giant resonance should be distorted, especially if the single-particle energy lies in the vicinity of the particle threshold. Moreover, what is often measured is the *S*-matrix strength function which can be very different from the *R*-matrix one. These questions have been investigated in the past by McVoy and collaborators [3]), by Schäfer [4]) and recently by Lane [5]). Their results can be summarized as follows: (i) the boundary condition mixing (due to different boundary conditions for the single-particle state and the complicated physical states into which it is mixed) can distort the giant resonance, especially for the neutron $l = 0$ wave [4]); (ii) the neutron $l = 0$ S-matrix strength function can exhibit a resonance below threshold only, while it is always decreasing above threshold – it displays a cusp at neutron threshold if the single-particle state is slightly unbound; (iii) the S-matrix strength function above neutron threshold is well described by the optical model, except in the vicinity of the optical model singularities, i.e. the zeroes of the optical model collision matrix [4]).

Here, we re-investigate this question. Its interest has been revived by recent work on the valency model of low-energy neutron radiative capture [6-8]). In this model, the partial radiative width is related to the neutron strength function. Moreover, the validity of this model critically depends on the value of the neutron strength function below neutron emission threshold [5, 8]).

More precisely the aim of this paper is twofold: (i) we investigate the properties of the neutron $l = 0$ strength function below threshold in the frame of the optical model. This has already been done by Goebel and McVoy who restricted themselves to the scattering length approximation [3]). Here we go beyond the latter approximation, perform full numerical calculations and arrive at somewhat different conclusions; (ii) we examine different possible definitions of the strength function and we compare them with each other and with some measurable quantities.

2. Different definitions of the strength function

In this section, we successively examine the definitions of the strength functions above and below threshold. We assume that the neutron elastic channel is the only physical channel.

2.1. ABOVE THRESHOLD

In the one channel case, we have for a given partial wave (we omit the l-index)

$$S = \frac{1+iK}{1-iK}. \tag{2.1}$$

A plausible form for the (real) K-function is

$$K = K^0 + \tfrac{1}{2}\sum_{\lambda} \frac{\Gamma_{\lambda n}}{E_\lambda - E}, \tag{2.2}$$

where all the quantities are real and where K^0 is a smoothly varying function. The optical S-function is defined by

$$S_{\text{opt}} = S(E+iI) = \langle S\rangle = \frac{1+i\langle K\rangle}{1-i\langle K\rangle}, \tag{2.3}$$

where the brackets mean that the quantity inside must be evaluated at the complex

energy $E+iI$. Taking account of the relation

$$S_{\text{opt}} = \exp(2i\delta_{\text{opt}}), \tag{2.4}$$

one gets

$$\text{Im}\,\langle K\rangle = \tfrac{1}{2}\pi \frac{\langle\Gamma_{\lambda n}\rangle}{D} = \text{Im}\tan\delta_{\text{opt}}. \tag{2.5}$$

For $l = 0$ neutron, it is of interest to consider the following quantity

$$s_K = \frac{1}{k}\frac{\langle\Gamma_{\lambda n}\rangle}{D} = \frac{2}{\pi k}\,\text{Im}\tan\delta_{\text{opt}}, \tag{2.6}$$

which we will refer to as the K-matrix strength function and which at low energy is, up to a constant, equal to the reduced R-matrix strength function. In eq. (2.6), k is the neutron wave number.

The pole strength function of the S-matrix is also often considered. In the customary low-energy limit, one has

$$s_S = \frac{T}{2\pi k} = \frac{1-|S_{\text{opt}}|^2}{2\pi k}. \tag{2.7}$$

This relation has to be regarded as a convenient definition rather than a true mathematical relation. Indeed, there is a problem related to the proper definition of the S-matrix strength function. If the strength function is taken to be, as in the K-matrix case, related to the poles strength function, then relation (2.7) is certainly not correct since the residues are complex. The alternative is then to consider, instead of s_S, the quantity

$$g_M = \frac{1}{k}\frac{\langle g^2\rangle}{D}, \tag{2.8}$$

where the g^2 are the residues of the S-matrix. Moldauer has shown that [9])

$$g_M = \frac{1}{2\pi k}\left(\frac{1}{S^*_{\text{opt}}} - S_{\text{opt}}\right). \tag{2.9}$$

Finally, one can consider the strength associated with the total width ω_λ (minus twice the imaginary part of the pole energy)

$$s_M = \frac{1}{k}\frac{\langle\omega_\lambda\rangle}{D}. \tag{2.10a}$$

It has been shown [10]) that the relation

$$s_M = -\frac{1}{\pi k}\ln|S_{\text{opt}}| = \frac{2}{\pi k}\,\text{Im}\,\delta_{\text{opt}}, \tag{2.10b}$$

holds exactly, in the one-channel case, although its validity has been checked successfully in model calculations involving many channels. In the following, we shall mainly consider s_S as defined in (2.7), paying some attention to the quantities (2.8) and (2.10). This is also the attitude adopted in ref. [3]).

We also consider a definition given by Porter [11]), which involves other optical-model quantities than phase shifts:

$$s_P = -\frac{2}{k}\int_0^\infty \mathrm{d}r\, |u_{\mathrm{opt}}|^2 W(r)\mathrm{d}r, \tag{2.11}$$

where $W(r)$ is the imaginary potential and where u_{opt} is the optical wave function normalized as

$$u_{\mathrm{opt}} \approx \sqrt{2/\pi\hbar v}\,\sin(kr+\delta_{\mathrm{opt}}). \tag{2.12}$$

The relation $s_P = s_K$ $(k = 0)$ has been proved by Porter [11]) to hold exactly at zero energy. In the appendix, we show how it transforms at finite energy.

2.2. BELOW THRESHOLD

Here, relation (2.1) has to be changed if one requires the reality of K. A possible choice is to associate K with the linearly independent wave functions which behave asymptotically like $\cosh \kappa r$ and $\sinh \kappa r$. Then the solution u_E to a $l = 0$ one-channel problem for $E < 0$ behaves like

$$u_E \approx \sinh \kappa r + K \cosh \kappa r, \tag{2.13}$$

where κ is the absolute value of the wave number. The matrix K is then real and one has

$$S = \frac{1-K}{1+K}; \tag{2.14}$$

changing (2.1) into (2.14) is in keeping with R-matrix theory, where one should have (except for the hard-sphere phase shift) $K = kR$, and where one changes k in $i\kappa$ in passing below threshold. Because of the analytic property of S, K can keep the same form (2.2), where the poles E_λ which are lying below threshold are now of interest.

To relate the strength function to the optical-model quantities, we make the usual assumption

$$S_{\mathrm{opt}} = \langle S \rangle. \tag{2.15}$$

Furthermore, we adopt the following parametrization of S_{opt}

$$S_{\mathrm{opt}} = \frac{1-\tan\delta_{\mathrm{opt}}}{1+\tan\delta_{\mathrm{opt}}}, \tag{2.16}$$

where δ_{opt} is determined by assuming the following behaviour for the negative energy optical model standing wave

$$u_{\mathrm{opt}} \approx \cosh \kappa r \sin\delta_{\mathrm{opt}} + \sinh \kappa r \cos\delta_{\mathrm{opt}}, \tag{2.17}$$

where $E = -\hbar^2\kappa^2/2m$.

Then, the K-matrix strength function reads

$$s_K = \frac{1}{\kappa}\frac{\langle \Gamma_{\lambda n}\rangle}{D} = \frac{2}{\pi k}\,\mathrm{Im}\tan\delta_{\mathrm{opt}}. \tag{2.18a}$$

The K-matrix is closely related to the R-matrix. This relation is particularly simple in a one-channel situation. Hence the following relation connects the quantity (2.18a) and the conventional R-matrix strength function with zero boundary condition s_R $(B = 0)$:

$$s_R(B=0) = \frac{s_K}{2a}\frac{1}{|\cosh \kappa a + \tan\delta_{\mathrm{opt}}\sinh \kappa a|^2}, \tag{2.18b}$$

where a is the channel radius.

The S-matrix strength function can be obtained as follows. A plausible structure of S below threshold is

$$S = S^0 + \sum_\lambda \frac{\omega_{\lambda n}}{E-e_\lambda}, \tag{2.19}$$

where all quantities are real and where S^0 is a smoothly varying function. We have

$$\langle S \rangle = S^0 + \mathscr{R} - i\pi\langle \omega_{\lambda n}\rangle/D. \tag{2.20}$$

Hence, the S-matrix strength function is

$$s_S = \frac{1}{\kappa}\frac{\langle \omega_{\lambda n}\rangle}{D} = -\frac{1}{\pi k}\,\mathrm{Im}\,S_{\mathrm{opt}}. \tag{2.21}$$

Using eqs. (2.21) and (2.16), it is easy to obtain

$$s_S = \frac{2}{\pi\kappa}\frac{\mathrm{Im}\tan\delta_{\mathrm{opt}}}{|1+\tan\delta_{\mathrm{opt}}|^2}. \tag{2.22a}$$

The poles of the S-matrix are connected with the physical states at negative energy, as are the Kapur-Peierls states. Therefore, a relation exists between the quantity s_S and the so-called Kapur-Peierls strength function s_{KP}. It has the form

$$s_S = 2ae^{2\kappa a}s_{\mathrm{KP}}, \tag{2.22b}$$

where a is the channel radius.

A priori, the quantity (2.22a) has no upper bound. Thus it could be useful to consider a modification of eq. (2.21)

$$s_N = -\frac{1}{\pi k}\,a\tan\left[\frac{\mathrm{Im}\langle S\rangle}{\mathrm{Re}\langle S\rangle}\right], \tag{2.23}$$

which reduces to s_S in the limit of vanishing $\langle \omega_{\lambda n}\rangle/D$. With the help of eq. (2.16), it is straightforward to get

$$s_N = \frac{1}{\pi k} \arctan \left\{ \frac{2 \operatorname{Im} \tan \delta_{opt}}{1-|\tan \delta_{opt}|^2} \right\}. \quad (2.24)$$

Finally, the Porter formula (2.11) can be extended below threshold, where u_{opt} has the asymptotic form (2.17).

2.3. RELATION BETWEEN THESE DEFINITIONS AND MEASURABLE QUANTITIES

Above threshold, the usually measured quantity is close to s_S. However, s_K is often extracted from experiment (for instance, when some R-matrix fit is performed). Both quantities have thus a physical meaning above threshold.

Below threshold, the K-matrix strength function has no direct physical meaning. The poles of the K-matrix have little relation with the physical bound states. Rather, the latter are connected with the quantity s_S which is proportional to the single-particle strength function measured in (d, p) reactions.

Three problems arise from the introduction of the quantities s_P and s_N and are investigated in the next sections: (i) Is s_P connected to s_K or s_S? (ii) How is this connection affected by the nature of the absorption (volume or surface)? (iii) How different is s_N from s_S?

3. Low-momentum expansions

In this section, we derive some expressions involving power series in the momentum, which are very useful in understanding the behaviour of different quantities defined in sect. 2.

We start with the scattering length expansion ($k > 0$),

$$k \cot \delta_{opt} = -a^{-1} + \tfrac{1}{2} r_0 k^2 + \ldots, \quad (3.1)$$

or

$$K = \tan \delta_{opt} = -ak + \rho_0 k^3 + \ldots, \quad (3.2)$$

where

$$\rho_0 = -\tfrac{1}{2} a^2 r_0. \quad (3.3)$$

Here, the scattering length

$$a = a_s - i\gamma_0 \quad (3.4)$$

and the effective range r_0 are complex quantities. The quantity γ_0 is related to the low energy K-matrix strength function since, with the help of eq. (2.6), we have

$$s_K = \frac{2}{\pi} \gamma_0 + \frac{2}{\pi} (\operatorname{Im} \rho_0) k^2 + \ldots. \quad (3.5)$$

Eq. (2.13) shows that the K-matrix below threshold is obtained from its value above threshold by replacing k by $i\kappa$ and by multiplying by $-i$. Hence, below threshold ($\kappa > 0$), we have

$$\tan \delta_{opt} = -a\kappa - \rho_0 \kappa^3 + \ldots \quad (3.6)$$

Eqs. (3.5), (3.6) and (2.18) give for $E = k^2 > 0$ or $E = (i\kappa)^2 < 0$, ($\hbar = m = 1$),

$$s_K = \frac{2}{\pi} \gamma_0 + \frac{2}{\pi} E \operatorname{Im} \rho_0 + \ldots. \quad (3.7)$$

This relation shows that the K-matrix strength function and its derivative are continuous across threshold.

The power series expansion of the optical phase shift starts as

$$\delta_{opt} = -ak + (\rho_0 + \tfrac{1}{3} a^3) k^3 + \ldots. \quad (3.8)$$

For $E > 0$, it is easy to get from eq. (2.7)

$$s_S = \frac{2}{\pi} \gamma_0 - \frac{4\gamma_0^2}{\pi} k + \frac{2}{\pi} [\operatorname{Im} (\rho_0 + \tfrac{1}{3} a^3) + \tfrac{8}{3} \gamma_0^2] k^2 + \ldots. \quad (3.9)$$

For negative energies, we have, taking eq. (3.6) into account, the following expansions

$$|1 + \tan \delta_{opt}|^2 = 1 - 2a_s \kappa + |a|^2 \kappa^2 - 2(\operatorname{Re} \rho_0) \kappa^3 + \ldots, \quad (3.10)$$

$$s_S = \frac{2}{\pi} \gamma_0 + \frac{4a_s}{\pi} \gamma_0 \kappa + \frac{2}{\pi} [4a_s^2 \gamma_0 - |a|^2 \gamma_0 - \operatorname{Im} \rho_0] \kappa^2 + \ldots. \quad (3.11)$$

From these two relations, it is evident that s_S has some special behaviour at threshold, which is characterized by the ratio

$$r = \gamma_0 / a_s.$$

Let us give here the expansions for the quantities g_M (eq. (2.8)) and s_M (eq. (2.10))

$$g_M = \frac{2}{\pi} \gamma_0 + \frac{2}{\pi} \operatorname{Im} (\rho_0 + a^3) k^2 + \ldots + i \left[-\frac{4}{\pi} a_s \gamma_0 k + \ldots \right], \quad (3.12)$$

$$s_M = \frac{2}{\pi} \gamma_0 + \frac{2}{\pi} \operatorname{Im} (\rho_0 + \tfrac{1}{3} a^3) k^2 + \ldots. \quad (3.13)$$

We note that, up to second order, the real part of g_M and s_M are very close to each other and also very close to s_K. The imaginary part of s_M tends towards zero at threshold.

The same expansion as (3.11) for s_N is

$$s_N = \frac{2}{\pi} \gamma_0 + \frac{2}{\pi} [\operatorname{Im} \rho_0 - |a|^2 \gamma_0 - \tfrac{8}{3} \gamma_0^3] \kappa^2 + \ldots, \quad (3.14)$$

which contains no term linear in κ.

It is shown in the appendix that the low-energy expansions of s_P are

$$s_P = \frac{2}{\pi}\gamma_0 - \frac{2}{\pi}(-\text{Im}\,\rho_0 + \gamma_0(a_s^2 - \gamma_0^2))k^2 + \ldots, \tag{3.15}$$

if $E > 0$, and

$$s_P = \frac{2}{\pi}\gamma_0 - \frac{2}{\pi}(\text{Im}\,\rho_0 + \gamma_0(a_s^2 - \gamma_0^2))k^2 + \ldots, \tag{3.16}$$

if $E < 0$. They indicate that s_P is continuous through threshold, and that its derivative is discontinuous, but remains finite on both sides. These conclusions are supported by our calculations below.

4. Numerical results

We show in figs. 1 to 4 some results of the calculation of the different expressions of the neutron strength function using a Saxon-Woods potential with volume absorption. The parameters are $V = 42.8$ MeV, $r_0 = 1.30$ fm, $a = 0.69$ fm and $W = 3.36$ MeV. This potential has been used in relation with the study of the valence capture model in the 3s region [8]). We are interested in the same region here. For this potential, the 3s level crosses the threshold around the critical value $A_c \approx 58$.

We have also performed a similar calculation with a surface absorption optical model potential [12]). The results are not shown, since they are qualitatively the same as those described below. We first comment about the K-matrix strength function.

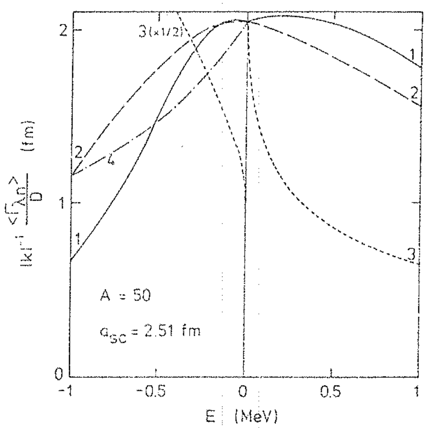

Fig. 1. Calculation of the different expressions of the $l = 0$ neutron strength function for $A = 50$. The parameters of the optical potential are given in the text. Curves 1 to 4 correspond to s_P [eq. (2.11)], s_K [eqs. (2.6) and (2.18)] s_S [eqs. (2.7) and (2.21)] and s_N [eq. (2.23)], respectively.

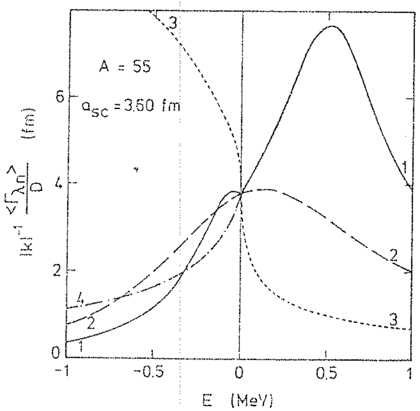

Fig. 2. Same as fig. 1 for $A = 55$.

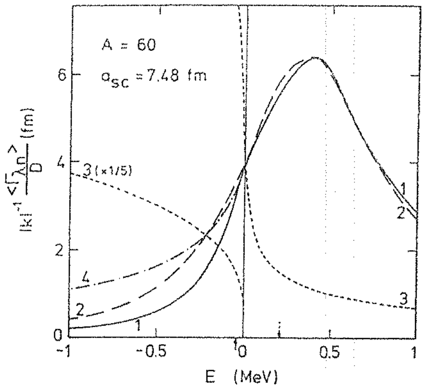

Fig. 3. Same as fig. 1 for $A = 60$. The arrow in full line indicates the position of the 3s level in the real part of the optical potential. The dotted arrow indicates the energy where the real part of the optical potential crosses $\frac{1}{2}\pi$ downwards.

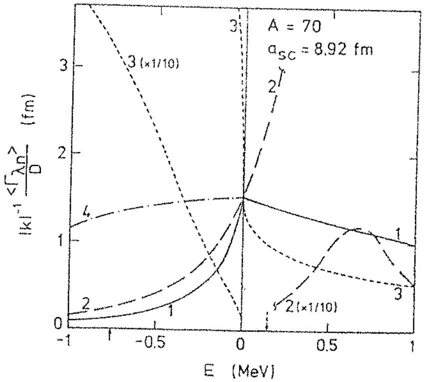

Fig. 4. Same as fig. 3 for $A = 70$.

The most remarkable result consists in the appearance of bumps at positive energy and which, roughly speaking, moves upwards in energy when A increases. This result can be related to the following general property of realistic optical model potentials (i.e. with the right absorption): the real part of their corresponding scattering length a_s (eq. (3.4)), is *always* positive. For a real potential, the scattering length is negative for $A < A_c$ (the critical value), and positive for $A > A_c$, with a discontinuity at $A = A_c$, and tends approximately to the well radius when A is going away from A_c. For a realistic optical-model potential, the discontinuity disappears, and a_s varies gently with A about the well radius; it thus remains positive. Looking at eqs. (3.3) and (3.5), and taking account of the fact that r_0 is practically equal to the well radius, we see that $\text{Im}\,\rho_0$ is in general positive. Hence, in general, s_K is increasing with E starting from threshold.

One can also show that the big bumps observed for $A = 60$ and 70 are a manifestation of a so-called echo state (characterized by the fact that the real part of the optical phase shift decreases through $\frac{1}{2}\pi$). Indeed, starting from eq. (2.6), and using the relation $\delta_{\text{opt}} = \delta_R + i\delta_I$, it is easy to get

$$s_K = \frac{2}{\pi k}\tanh\delta_I \frac{1+\tan^2\delta_R}{1+\tan^2\delta_R\tanh^2\delta_I}. \tag{4.1}$$

Now, $\tanh\delta_I \approx \delta_I \approx \gamma_0 k < 1$, even when $\delta_R \approx -\frac{1}{2}\pi$, as we checked numerically. For the latter value, one has

$$s_K \approx \frac{2\gamma_0}{\pi}\frac{1}{\gamma_0^2 k^2} > s_K(k=0). \tag{4.2}$$

For $A < 60$, we did not observe an echo state, at least within the range of energy we investigate. However, for all values of A, the real part of the optical phase shift is always decreasing. The scattered part of the optical wave always precedes the incident part, for realistic optical-model potentials.

The S-matrix strength function s_S displays an inflexion point at threshold for all values of A. A cusp has never appeared, contrarily to what was suggested by Goebel and McVoy [3]). The reason is that the real part of the optical-model scattering length is always positive. Taking this observation into account, we agree with McVoy and Goebel's result, which implies that a positive scattering length† leads to an inflexion point. We have checked that this inflexion point is, for A less than the critical value, a consequence of the rather strong absorption. In fig. 5, we artificially reduce the amplitude of the imaginary potential ($W = 0.5$ MeV) in order to get a *negative* scattering length for $A = 55$. The situation is dramatically different from what is shown in fig. 2. In particular, the strength function now shows a marked cusp at threshold.

We also recover, in this calculation, two well-known properties of s_S: (i) it is decreasing with energy above threshold, which results from eq. (3.9); (ii) it is always smaller than s_K above threshold. The latter property can be demonstrated with the help of eqs. (2.4), (2.5) and (2.7), which yield

$$s_S = \frac{2}{\pi k} \frac{\mathrm{Im}\tan\delta_{\mathrm{opt}}}{|1-i\tan\delta_{\mathrm{opt}}|^2}. \tag{4.3}$$

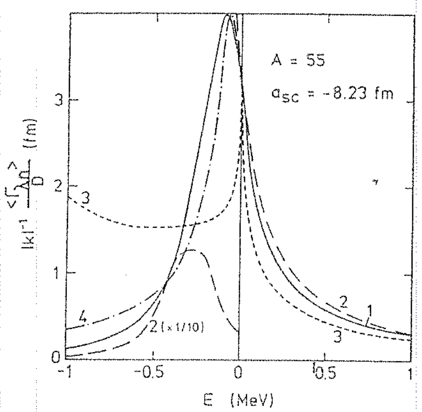

Fig. 5. Same as fig. 2 for $A = 55$ with a reduced imaginary potential to make the scattering length negative.

† Note that the scattering length, as defined in ref. [3]), has the sign opposite to the usual definition, which is adopted here.

On the other hand, eq. (2.3) gives

$$\tan\delta_{\mathrm{opt}} = -i\frac{S_{\mathrm{opt}}-1}{S_{\mathrm{opt}}+1}, \tag{4.4}$$

or

$$1-i\tan\delta_{\mathrm{opt}} = \frac{2}{S_{\mathrm{opt}}+1}. \tag{4.5}$$

Finally, we get from eqs. (4.3), (4.5) and (2.6)

$$s_S = \tfrac{1}{4}s_K|S_{\mathrm{opt}}+1|^2. \tag{4.6}$$

It can be verified by the same trick that this relation also holds below threshold. Above threshold, $|S_{\mathrm{opt}}| \leqq 1$, which implies $s_S \leqq s_K$.

A very intriguing feature of our results is the continuous rise of s_S as the energy becomes more and more negative. This rise seems to persist even when the energy is below the unperturbed single-particle energy (see e.g. fig. 3). The quantity $\mathrm{Re}\,\delta_{\mathrm{opt}}$ seems to tend rapidly to $-\frac{1}{4}\pi$ as the energy becomes negative, which makes s_S very large (see eq. (2.22)). This seems quite general, almost independent of the location of the single-particle state. The large values of s_S for negative energy have to be put in parallel with eqs. (2.18b) and (2.22b). The fast variation of the factors involving κ_a should then indicate that s_{KP} and s_R ($B = 0$) are slowly varying functions at negative energies. Two effects may modify the variation of s_S: (i) the imaginary part of the potential must decrease with energy, since it should vanish at the Fermi level; (ii) the antisymmetrization principle which requires the wave function to be orthogonal to the 2s orbital in the real part of the optical potential, which can be considered as fully occupied in our case. For a real potential, the scattering state is automatically orthogonal to the 2s state, at positive energy. This orthogonality is not ensured for a complex potential, especially if the energy is more and more negative. It is, however, difficult to guess whether this effect reduces the strength function s_S.

We make a final remark on this S-matrix strength function. Although definition (2.21) is the most acceptable one for s_S below threshold, the latter quantity is not the analytic continuation (in the k-variable) of the definition of s_S above threshold, i.e. the quantity (2.7), as it can be seen by comparing the expressions (3.9) and (3.11). The same remark applies to the work by Goebel and McVoy. This fact is confirmed by eq. (4.3), since even if a function is analytic, its modulus is not analytic. However, we emphasize that if one could find a definition of s_S which should be the analytic continuation of (3.9) below threshold, one should get the same qualitative behaviour of s_S at threshold, since a_s has the same sign as γ_0.

The quantity s_N (for $E < 0$) has no special properties except the expected ones. It seems, however, to decrease more slowly when E decreases, for large A. Let us finally turn to the Porter definition (eq. (2.11)). We see that s_P tends to s_S and s_K at zero energy, as it should [11]). It is, however, difficult to say whether s_P is close to

s_S or to s_K. The only thing we can say is that s_P is rather closer to s_K at negative energy. We did also some calculations with Moldauer's optical potential [12]), which contains a surface absorption. The results are not displayed here, but the conclusion is the same, namely that for $E < 0$, s_P is rather close to s_K, while at positive energy, it is sometimes closer to s_K and sometimes closer to s_S.

Let us finally notice that the quantity which is called strength function in the literature is proportional to $s_K(0) = s_S(0)$, with constant of proportionality equal to $2.1442 \times 10^{-4}\ \text{fm}^{-1}$. It can be seen from figs. 1 to 4 that we recover the giant resonance in the A-dependence of the strength function.

5. Conclusions

The optical-model picture of the fragmentation of the $l = 0$ single-particle state near threshold leads to the following conclusions:

(i) Above threshold, the K-matrix strength function has usually a bump, which can be related to an "echo state", always present with a realistic absorption.

(ii) The S-matrix strength function displays an inflexion point at threshold. We have shown that this is related to the observed fact that the real part of the scattering length for a realistic optical potential is positive. If it were negative, a cusp would appear instead of an inflexion point.

(iii) The Porter definition is in general quite different from the S- or K-matrix strength function, except at zero energy, where all three quantities are equal.

From a physical point of view, the S-matrix strength function seems to be the more relevant quantity below threshold. In this region, it increases with a rate related to the optical-model scattering length. In the valence capture model, the photon strength function is related to the neutron strength function by a factor which decreases as the energy becomes negative. This energy dependence is also governed by the scattering length [8]). Hence, it is expected that the photon strength function will exhibit different features near neutron threshold, depending on how A compares with the critical value. This question is investigated in a forthcoming paper.

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Appendix

Let us assume a local optical potential $V+iW$ and let u be a regular solution of the equation

$$u''+\left[k^2-\frac{2M}{\hbar^2}(V+iW)\right]u = 0. \tag{A.1}$$

We normalize u by the requirement that it tends asymptotically to a given function $v(r)$ solution of

$$v''+k^2v = 0. \tag{A.2}$$

Then u obeys the conditions

$$u(0) = 0, \qquad u(r) \underset{r\to\infty}{\to} v(r). \tag{A.3}$$

It is easy to get

$$\operatorname{Im} u^*u'' = +\frac{2M}{\hbar^2}(u^*Wu), \tag{A.4}$$

$$\operatorname{Im} v^*v'' = 0. \tag{A.5}$$

Subtracting (A.5) from (A.4), integrating from zero to infinity and using

$$f^*f'' = (f^*f')-|f'|^2, \tag{A.6}$$

we get

$$\operatorname{Im}\{u^*u'|_0^\infty - v^*v'|_0^\infty\} = +\frac{2M}{\hbar^2}\int_0^\infty u^*Wu\,dr, \tag{A.7}$$

or

$$+\operatorname{Im}\{v^*(0)v'(0)\} = \frac{2M}{\hbar^2}\int_0^\infty u^*Wu\,dr. \tag{A.8}$$

Let us take, for instance, $v = \sin(kr+\delta_{opt})$. Hence,

$$+\operatorname{Im}\{k(\sin\delta_{opt})^*\cos\delta_{opt}\} = \frac{2M}{\hbar^2}\int u^*Wu\,dr. \tag{A.9}$$

Using an optical wave u_{opt} with a different normalization

$$u_{opt} \approx \sqrt{2/\pi\hbar v}\,\sin(kr+\delta_{opt}), \tag{A.10}$$

(A.9) becomes

$$+\operatorname{Im}\{(\sin\delta_{opt})^*\cos\delta_{opt}\} = \pi\int_0^\infty |u_{opt}|^2W\,dr. \tag{A.11}$$

At low energy, $\sin\delta_{opt} \approx \tan\delta_{opt}$, $\cos\delta_{opt} \approx 1$, and using (2.6), we have

$$\frac{1}{k}\frac{\langle\Gamma_{\lambda n}\rangle}{D} \approx -\frac{2}{k}\int_0^\infty |u_{opt}|^2W\,dr. \tag{A.12}$$

This is definition (2.11). With the help of eqs. (3.8) and (3.4) we have the exact result

$$\frac{2}{\pi}\gamma_0 = \lim_{k\to 0}\left(-\frac{2}{k}\right)\int_0^\infty |u_{opt}|^2W\,dr. \tag{A.13}$$

We recover the relation given by Porter [11]). It shows that at zero energy, the expression (2.11) for the strength function is identical to the S- or K-matrix strength function.

Let us turn to the low-momentum expansions. From eq. (3.8) we have for $E > 0$

$$(\sin \delta_{\rm opt})^* = -a^*k + (\rho_0^* + \tfrac{1}{6}a^{*3})k + \ldots, \tag{A.14}$$

$$\cos \delta_{\rm opt} = 1 - \tfrac{1}{2}a^2k + \ldots. \tag{A.15}$$

Eqs. (A.11) and (2.11) yield

$$s_{\rm P} = \frac{2\gamma_0}{\pi} - \frac{2}{\pi}[-\operatorname{Im} \rho_0 + \gamma_0(a_s^2 - \gamma_0^2)]k^2 + \ldots. \tag{A.16}$$

It can be verified, using eq. (3.6), that the similar expansion for negative energy is

$$s_{\rm P} = \frac{2}{\pi}\gamma_0 - \frac{2}{\pi}[+\operatorname{Im} \rho_0 + \gamma_0(a_s^2 - \gamma_0^2)]k^2 + \ldots. \tag{A.17}$$

References

1) A. M. Lane, R. G. Thomas and E. P. Wigner, Phys. Rev. **98** (1955) 693
2) G. E. Brown, Rev. Mod. Phys. **31** (1959) 893
3) K. W. McVoy, Nucl. Phys. **A115** (1968) 481, 495;
C. J. Goebel and K. W. McVoy, Nucl. Phys. **A115** (1968) 504
4) L. Schäfer, Nucl. Phys. **A188** (1972) 577
5) A. M. Lane, Phys. Lett. **B50** (1974) 207
6) A. M. Lane and S. F. Mughabghab, Phys. Rev. **C10** (1974) 412;
A. M. Lane, Proc. 2nd Int. Symp. on neutron gamma-ray spectroscopy and related topics, RCN, Petten, The Netherlands, 1975, p. 31
7) R. F. Barrett and T. Terasawa, Nucl. Phys. **A240** (1975) 445
8) J. Cugnon and C. Mahaux, Proc. 2nd Int. Symp. on neutron gamma-ray spectroscopy and related topics, RCN, Petten, The Netherlands, 1975, p. 81;
J. Cugnon and C. Mahaux, Ann. of Phys. **94** (1975) 128
9) P. A. Moldauer, Phys. Rev. Lett. **19** (1967) 1047
10) P. A. Moldauer, Phys. Rev. **177** (1968) 1841
11) C. E. Porter, Phys. Rev. **100** (1963) 65
12) P. A. Moldauer, Nucl. Phys. **47** (1963) 65