PROPERTIES OF AN EQUATION WITH A NEGATIVE COLLECTIVE MASS

J. CUGNON*

*Institut de Physique Nucléaire, Division de Physique Théorique**, 91406 - Orsay, France*

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The properties of a Schrödinger equation involving a variable collective mass, which can be negative, are investigated within a simple model. In particular, it is shown that such an equation can yield a ground state below the minimum of the potential.

Recently, several authors [1–5] have pointed out that negative collective mass parameters can occur in some theories of the collective motion. This raises the following question: does a negative collective mass have a physical meaning? Our purpose here is to investigate this question in the framework of a simple model. We study a Schrödinger equation with a variable mass parameter, which can be negative and we show that the energy spectrum and the wave functions are physically acceptable. We recall that the required properties for the wave function are [6] : i) square integrability, ii) uniformity, iii) the possibility of defining a probability density and a probability current, which are integrable upon any interval of the domain of variation of the position variable, and which satisfy the continuity condition. In general, the last requirement is presented in the following stronger formulation: the probability density and the probability current must be finite everywhere. However, since any measurement involves a non zero interval of the position variable, and not a geometrical point, one may require only that the probability density and the probability current must be integrable over any non-zero interval [6], as we have announced above.

We will also show that the ground state yielded by the Schrödinger equation with negative mass may lie below the minimum of the potential. This contrasts with the situation where a positive mass (e.g. the cranking mass) is used together with a collective potential given a constrained Hartree-Fock (HF) method (as e.g. in fission). When the mass is positive everywhere, the lowest state could never be below the minimum of the potential, and is generally quite above. This, however, is in contradiction with the fact that the HF minimum is an upper bound for the ground state energy, the difference being precisely due to the correlations brought in by the collective degree freedom.

Here, we investigate the properties of the equation

$$\left[-\frac{\hbar^2}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}f(\alpha)\frac{\mathrm{d}}{\mathrm{d}\alpha}+V(\alpha)\right]\psi(\alpha)=E\,\psi(\alpha), \tag{1}$$

where α is the collective variable, and where the function $f(\alpha)$ is allowed to be negative for some values of α. We will take the simplest model in order to exhibit the general features associated with the negative mass. For the potential, we take a square well: $V(\alpha)=0$ for $0<\alpha<a$, and $V(\alpha)=\infty$ elsewhere. More precisely we allow the system to move between 0 and a only.

The mass could change from positive to negative values by continuity. We, however, disregard this case here for two reasons: a) it is straightforward to see that this case is equivalent to the case of an energy dependent potential, whose physical meaning is well established. b) in each of the example cited above [1–5] the mass is not continuous through a changement of sign. Rather, its inverse is continuous and the mass changes of sign by passing from $+\infty$ to $-\infty$. We thus take a continuous function $f(\alpha)$. The simplest form is a linear one:

$$f(\alpha)=\frac{1}{M_0}\,\frac{\alpha-\alpha_0}{a-\alpha_0}. \tag{2}$$

The quantity M_0 plays the role of a reference mass,

* On leave of absence from the University of Liège.

** Laboratoire associé au C.N.R.S.

since, whatever the value of α_0 is, – and we allow it to change – the mass is always equal to M_0 for $\alpha = a$. By using the reduced variable $x = a^{-1}(\alpha - \alpha_0)$, eq. (1) can be written as:

$$\left(x \frac{d^2}{dx^2} + \frac{d}{dx} + \gamma\right) \psi = 0, \tag{3}$$

where

$$\gamma = \frac{E}{\hbar^2/2M_0\, a(a-\alpha_0)} . \tag{4}$$

This differential equation can be solved by standard techniques [7]. The general solution is a linear combination of two linearly independent solutions whose forms are:

$$y_1 = \sum_{n=0}^{\infty} a_n x^n , \tag{5}$$

$$y_2 = \sum_{n=0}^{\infty} b_n x^n + y_1 \ln x , \tag{6}$$

where

$$a_n = -\frac{\gamma}{n^2} a_{n-1}, \qquad a_0 = 1, \tag{7}$$

$$b_n = -\frac{1}{n^2}\left[\gamma\, b_{n-1} + 2n\, a_n\right], \qquad b_0 = 1. \tag{8}$$

Requiring the wave function to vanish at $\alpha = 0$ and $\alpha = a$ completely determines the solution and generates a spectrum.

We see from (5), that the wave function has a logarithmic singularity, but this is not critical, since such a singularity is square integrable: the wave function thus possesses property i) cited above. It has evidently property ii) also. Moreover, it is easy to verify that the probability density and the probability current, as defined as usual, are integrable over any non-zero interval. The continuity equation is a trivial consequence of the self-adjointness of eq. (1). So, the wave function possesses property iii).

We analyse now the energy spectrum generated by our model. The variation of the spectrum versus α_0 is shown in fig. 1. As α_0 goes to $\pm\infty$, one retrieves the original spectrum of a square well with constant mass M_0. As α_0 increases from $-\infty$ to 0, the variable mass is larger than the reference mass M_0 and the spectrum lowers. This effect has been shown by Hofmann and

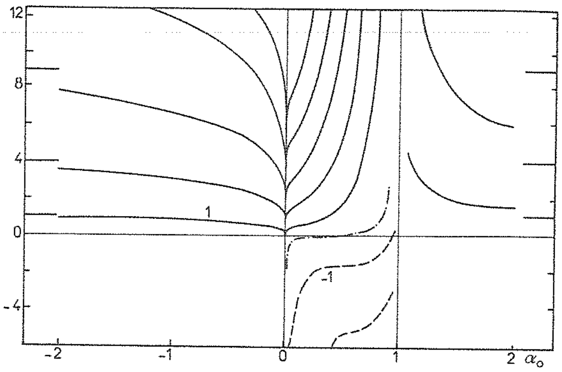

Fig. 1. Variation of the spectrum with the parameter α_0 for the model described in the text. The horizontal lines on botl sides of the figure show the spectrum for the constant references mass M_0. The units for the x-axis is the length a of the interval. The energy is expressed in $\hbar^2(2M_0a^2)^{-1}$ units.

Dietrich [8]. On the other hand, as α_0 decreases fron $+\infty$ to α, the variable mass is smaller than the reference mass M_0, which raises the spectrum.

When α_0 lies between 0 and a, the mass is negative somewhere in the interval and the situation is change dramatically. A part of the spectrum has a one-to-on correspondence with the spectrum obtained with a non-negative mass. This is true as far as the energy is concerned, when α_0 goes from negative to positive values. However, the wave function has one more node when α_0 lies in the interval $[0, a]$. For instance, the state which corresponds to the lowest state for α_0 negative, has a wave function with one mode if $0 < \alpha_0 < a$, while it has no mode if $\alpha_0 < 0$ (or $\alpha_0 > a$) as is shown in fig. 2. This state which can be interpreted as the ground state for a positive mass ($\alpha_0 < 0$) cannot keep this property as the mass is negative somewhere ($0 < \alpha_0 < a$). On the contrary, the state represented by the dot-and-dashed in fig. 1, and whic does not exist for positive mass, can be interpreted as the ground state. It has the required properties for that i) its energy can be negative, i.e., below the minimum of the potential, as least for some values of α_0. This is in keeping with the discussion above ii) its wave function has no node, as shown in fig. 2. Classifying the upper states following the number of nodes of their wave function, they can be regarded as one-, two-, three-... phonon states, although the work "phonon" is improperly used here, since the collective motion is not harmonic.

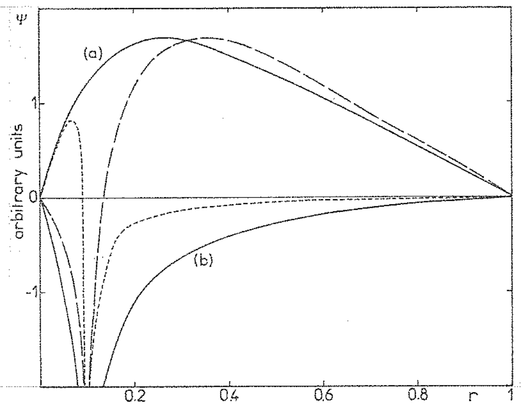

Fig. 2. The full curve (a) represents the wave function of the state labelled (1) in fig. 1 for $\alpha_0 = -0.05$. The other curves refer to $\alpha_0 = 0.05$ and to the state labelled (1) in fig. 1 (long dashes), to the ground state (full curve (b)) and to the state labelled (−1) in fig. 1 (small dashes), respectively.

Finally the lower states, indicated by the dashes in fig. 1 can be regarded as unphysical, because they are lying below the ground state. They have, however, an interesting property. Their wave function has nodes: one, two, three,... nodes starting from the highest level. In other words they have the same number of nodes as their symmetrical with respect to the ground state (see fig. 2). This situation bears a resemblance with the RPA. The resemblance is particularly striking when α_0 is at the middle of the interval. Then the ground state is exactly at zero energy and the positive and negative levels are exactly symmetrical to each other with respect to the ground state.

In conclusion, we have shown that a Schrödinger equation involving a variable collective mass, which is negative for some values of the position variable have interesting properties. In particular, the wave function are physically acceptable. Another interesting effect is that this equation can yield a ground state below th minimum of the potential.

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