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NATURAL STRAINS AND STRESSES FOR TRAPEZOIDAL STRUCTURES ANALYSIS

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FOREWORD

This report was prepared by the Aeronautics and Space Laboratory, University of Liege, Belgium, under Contract AF61(052)-892, Project No. 1467, "Structural Analysis Methods", Task No. 146705, "Automatic Computer Methods of Analysis for Flight Vehicle Structures". The work was administered under the direction of the Air Force Flight Dynamics Laboratory by Mr. James R. Johnson, Project Engineer, and through the European Office of Aerospace Research (OAR), United States Air Force.

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This report has been reviewed and is approved.

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ABSTRACT

This paper presents a tool to study the tapered structures by the direct stiffness method. It appears that a natural coordinate system is convenient to simplify the boundary conditions and in consequence the analytic calculations. The development of a particular element is taken as example and illustrated in details: the tapered spar. By means of the special coordinate system the conformity with the already developed cover sheets (quadratic elements, linear elements) is easily realized.

CONTENTS

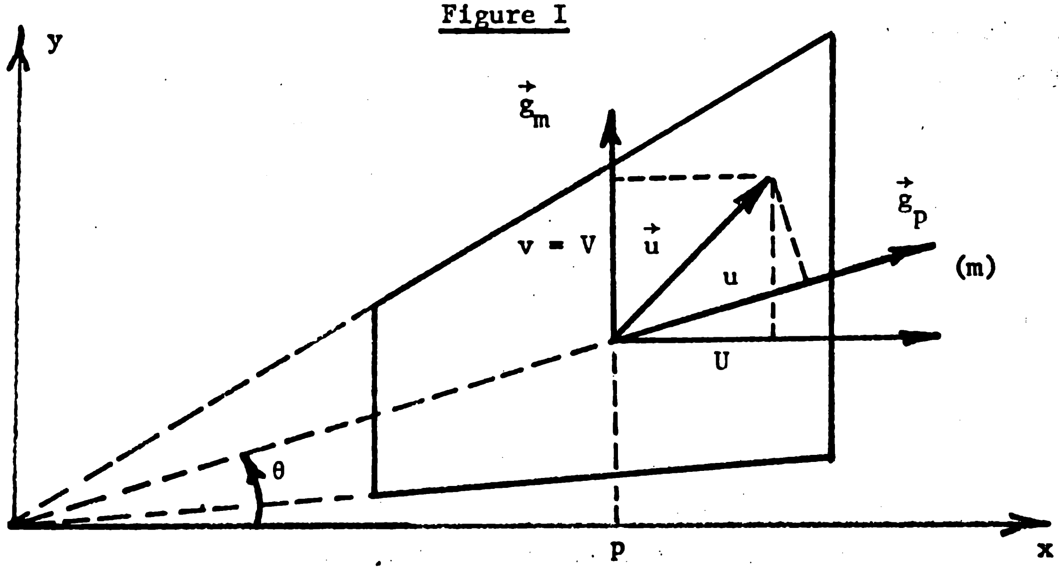
SYMBOLS.

x , y	cartesian coordinate system
p , m	special coordinate system
X , Y	non dimensional coordinate system
U , V	cartesian displacement components
u , v	normal projections of the displacement vector
u_p , u_m	covariant displacement components
u^p , u^m	contravariant displacement components
ε_x ε_y γ_{xy}	cartesian strain components
ε_p ε_m γ_{mp}	special strain components
E_p E_m G_{mp}	physical strain components
σ_x σ_y τ_{xy}	cartesian stress components
σ_p σ_m τ_{mp}	special stress components
S_p S_m T_{mp}	physical stress components
q	generalized displacements sequence
α	parameters sequence
T'	transpose matrix of the matrix T
K	stiffness matrix
E	Young modulus
ν	Poisson ratio
t	panel thickness

I. Elasticity equations in natural coordinates for a trapezoidal panel.

To simplify the boundary conditions, the special coordinate system (p, m) of figure I is adopted.

Figure I



Its relation to the cartesian system (x, y) is given by

$$x = p \qquad p = x$$
$$y = p\,m \qquad m = y/x \ (= \tan\theta) \qquad \text{(I.I)}$$

The local base vectors are

$\vec{g}_p$ of cartesian components $\dfrac{\partial x}{\partial p} = 1$ and $\dfrac{\partial y}{\partial p} = m$

$\vec{g}_m$ $\dfrac{\partial x}{\partial m} = o$ and $\dfrac{\partial y}{\partial m} = p$

They define the contravariant components (u^p, u^m) of the displacement vector by

$$\vec{u} = u^p\,\vec{g}_p + u^m\,\vec{g}_m$$

and the covariant components

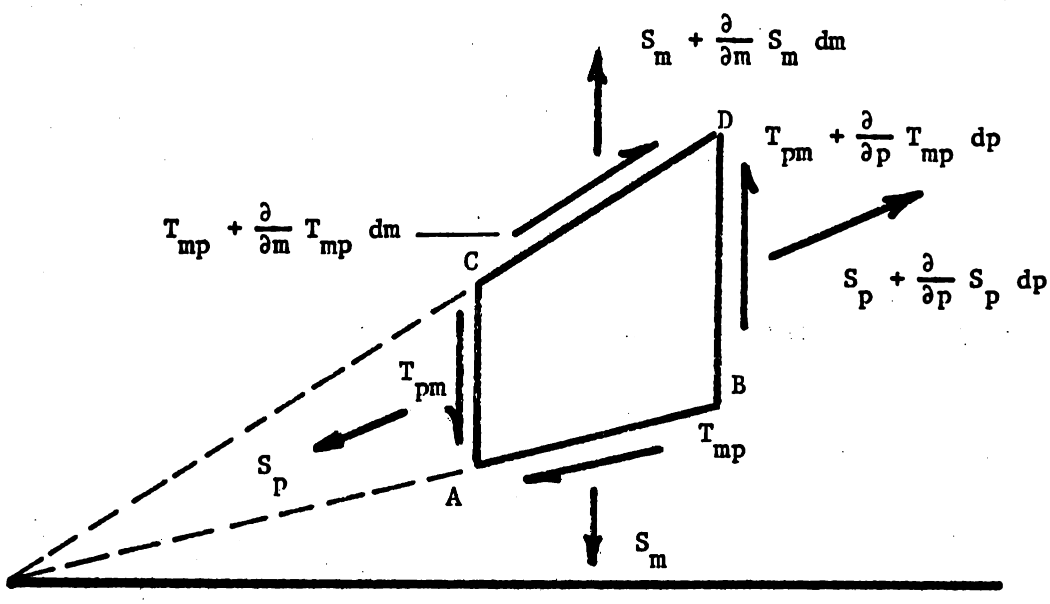
$$u_p = \vec{u} \cdot \vec{g}_p \qquad\qquad u_m = \vec{u} \cdot \vec{g}_m$$

For displacement models, where conformity with other elements is expressed by identification of local interface displacements, it seems convenient to use the simple normal projections (u, v) on the local directions provided by the base vectors. They are directly related to the covariant components by dividing those by the lengths of the base vectors

$$u = \frac{u_p}{\sqrt{1 + m^2}} \qquad\qquad v = \frac{u_m}{p} \tag{I.2}$$

The general formalism of tensor calculus can be used to derive the elasticity equations in the metric provided by the base vectors [1]. A much less cumbersome and safer procedure consists in returning to basic principles. Consider the virtual work done by stresses in a small increment of the field of displacements. Denote by S_m, T_{mp}, T_{pm} and S_p the stresses resolved in the directions of the base vectors (fig. 2) and defined per unit length of the edges of the surface element.

Figure 2



Along AB the virtual work is

$$- (T_{mp} \delta u + S_m \delta v) \sqrt{1 + m^2}\, dp$$

Adding the virtual work along CD , this becomes

$$\frac{\partial}{\partial m} (\sqrt{1 + m^2}\, T_{mp} \delta u + \sqrt{1 + m^2}\, S_m \delta v) dm\, dp$$

Similarly we have along AC

$$- (S_p \delta u + T_{pm} \delta v) p\, dm$$

and, after adding the contribution of BD ,

$$\frac{\partial}{\partial p} (p\, S_p \delta u + p\, T_{pm} \delta v) dm\, dp$$

Dividing the total work by the surface pdpdm of the element we obtain the increase in strain energy per unit area

$$\delta W = \frac{1}{p} \frac{\partial}{\partial m} (\sqrt{1 + m^2}\, T_{mp} \delta u + \sqrt{1 + m^2}\, S_m \delta v) + \frac{1}{p} \frac{\partial}{\partial p} (p\, S_p \delta u + p\, T_{pm} \delta v) \qquad (I.3)$$

The equilibrium equations are now found by expressing that $\delta W = o$ for incremental displacements corresponding to rigid body motions.
In a horizontal translation $\delta u = \frac{1}{\sqrt{1 + m^2}} \delta h$, $\delta v = o$

this produces the equilibrium equation

$$\frac{\partial}{\partial m} T_{mp} + \frac{1}{\sqrt{1+m^2}} \frac{\partial}{\partial p} (p\, S_p) = o \qquad (I.4)$$

The vertical translation $\delta u = \frac{m}{\sqrt{1+m^2}} \delta f$, $\delta v = \delta f$ produces

$$\frac{\partial}{\partial m} (m\, T_{mp} + \sqrt{1 + m^2}\, S_m) + \frac{\partial}{\partial p} (\frac{mp}{\sqrt{1+m^2}} S_p + p\, T_{pm}) = o$$

The rotation $\delta u = o$; $\delta v = p\, \delta\omega$ about the apex

$$\frac{\partial}{\partial m} (p \sqrt{1 + m^2}\, S_m) + \frac{\partial}{\partial p} (p^2\, T_{pm}) = o \qquad (I.5)$$

Considering (I.4) and (I.5), the second equilibrium equation can be seen to reduce to the reciprocity relation

$$T_{pm} = T_{mp} \qquad (I.6)$$

One of the possible ways to introduce an Airy function $A(p, m)$ in order to satisfy the equilibrium equations is as follows

$$S_p = \frac{\sqrt{1+m^2}}{p} \frac{\partial^2 A}{\partial m^2} \quad , \quad T_{mp} = - \frac{\partial^2 A}{\partial m \partial p} \quad , \quad S_m = \frac{1}{p\sqrt{1+m^2}} \frac{\partial}{\partial p} (p^2 \frac{\partial A}{\partial p}) \qquad (I.7)$$

We now simplify the general formula (I.3) by use of the equilibrium equations; it reduces to

$$\delta W = \frac{1}{p} \{ p\, S_p \frac{\partial}{\partial p} (\delta u) + p^2\, T_{mp} \frac{\partial}{\partial p} (\frac{\delta v}{p}) + T_{mp} \frac{\partial}{\partial m} (\sqrt{1+m^2}\, \delta u) + \sqrt{1+m^2}\, S_m \frac{\partial}{\partial m} (\delta v)$$

The coordinates (p, m) being considered as lagrangian, the partial derivative operators commute with the operator δ and the result can be written in the form

$$\delta W = S_p\, \delta E_p + T_{mp}\, \delta G_{mp} + S_m\, \delta E_m \qquad (I.8)$$

introducing the following definitions of strains

$$E_p = \frac{\partial}{\partial p} u$$

$$G_{mp} = p \frac{\partial}{\partial p} (\frac{v}{p}) + \frac{1}{p} \frac{\partial}{\partial m} (u \sqrt{1 + m^2}) \qquad (I.9)$$

$$E_m = \frac{\sqrt{1+m^2}}{p} \frac{\partial}{\partial m} v$$

Equation (I.8) contains the statement that the energy W per unit area is a function of those strains and implies the general form of the stress-strain realtions

$$S_p = \frac{\partial W}{\partial E_p} \quad , \quad T_{mp} = \frac{\partial W}{\partial G_{mp}} \quad , \quad S_m = \frac{\partial W}{\partial E_m} \qquad (I.10)$$

The elasticity equations which have just been derived are not those of the general theory in curvilinear coordinates, which expresses the strains as covariant derivatives

$$\varepsilon_p = u_{p|p} \quad , \quad \gamma_{mp} = u_{p|m} + u_{m|p} \quad , \quad \varepsilon_m = u_{m|m} \tag{I.11}$$

of the covariant displacement components. To obtain those, the definition of stresses should be altered as follows :

$$\sigma_p = \frac{S_p}{\sqrt{1+m^2}} \qquad \tau_{mp} = \frac{T_{mp}}{p} \qquad \sigma_m = \frac{\sqrt{1+m^2}}{p^2} S_m \tag{I.12}$$

This definitionis justified by considering the new virtual work expressions

$$-(\tau_{mp}\,\delta u_p + \sigma_m\,\delta u_m)p\,dp \qquad \text{along} \quad AB$$

$$-(\sigma_p\,\delta u_p + \tau_{pm}\delta u_m)p\,dm \qquad \text{along} \quad AC$$

where the factor p , outside of the brackets, represents the square root of the determinant of the fundamental metric tensor

$$\begin{matrix} \vec{g}_p \cdot \vec{g}_p = 1 + m^2 & \vec{g}_p \cdot \vec{g}_m = m\,p \\ \vec{g}_m \cdot \vec{g}_p = m\,p & \vec{g}_m \cdot \vec{g}_m = p^2 \end{matrix} \tag{I.13}$$

When the new form of the strain-energy increase

$$\delta W = \frac{\partial}{\partial m}(\tau_{mp}\,\delta u_p + \sigma_m\,\delta u_m) + \frac{1}{p}\frac{\partial}{\partial p}(p\,\sigma_p\,\delta u_p + p\,\tau_{pm}\,\delta u_m)$$

is processed as before, there come as equilibrium equations

$$\frac{\partial}{\partial m}\tau_{mp} + \frac{\partial}{\partial p}\sigma_p + \frac{1}{p}\sigma_p = o \tag{I.14}$$

$$\tau_{mp} = \tau_{pm} \tag{I.15}$$

$$3\,\tau_{mp} + p\left(\frac{\partial}{\partial m}\sigma_m + \frac{\partial}{\partial p}\tau_{mp}\right) = o \tag{I.16}$$

and as definition of strains

$$\varepsilon_p = \frac{\partial}{\partial p}u_p \quad , \quad \gamma_{mp} = p^2\frac{\partial}{\partial p}\left(\frac{u_m}{p^2}\right) + \frac{\partial}{\partial m}u_p \quad , \quad \varepsilon_m = \frac{\partial}{\partial m}u_m \tag{I.17}$$

which are the explicit forms the covariant derivatives in (I.II). The stress-strain relations are, as usual

$$\sigma_p = \frac{\partial W}{\partial \varepsilon_p} \quad , \quad \tau_{mp} = \frac{\partial W}{\partial \gamma_{mp}} \quad , \quad \sigma_m = \frac{\partial W}{\partial \varepsilon_m} \tag{I.I8}$$

Between the strain definitions (I.I7) and (I.9) we have the correspondance, easily derived from (I.2),

$$\varepsilon_p = \sqrt{1 + m^2}\, E_p \quad , \quad \gamma_{mp} = p\, G_{mp} \quad , \quad \varepsilon_m = \frac{p^2}{\sqrt{1+m^2}} E_m \tag{I.I9}$$

In practice the definitions (I.I7) are somewhat easier to use because they do not involve the factor $\sqrt{1+m^2}$. It will also be realized later that the structure of displacement assumptions is easier to express in covariant components.

2. Explicit stress-strain relations.

For an isotropic sheet of material in a state of plane stress the energy per unit area can be expressed in terms of invariants θ_1 and θ_2 of the strain tensor as follows

$$W = \frac{Et}{4} \left(\frac{1}{(1-\nu)} \theta_1^2 + \frac{1}{1+\nu} \theta_2 \right) \tag{2.I}$$

E is Young's modulus, ν Poisson's ratio and t , not necessarily constant, the sheet thickness. In cartesian coordinates

$$\theta_1 = \varepsilon_x + \varepsilon_y$$

is the first fundamental invariant of the strain tensor;

$$\theta_2 = (\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2$$

the second invariant of the deviator. One possibility to obtain their expressions in terms of the strains (I.I7) is to derive the relationship between (I.I7) and cartesian strains (Appendix A); another and more satisfactory procedure from the point of view of principles is to apply intrinsic definitions of the invariants in terms of contour integrals (Appendix B). The result is

$$\theta_1 = \varepsilon_p + \frac{1+m^2}{p^2}\,\varepsilon_m - \frac{m}{p}\,\gamma_{mp} \tag{2.2}$$

$$\theta_2 = \left(\varepsilon_p + \frac{m^2-1}{p^2}\,\varepsilon_m - \frac{m}{p}\,\gamma_{mp}\right)^2 + \frac{1}{p^2}\left(\gamma_{mp} - \frac{2m}{p}\,\varepsilon_m\right)^2 \tag{2.3}$$

$$= \theta_1^2 + \frac{1}{p^2}\left(\gamma_{mp}^2 - 4\,\varepsilon_m\,\varepsilon_p\right)$$

and the isotropic sheet has a strain energy per unit area equal to

$$W = \frac{Et(p,m)}{4(1-\nu^2)}\left\{ 2\left(\varepsilon_p + \frac{1+m^2}{p^2}\,\varepsilon_m - \frac{m}{p}\,\gamma_{mp}\right)^2 + (1-\nu)\,\frac{\gamma_{mp}^2 - 4\,\varepsilon_m\,\varepsilon_p}{p^2} \right\} \tag{2.4}$$

Application of the formulas (I.18) furnishes the following explicit isotropic stress-strain relations

$$\begin{vmatrix} \sigma_p \\ \sigma_m \\ \tau_{mp} \end{vmatrix} = M \begin{vmatrix} \varepsilon_p \\ \varepsilon_m \\ \gamma_{mp} \end{vmatrix} \tag{2.5}$$

$$M = \frac{Et}{1-\nu^2}\begin{vmatrix} 1 & \dfrac{m^2+\nu}{p^2} & -\dfrac{m}{p} \\[2ex] \dfrac{m^2+\nu}{p^2} & \left(\dfrac{1+m^2}{p^2}\right)^2 & -\dfrac{m(1+m^2)}{p^3} \\[2ex] -\dfrac{m}{p} & -\dfrac{m(1+m^2)}{p^3} & \dfrac{1-\nu+2\,m^2}{2\,p^2} \end{vmatrix} \tag{2.6}$$

The inverse relation

$$\begin{vmatrix} \varepsilon_p \\ \varepsilon_m \\ \gamma_{mp} \end{vmatrix} = M^{-1} \begin{vmatrix} \sigma_p \\ \sigma_m \\ \tau_{mp} \end{vmatrix} \tag{2.7}$$

is governed by the reciprocal matrix

$$M^{-1} = \frac{1}{Et} \begin{vmatrix} (1+m^2)^2 & p^2(m^2-\nu) & 2\ pm(1+m^2) \\ p^2(m^2-\nu) & p^4 & 2\ mp^3 \\ 2\ mp(1+m^2) & 2\ mp^3 & 2\ p^2(1+\nu+2\ m^2) \end{vmatrix} \tag{2.8}$$

The choice of coordinates also lends itself naturally to the anisotropic case represented by an isotropic sheet or matrix reinforced by stringers or fibers in the directions m = constant and p = constant .

Along m = constant the direct strain is

$$\frac{1}{\sqrt{1+m^2}} \frac{\partial u}{\partial p} = \frac{\varepsilon_p}{1+m^2} \tag{2.9}$$

Any fiber of transverse section A_f carries a load

$$E_f\ A_f \frac{\varepsilon_p}{1+m^2}$$

which is in the nature of a concentrated S_p load. If we take the fiber section A_p , in the oblique plane p = constant , we should replace $A_f = A_p/\sqrt{1+m^2}$. Thus if a_p denotes the area of fiber (or stringer) reinforcement per unit length in a plane p = constant , its contribution to S_p is $E_f\ a_p \frac{\varepsilon_p}{(1+m^2)^{3/2}}$.

Its contribution to σ_p is

$$(\sigma_p)_{reinf} = E_f\ a_p \frac{\varepsilon_p}{(1+m^2)^2} \tag{2.10}$$

Similarly the direct strain along p = constant is

$$\frac{1}{p} \frac{\partial v}{\partial m} = \frac{\varepsilon_m}{p^2} \tag{2.11}$$

and a fiber of normal section A_f carries a load

$$E_f\ A_f \frac{\varepsilon_m}{p^2}$$

in the nature of a concentrated S_m load. If the section is measured in the oblique plane m = constant A_f must be replaced by $A_m/\sqrt{1+m^2}$. Thus if a_m denotes the area of reinforcing members per unit length in a m = constant plane, the contribution to S_m is $E_f\, a_m \dfrac{\varepsilon_m}{p^2\sqrt{1+m^2}}$.

The contribution to σ_m is

$$(\sigma_m)_{reinf} = E_f\, a_m \frac{\varepsilon_m}{p^4} \tag{2.I2}$$

Confirmation of the results (2.I0) and (2.I2) is found by examination of the behaviour of the isotropic sheet itself under special states of stress. When $\sigma_m = o$ and $\tau_{mp} = o$ the sheet behaves as a group of m = constant fibers; we find from (2.7) and (2.8)

$$\varepsilon_p = \frac{(1+m^2)^2}{Et}\,\sigma_p$$

which agrees with (2.I0) for $E_f = E$ and $a_p = t$.
When $\sigma_p = o$ and $\tau_{mp} = o$ the sheet behaves as a group of p = constant fibers and we find

$$\varepsilon_m = \frac{p^4}{Et}\,\sigma_m$$

which agrees with (2.I2) for $E_f = E$ and $a_m = t$.

3. Conforming interface connections.

We begin by investigating the nature of conforming interface connections along a slanting edge m = constant of the trapezoidal element. Let $\vec{a} = \vec{g}_p / |\vec{g}_p|$ denote the unit vector along the edge. Similarly, using the contravariant base vector $\vec{g}^m$ defined in Appendix B, we introduce the unit vector $\vec{b} = \vec{g}^m / |\vec{g}^m|$ normal to the edge in the plane of the element. Finally let $\vec{c}$ denote the unit vector normal to the plane. Conformity requires that the displacement vector u be the same on the edge for all elements meeting there.
In the trapezoidal panel the projections of $\vec{u}$ on the cartesian vectors $(\vec{a}, \vec{b}, \vec{c})$ are

$$u = \frac{u_p}{\sqrt{1+m^2}} \qquad \text{on } \vec{a}$$

$$v' = \frac{1}{|\vec{g}^m|} \vec{u} \cdot \vec{g}^m = \frac{\sqrt{1+m^2}}{p} u_m - \frac{m}{\sqrt{1+m^2}} u_p \qquad \text{on } \vec{b}$$

$$w \qquad \text{on } \vec{c}$$

w is indeterminate, because we consider only a membrane state of stress.
If the connected element is a spar flange, supposedly devoid of bending rigidity, its axial displacement must coincide with u.
If another trapezoidal panel is connected in the same plane and has the same focus O and the same reference orientation from which to measure θ or m, u_p and u_m must be continuous at the interface. If the focus is different but the reference orientation the same, account must be taken of the discontinuity in the coordinate p across the interface. In such a case the continuity of u at the interface is equivalent to that of u_p but the continuity of v' requires that of $\frac{u_m}{p}$.
The most interesting case is that where the other membrane element lies in a different plane; as in the example of tapered and swept box beams. Here the only requirement is the continuity of u. The continuity conditions in the plane $(\vec{b}, \vec{c})$ can always be enforced by adjusting the, otherwise indeterminate, values of the w components of the elements.
A more complicated situation arises when there are several membrane elements connected in different planes.
The situation is depicted on figure 3 in the plane normal to the edge. The plane of a membrane element of index i is traced by the angle α_i with respect to a conventional reference plane.
Then, if v' and w denote respectively the transverse components of the displacement vector with respect to the reference plane, we must have

$$v'_i = v' \cos \alpha_i + w \sin \alpha_i \qquad i = 1, 2 \ldots$$

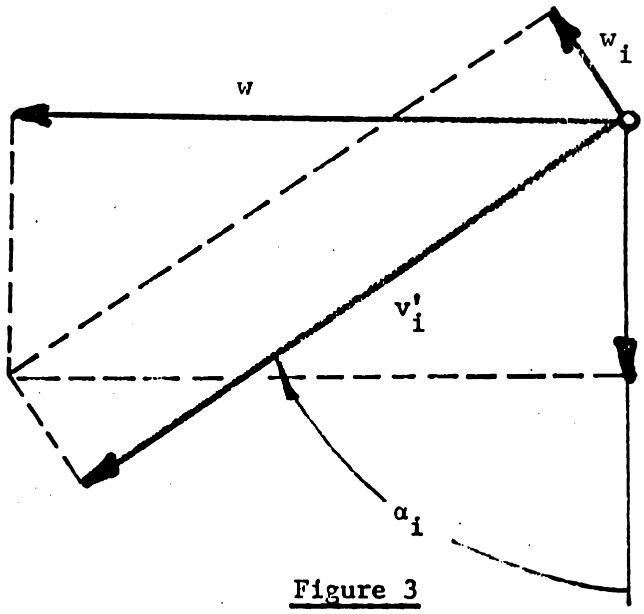


Figure 3

In the case of three membrane elements, the compatibility condition of those equations for the unknowns v' and w is

$$\begin{vmatrix} v'_1 & \cos\alpha_1 & \sin\alpha_1 \\ v'_2 & \cos\alpha_2 & \sin\alpha_2 \\ v'_3 & \cos\alpha_3 & \sin\alpha_3 \end{vmatrix} = o$$

or $$v'_1 \sin(\alpha_3 - \alpha_2) + v'_2 \sin(\alpha_1 - \alpha_3) + v'_3 \sin(\alpha_2 - \alpha_1) = o \qquad (3.I)$$

It is a constraint to be satisfied between the "in plane" components of displacement v' of the different membrane elements.

For $n > 3$ elements there are $n - 2$ independant conditions of this type.

In the applications we will restrict ourselves to the box beam situation, with or without spar flanges. To conform with our already developed skin elements of quadratic type the component u will be assumed to have parabolic variation along the edges m = constant. The same is true then of u_p considered as a function of p :

$$u_p = \alpha_1(m) + \alpha_2(m)p + \alpha_3(m)p^2 \qquad (3.2)$$

$$u = \frac{1}{\sqrt{1+m^2}} \{ \alpha_1(m) + \alpha_2(m)p + \alpha_3(m)p^2 \} \qquad (3.3)$$

We next investigate the nature of a conforming connection along an interface $p = \text{constant}$. The local cartesian unit vectors consist of $\vec{v}$ along the interface edge, $\vec{g}^p$ normal to it in the plane of the element and $\vec{c}$ again, normal to the plane. The projections of $\vec{u}$ are

$$v = \frac{u_m}{p} \qquad \text{along } \vec{v}$$

$$U = u_p \, \vec{g}^p \cdot \vec{g}^p + u_m \, \vec{g}^p \cdot \vec{g}^m = u_p - \frac{m}{p} u_m \qquad \text{along } \vec{g}^p$$

$$w \text{ indeterminate} \qquad \text{along } \vec{c}$$

If the connected element is a rib flange, devoid of bending rigidity, its axial displacement must coincide with v.

If another trapezoidal panel is connected in the same plane with the same focus O, u_p and u_m must be continuous. If the focus is different, as in the case of a "kinked" spar, account must be taken of the discontinuities of both m and p at the interface; accenting the values of the variables of the second element, the transition conditions based on the continuity of the cartesian projections are

$$\left.\begin{aligned} u'_m &= p' \frac{u_m}{p} \\ u'_p &= u_p + (m' - m) \frac{u_m}{p} \end{aligned}\right\} \text{ at interface} \qquad (3.4)$$

4. Simple models for tapered and swept spar element.

Preliminary stress assumption.

A transverse distribution of stresses or displacements is assumed, which reduces effectively the two-dimensional elasticity problem to the determination of functions of a single variable : the position coordinate x of a transverse section. This determination if effected by cutting the spar into consecutive segments, the finite elements, and adopting polynomial approximations to the unknown functions. The convergence obtained by reducing the length of the segments is not towards an exact solution of the two-dimensional elasticity problem but towards an exact solution of some "Engineering Beam Theory" related to the assumed transverse distribution. In more sophisticated models one allows for distortion of the cross-section and elasticity of the transverse fibers and the complexity of the models seems justified by the good correlation obtained between both the displacement and stress patterns within the equilibrium models on the one hand and the displacement models on the other hand [2]. However, if one is not too much interested in the details of the stress distribution in the web but only in a correct representation of the spar as a connection member between other elements of the structure, sufficient accuracy is obtained with the cruder models. They converge in principle to the exact solution of the usual engineering beam theory. However, in the case of the displacement model that has inextensible transverse fibers, the moment of inertia of the web would be overestimated by a factor $(1-\nu^2)^{-1}$ if the initial de Saint-Venant type assumption

$$\sigma_y = 0 \tag{4.1}$$

were not introduced at the outset. Because of this mixture of stress and displacement assumptions this model is really hybrid and its lower bound properties only hold with respect to exact solutions of such beam theories that incorporate (4.1). The equilibrium model is also based on (4.1) but its upper bound properties hold with respect to any exact solution whether relying on (4.1) or not. In the presence of taper and sweep, the natural generalization of (4.1) is the assumption

$$\sigma_m = 0 \tag{4.2}$$

that will be retained in the sequel. It reduces the expression of the energy density, in its Clapeyron form, to

$$\hat{W} = \hat{\Phi} = \frac{1}{2} (\varepsilon_p \sigma_p + \gamma_{mp} \tau_{mp}) \tag{4.3}$$

From the isotropic stress-strain matrix (2.8) we obtain in view of (4.2)

$$Et\, \varepsilon_p = (1+m^2)^2 \sigma_p + 2\, pm(1+m^2)\tau_{mp} \tag{4.4}$$

$$Et\, \gamma_{mp} = 2\, pm(1+m^2)\sigma_p + 2\, p^2(1+\nu+2m^2)\tau_{mp} \tag{4.5}$$

This, substituted into (4.3), furnishes the complementary form of the energy density that is of interest for the equilibrium model

$$\mathcal{S} = \frac{1}{2\ Et} \{ (1+m^2)^2 \ \sigma_p^2 + 4 \ pm(1+m^2)\sigma_p \ \tau_{mp} + 2 \ p^2(1+\nu+2m^2)\tau_{mp}^2 \ \} \tag{4.6}$$

Solving equations (4.4) and (4.5) for the stresses :

$$\sigma_p = \frac{Et}{(1+\nu) \ p(1+m^2)^2} \{ \ p(1+\nu+2m^2)\varepsilon_p - m(1+m^2)\gamma_{mp} \ \} \tag{4.7}$$

$$\tau_{mp} = \frac{Et}{2(1+\nu) \ p^2(1+m^2)} \{ \ (1+m^2)\gamma_{mp} - 2 \ mp \ \varepsilon_p \ \} \tag{4.8}$$

and substituting into (4.3) we obtain the energy density of interest for the displacement model

$$\hat{W} = \frac{Et}{2(1+\nu)} \{ \frac{1 + \nu + 2 \ m^2}{(1+m^2)^2} \varepsilon_p^2 - \frac{2 \ m}{p(1+m^2)} \gamma_{mp} \ \varepsilon_p + \frac{1}{2p^2} \gamma_{mp}^2 \ \} \tag{4.9}$$

For a spar flange we can derive from the direct strain (2.9) the energy per unit p length

$$w = \frac{1}{2} \frac{EA}{(1+m^2)^2} \varepsilon_p^2 \tag{4.I0}$$

where A is the flange area in a p = constant section and is related to the area S normal to the flange axis by

$$A = \sqrt{1+m^2} \ S \tag{4.II}$$

We note that the load L obtained by differentiation of this energy per unit p length

$$L = \frac{\partial w}{\partial \varepsilon_p} = \frac{EA}{(1+m^2)} \varepsilon_p \tag{4.I2}$$

can be put in a form consistent with the first of the stress definitions (I.I2)

$$L = \frac{N}{\sqrt{1+m^2}}$$

by considering the usual normal load N in the section normal to the flange axis

$$N = E \ S \ \frac{\varepsilon_p}{1+m^2} \tag{4.I3}$$

The complementary form of the strain energy per unit p length of flange is from (4.I2)

$$\phi = \frac{1}{2}\frac{L^2(1+m^2)^2}{EA} \tag{4.I4}$$

with

$$\frac{\partial\phi}{\partial L} = \varepsilon_p \tag{4.I5}$$

Displacement model assumptions.

The trapezoidal panel is defined by the four parameters a , b , p_o , h , illustrated on figure 4.

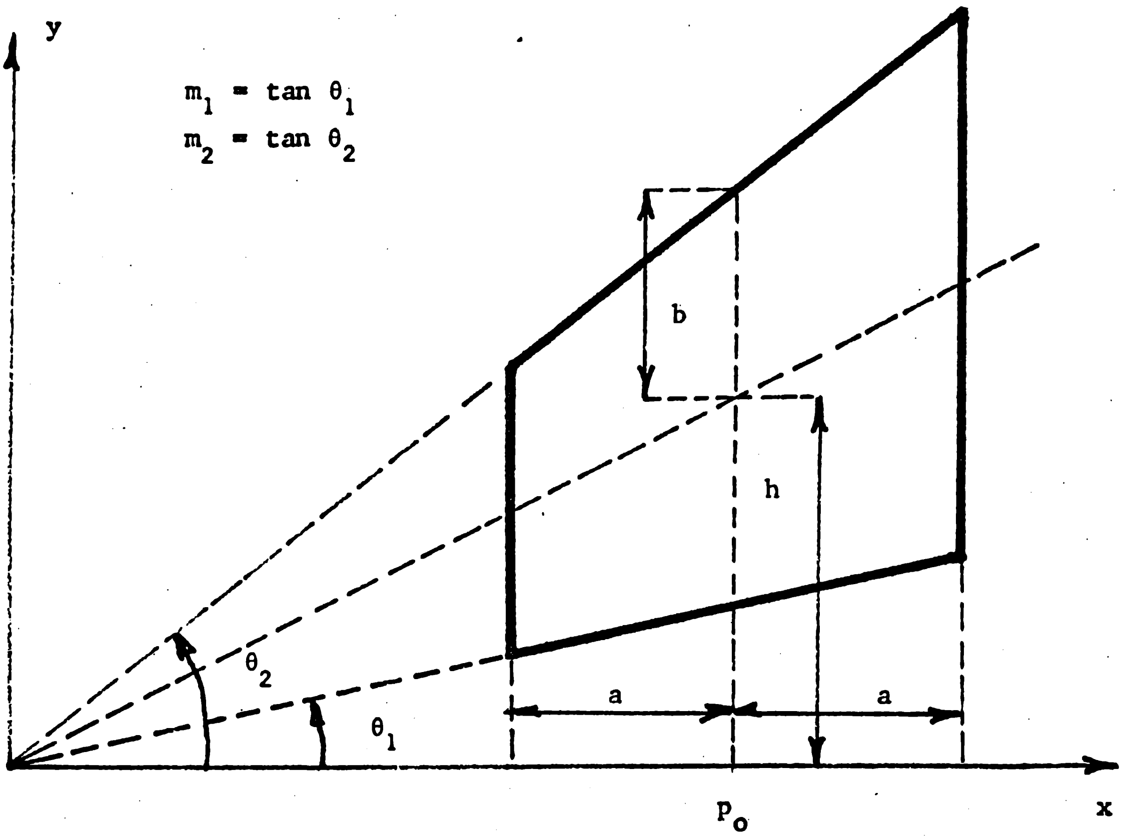


Figure 4

For the simplicity of the analytical computations we introduce 3 non-dimensional parameters

$$\lambda = \frac{b}{a} \qquad \text{defining the aspect-ratio}$$

$$\beta_o = \frac{a}{p_o} = \frac{1}{t_o} \qquad \text{defining the taper} \tag{4.16}$$

$$\alpha_o = \frac{h}{b}$$

β_o varies between o and 1 .

Another natural parameter appears in the subsequent calculations : m_o , related to the previous parameters by the relations

$$m_o = \frac{m_2 - m_1}{2} = + \frac{b}{p_o} \tag{4.17}$$

$$m_o\ t_o = \frac{m_o}{\beta_o} = \lambda \tag{4.18}$$

Consider now the non-dimensional system of coordinates

$$X = \frac{p - p_o}{a} \tag{4.19}$$

$$Y = \frac{m}{m_o} - \alpha_o = \frac{mp_o - h}{b} \tag{4.20}$$

When p varies between $p_o - a$ and $p_o + a$

$$X \text{ varies between } -1 \text{ and } +1 \tag{4.21}$$

When m varies between m_1 and m_2 ,

$$Y \text{ varies also between } -1 \text{ and } +1 \tag{4.22}$$

The surface element is :

$$dA = pdpdm = ab(1 + \beta_o X)dXdY = a^2Z\ dXdY \tag{4.23}$$

where $$Z = m_o X + \lambda \tag{4.24}$$

Because of taper, sweep and unequal flange areas we lack the symmetry required for distinguishing between bending and extension. They must be handled simultaneously. We also require conformity along the flange axes $m = m_1$ and $m = m_2$ with cover sheet elements having parabolic distributions in the edge displacements. This will be implemented by a displacement field u or $u_p = u\sqrt{1+m^2}$ of the second degree in the coordinate p. At an interface p = constant we must have continuity of the cartesian components U and V. Extending the concept of inextensible vertical fibers we take V to depend only on p or X and the same will then be true of $u_m = pV$. We will also assume the cross-section to remain plane by taking U to be linear in y. Then $u_p = U + mV$ will be linear in m or Y and (see fig. 4) the field u_p will be completely determined by six local values along the flange axes :

$$u_p = \alpha_1 + \alpha_2 X + \alpha_3 X^2 + Y(\alpha_4 + \alpha_5 X + \alpha_6 X^2) \qquad (4.25)$$

From the relations (I.I7) the strains are :

$$\varepsilon_p = \frac{1}{a}\left(\alpha_2 + 2\,\alpha_3 X + Y(\alpha_5 + 2\,\alpha_6 X) \right)$$

$$\gamma_{mp} = p^2 \frac{\partial}{\partial p}\left(\frac{u_m}{p^2}\right) + \frac{1}{m_o}(\alpha_4 + \alpha_5 X + \alpha_6 X^2)$$

in matrix notations we way write :

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad (4.26)$$

where

$$\varepsilon' = (\varepsilon_p\,,\ \gamma_{mp})$$

$$\varepsilon_1 = D\,\alpha \qquad (4.27)$$

$$\varepsilon_2' = \left| \;o \qquad p^2 \frac{\partial}{\partial p}\left(\frac{u_m}{p^2}\right) \right| \qquad (4.28)$$

and

$$D = \begin{vmatrix} o & \frac{1}{a} & \frac{2X}{a} & o & \frac{Y}{a} & \frac{2XY}{a} \\ o & o & o & \frac{1}{m_o} & \frac{X}{m_o} & \frac{X^2}{m_o} \end{vmatrix} \qquad (4.29)$$

$$\alpha' = | \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 |$$

The strain energy density (4.7) takes the following form :

$$W = \frac{1}{2} \varepsilon' N \varepsilon = W_1 + W_2 \tag{4.30}$$

where

$$N = \frac{Et}{(1+\nu)(1+m^2)^2} \begin{vmatrix} 1 + \nu + 2m^2 & -m(1+m)p^{-1} \\ -m(1+m)p^{-1} & \frac{1}{2}(1+m^2)^2 p^{-2} \end{vmatrix} \tag{4.31}$$

and

$$W_1 = \frac{1}{2} \varepsilon_1' N \varepsilon_1 \tag{4.32}$$

$$W_2 = \frac{1}{2} \varepsilon_2' N \varepsilon_2 + \varepsilon_2' N \varepsilon_1 \tag{4.33}$$

The first part of the strain energy

$$U_1 = \iint W_1 \, dA$$

can be written in terms of the parameters α_i

$$U_1 = \frac{1}{2} \alpha' (B_1^{*} + B_1^{**}) \alpha \tag{4.34}$$

where B_1^{*}, B_1^{**} are displayed in analytical forms in figures 5 and 6.
In those matrices we denote

$$A(m_o) = \int_{-1}^{1} \frac{dY}{(1+m_o^2)^2} = \frac{1+2m_o^2(1-\alpha_o^2)}{2\big(r(m_o)s(m_o)\big)} + \frac{1}{m_o}\big(\text{arctg } m_o(\alpha_o+1) - \text{arctg } m_o(\alpha_o-1) \big)$$

$$(4.35)$$

$$B(m_o) = \frac{1}{m_o^2} \int_{-1}^{+1} \frac{m^2 dY}{(1+m^2)^2} = \frac{1}{m_o^2} \left(\frac{1-m_o^2(\alpha_o^2-1)}{r(m_o)s(m_o)} \right) + \frac{1}{2m_o^3} \big(\text{arctg } m_o(\alpha_o+1) - \text{arctg } m_o(\alpha_o-1) \big)$$

$$C(m_o) = \frac{1}{m_o} \int_{-1}^{+1} \frac{m dY}{(1+m^2)} = \frac{2\alpha}{r(m_o)s(m_o)}$$

$$D(m_o) = \frac{1}{m_o^4} \int_{-1}^{+1} \frac{m^4 dY}{(1+m^2)^2} = \frac{1}{m_o^5} \left\{ \frac{6m_o+2m_o^3(3\alpha_o^2+5)+4m_o^5(\alpha_o^2-1)^2}{r(m_o)s(m_o)} - \frac{3}{2}\left(\text{arctg}\, m_o(\alpha_o+1) - \text{arctg}\, m_o(\alpha_o-1) \right) \right\}$$

$$E(m_o) = \frac{1}{m_o^3} \int_{-1}^{+1} \frac{m^3 dY}{(1+m^2)^2} = \frac{1}{2m_o^4} \left\{ \ell n \frac{r(m_o)}{s(m_o)} - 4 \frac{m_o^2 \alpha_o}{r(m_o)s(m_o)} \right\}$$

$$F(m_o) = \int_{-1}^{+1} \frac{Y^2 dY}{(1+m^2)^2} = B(m_o) + \alpha_o^2 A(m_o) - 2\alpha_o C(m_o)$$

(4.35)

$$G(m_o) = \int_{-1}^{+1} \frac{Y^2 m^2}{(1+m^2)^2} dY = m_o^2 \left(D(m_o) + \alpha_o^2 B(m_o) - 2\alpha_o E(m_o) \right)$$

$$H(m_o) = \frac{1}{m_o} \int_{-1}^{+1} \frac{m dY}{1+m^2} = \frac{1}{2m_o^2} \ell n \frac{r(m_o)}{s(m_o)}$$

$$I(m_o) = \frac{1}{m_o^2} \int_{-1}^{+1} \frac{m^2 dY}{1+m^2} = \frac{1}{m_o^3} \left\{ 2m_o - \left(\text{arctg}\, m_o(\alpha_o+1) - \text{arctg}\, m_o(\alpha_o-1) \right) \right\}$$

$$K(m_o) = \int_{-1}^{+1} \frac{mY}{1+m^2} = m_o^2 I(m_o) - \alpha m_o H(m_o)$$

while we also need the simple integrals

$$\int_{-1}^{+1} XZdZ = \frac{2}{3} m_o \qquad \int_{-1}^{+1} ZX^2 dX = \frac{2}{3}\lambda \qquad \int_{-1}^{+1} X^2 ZdZ = \frac{2}{3}\lambda$$

(4.36)

$$\int_{-1}^{+1} ZdX = 2\lambda \qquad \int_{-1}^{+1} X^2 dX = \frac{2}{3}$$

For $|\beta_o| < 1$ the expansions in power series of β_o are :

$$\frac{1}{2} f(\beta_o) = 1 + \frac{\beta_o^2}{3} + \frac{\beta_o^4}{5} + \dots$$

(4.37)

$$\frac{1}{2} g(\beta_o) = \frac{1}{3} + \frac{\beta_o^2}{5} + \frac{\beta_o^4}{7} + \dots$$

$$\frac{1}{2}\, h(\beta_o) = \frac{1}{5} + \frac{\beta_o^2}{7} + \frac{\beta_o^4}{9} + \ldots \tag{4.37}$$

Determination of $u_m(p)$.

The part of the strain energy density which depends on the unknown $u_m(p)$ is

$$W_2 = \frac{1}{2}\, \varepsilon_2'\, N\, \varepsilon_2 + \varepsilon_2'\, N\, D\, \alpha$$

From (4.28) and (4.3I) we have

$$\varepsilon_2'\, N = \frac{Et}{1+\nu}\, p\, \frac{\partial}{\partial p}\left(\frac{u_m}{p^2}\right) \left| \; -\frac{m}{1+m^2} \qquad \frac{1}{2p} \right| \tag{4.38}$$

Putting

$$q' = \left| \; -\frac{m}{1+m^2} \quad , \quad \frac{1}{2p} \right| D \tag{4.39}$$

$$\phi(X) = \frac{Z}{a^2}\, \frac{\partial}{\partial X}\left(\frac{u_m}{Z^2}\right) m_o \tag{4.40}$$

the remainder of the energy density may be written :

$$W_2 = \frac{Et}{1+\nu}\left(\frac{1}{4}\, \dot{\phi}^2(X) + \dot{\phi}\, q'\alpha \right) \tag{4.4I}$$

The best choice for $\phi(X)$ will be that which minimizes the energy

$$U_2 = a^2 \int_{-1}^{+1} \int_{-1}^{+1} W_2\, Z\, dX dY \quad \min$$

under the assumption that $u_m(X)$ is given for $X = -1$ and $X = +1$.

From (4.4I) this expression becomes under the assumption of constant thickness t

$$\int_{-1}^{+1} \left(\frac{1}{2}\, Z\, \dot{\phi}^2(X) + \dot{\phi}\, \gamma'(X)\, \alpha \right) dX \quad \min \tag{4.42}$$

where the vector $\gamma'(X)$ is :

$$\gamma'(X) = \int_{-1}^{+1} Zq'dY = \frac{1}{a} \left| \; o \qquad -m_o Z H(m_o) \qquad -2m_o H(m_o) XZ \qquad 1 \qquad -K(m_o)Z+X \qquad -2K(m_o)ZX+X^2 \right| \tag{4.43}$$

(4.42) furnishes the differential equation to be satisfied by $\Phi(X)$

$$\frac{d}{dX}\left(\gamma'(X)\alpha + Z\dot{\Phi}\right) = o$$

The first integral is

$$\dot{\Phi} = \frac{\frac{c_1}{a} - \gamma'(X)\alpha}{Z} \tag{4.44}$$

from (4.43) we find :

$$\dot{\Phi} = \frac{1}{aZ}\left(c_1 + m_o\, ZH(m_o)\alpha_2 + 2\, m_o\, H(m_o)XZ\, \alpha_3 - \alpha_4 + \alpha_5\left(K(m_o)Z-X \right)\right.$$
$$\left. + \alpha_6\left(2K(m_o)ZX - X^2 \right)\right) \tag{4.45}$$

Noting that :

$$p = p_o + aX = \frac{aZ}{m_o} \tag{4.46}$$

we obtain from (I.2) and (4.40) :

$$v = aZ \int^X \frac{\dot{\Phi}(X)}{Z}\, dX \tag{4.47}$$

To evaluate the integral (4.47) it is necessary to introduce the following functions :

$$\begin{aligned}
&\int_o^X \frac{dX}{Z} = \frac{1}{m_o}\,\ell n\,\frac{m_oX+\lambda}{\lambda} = \eta(X, m_o)\\
&\int_o^X \frac{XdX}{Z} = \frac{1}{m_o^2}\left(m_oX - \lambda\,\ell n\,\frac{m_oX+\lambda}{\lambda} \right) = \xi(X, m_o)\\
&\int_o^X \frac{XdX}{Z^2} = \frac{1}{m_o^2}\left(\ell n\,\frac{m_oX+\lambda}{\lambda} - \frac{\lambda\, m_oX}{m_oX+\lambda} \right) = \delta(X, m_o)\\
&\int_o^X \frac{X^2dX}{Z^2} = \frac{1}{m_o^3}\left(m_oX - 2\,\lambda\,\ell n\,\frac{m_oX+\lambda}{\lambda} + \frac{\lambda\, m_oX}{m_oX+\lambda} \right) = \mu(X, m_o)
\end{aligned} \tag{4.48}$$

So that the vertical displacement is :

$$v = Z \{ \frac{c_1 - \alpha_4}{\lambda} \frac{X}{m_o X + \lambda} + m_o H(m_o)\, \eta(X,m_o) X + 2 m_o H(m_o)\, \xi(X,m_o)$$

$$+ \left(K(m_o)\, \eta(X,m_o) - \delta(X,m_o) \right) \alpha_5 + \left(2K(m_o)\, \xi(X,m_o) - \mu(X,m_o) \right) \alpha_6 + c_2 \} \qquad (4.49)$$

It is easy to shown that

$$\left(\eta(X,m_o) \right)_{m_o \to o} = \frac{X}{\lambda}$$

$$\left(\xi(X,m_o) \right)_{m_o \to o} = \frac{1}{2} \frac{X^2}{\lambda}$$

$$\left(\delta(X,m_o) \right)_{m_o \to o} = \frac{X^2}{2\, \lambda^2}$$

$$\left(\mu(X,m_o) \right)_{m_o \to o} = \frac{1}{3} \frac{X^3}{\lambda^2}$$

The vertical deflection for a rectangular spar takes the form :

$$v = \frac{c_1 - \alpha_4}{\lambda} X + m_o H(m_o) X\, \alpha_2 + m_o H(m_o) X^2\, \alpha_3 + X \left(K(m_o) - \frac{X}{2\, \lambda} \right) \alpha_5$$

$$+ X^2 \left(K(m_o) - \frac{X}{3\, \lambda} \right) \alpha_6 + c_2 \qquad (4.50)$$

which is a cubic function of X , and the element is equivalent to the rectangular spar given by de VEUBEKE [3] .
From (4.4I), (4.42) and (4.44) the part of strain energy depending on $u_m(p)$ may be found

$$\frac{1+\nu}{Et} U_2 = \int_{-1}^{+1} a^2 \{ \dot{\Phi} \left(\frac{c_1}{a} - \frac{1}{2} Z\, \Phi(X) \right) \} dX = \frac{1}{2} \alpha'\ a^2\, B_2\, \alpha \qquad (4.5I)$$

where the matrix B_2 is given in figure 7.
The displacement obtained previously (2.25), (2.50) can be completely determined by (see fig. 8) this suitable sequence of generalized coordinates :

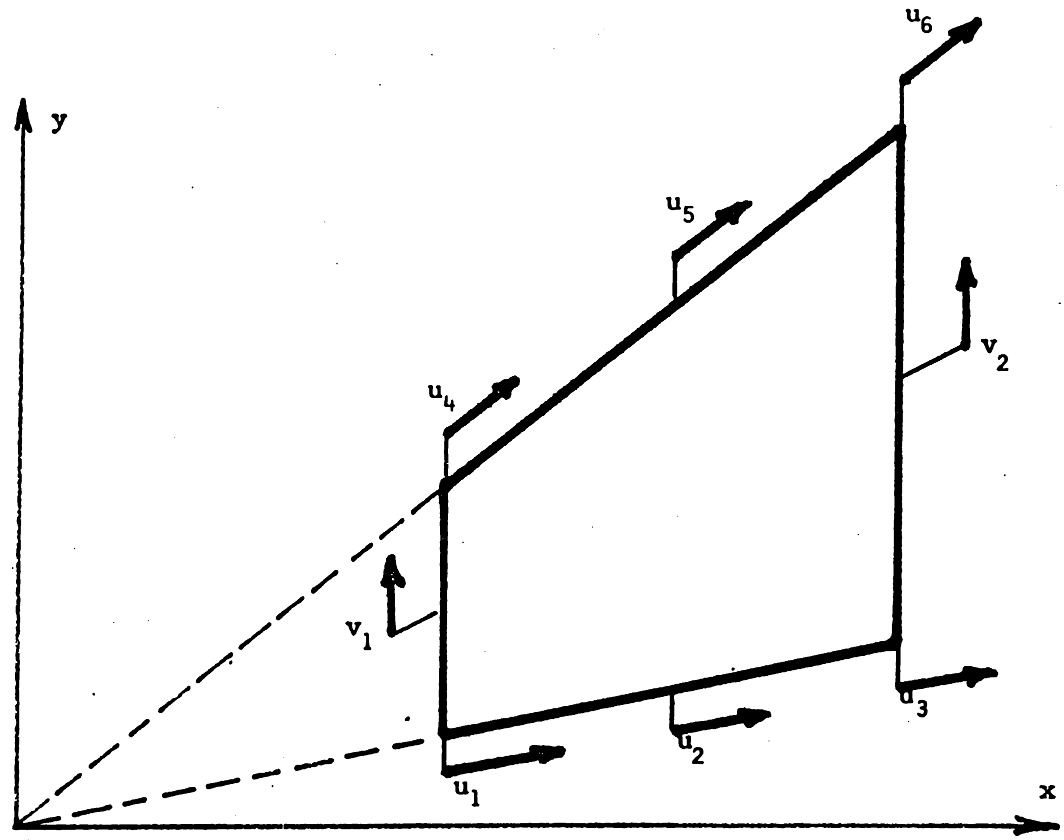


Figure 8

$$\begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \cdots \\ v_1 \\ v_2 \end{vmatrix} = \left|\begin{array}{c:c} & \\ & \\ U & O \\ & \\ & \\ \cdots & \cdots \\ T & V \end{array}\right| \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \cdots \\ c_1 \\ c_2 \end{vmatrix} \qquad (4.52)$$

The matrix relation (4.52) may be easily inverted

$$\begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \cdots \\ c_1 \\ c_2 \end{vmatrix} = \begin{vmatrix} U^{-1} & 0 \\ \cdots & \cdots \\ -VTU^{-1} & V^{-1} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \cdots \\ v_1 \\ v_2 \end{vmatrix} = Wq$$

where

$$U^{-1} = \frac{\sqrt{1+m_o^2}}{4} \begin{vmatrix} 0 & 2 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 \end{vmatrix}$$

$$V^{-1} = \begin{vmatrix} -\frac{1}{\lambda} & 1 \\ \frac{1}{\lambda} & 1 \end{vmatrix}$$

$$T' = \begin{vmatrix} 0 & 0 \\ m_o H\eta(-1,m_o) & m_o H\eta(1,m_o) \\ 2m_o H\xi(-1,m_o) & 2m_o H\xi(1,m_o) \\ \frac{1}{\lambda(-m_o+\lambda)} & -\frac{1}{\lambda(m_o+\lambda)} \\ K\eta(-1,m_o)-\delta(-1,m_o) & K\eta(1,m_o)-\delta(-1,m_o) \\ 2K\xi(-1,m_o)-\mu(-1,m_o) & 2K\xi(1,m_o)-\mu(1+m_o) \end{vmatrix}$$

The stiffness matrix turns out to be

$$K = \frac{Et \quad W'}{2(1+\nu)} \left(B_1^{*} + B_1^{**} + B_2\right) W$$

Using the formulae (A.5) the cartesian stresses are calculated in the local axes

$$\begin{vmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{vmatrix} = \frac{Et}{(1+\nu)a} N^{*} S \; Wq$$

where :

$$N^{*} = \begin{vmatrix} \frac{1+\nu+m^2}{1+m^2} & -\frac{m}{Z} \\ \frac{m(\nu+m^2)}{1+m^2} & \frac{1-m^2}{2Z} \\ -\frac{m(1-\nu)}{1+m^2} & \frac{m}{Z} \end{vmatrix}$$

$$S = \begin{vmatrix} 0 & 1 & 2X & 0 & Y & 2KY & 0 & 0 \\ 0 & m_o ZH & 2m_o HXZ & 0 & KZ & 2KZX & m_o & 0 \end{vmatrix}$$

Relations between natural and cartesian stresses and strains.

If U and V denote the cartesian components of displacement, we have from the cartesian components of the contravariant base vectors (see Appendix B).

$$U = u_p - \frac{m}{p} u_m \qquad V = \frac{1}{p} u_m \qquad (A.I)$$

while the operators

$$\frac{\partial}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial}{\partial p} + \frac{\partial m}{\partial x}\frac{\partial}{\partial m} = \frac{\partial}{\partial p} - \frac{m}{p}\frac{\partial}{\partial m}$$

$$\frac{\partial}{\partial y} = \frac{\partial p}{\partial y}\frac{\partial}{\partial p} + \frac{\partial m}{\partial y}\frac{\partial}{\partial m} = \frac{1}{p}\frac{\partial}{\partial m}$$

Hence

$$\varepsilon_x = \frac{\partial U}{\partial x} = \left(\frac{\partial}{\partial p} - \frac{m}{p}\frac{\partial}{\partial m}\right)\left(u_p - \frac{m}{p} u_m\right)$$

$$\varepsilon_y = \frac{\partial V}{\partial y} = \frac{1}{p}\frac{\partial}{\partial m}\left(\frac{1}{p} u_m\right)$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{1}{p}\frac{\partial}{\partial m}\left(u_p - \frac{m}{p} u_m\right) + \left(\frac{\partial}{\partial p} - \frac{m}{p}\frac{\partial}{\partial m}\right)\frac{1}{p} u_m$$

or, finally

$$\begin{vmatrix} \varepsilon_x \\ \gamma_{xy} \\ \varepsilon_y \end{vmatrix} = \begin{vmatrix} 1 & -\frac{m}{p} & \frac{m^2}{p^2} \\ o & \frac{1}{p} & -\frac{2m}{p^2} \\ o & o & \frac{1}{p^2} \end{vmatrix} \begin{vmatrix} \varepsilon_p \\ \gamma_{mp} \\ \varepsilon_m \end{vmatrix} \qquad (A.2)$$

on the other hand, from direct equilibrium considerations,

$$\begin{vmatrix} S_p \\ T_{mp} \\ S_m \end{vmatrix} = \begin{vmatrix} \sqrt{1+m^2} & o & o \\ -m & 1 & o \\ \frac{m^2}{\sqrt{1+m^2}} & -\frac{2m}{\sqrt{1+m^2}} & \frac{1}{\sqrt{1+m^2}} \end{vmatrix} \begin{vmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{vmatrix}$$

and, considering the definitions (I.I2)

$$\begin{vmatrix} \sigma_p \\ \tau_{mp} \\ \sigma_m \end{vmatrix} = \begin{vmatrix} 1 & o & o \\ -\frac{m}{p} & \frac{1}{p} & o \\ \frac{m^2}{p^2} & -\frac{2\,m}{p^2} & \frac{1}{p^2} \end{vmatrix} \begin{vmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{vmatrix} \qquad (A.3)$$

The inverse relations to (A.2) and (A.3) are respectively

$$\begin{vmatrix} \varepsilon_p \\ \gamma_{mp} \\ \varepsilon_m \end{vmatrix} = \begin{vmatrix} 1 & m & m^2 \\ o & p & 2\,mp \\ o & o & p^2 \end{vmatrix} \begin{vmatrix} \varepsilon_x \\ \gamma_{xy} \\ \varepsilon_y \end{vmatrix} \qquad (A.4)$$

$$\begin{vmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{vmatrix} = \begin{vmatrix} 1 & o & o \\ m & p & o \\ m^2 & 2\,mp & p^2 \end{vmatrix} \begin{vmatrix} \sigma_p \\ \tau_{mp} \\ \sigma_m \end{vmatrix} \qquad (A.5)$$

The matrices in (A.2) and (A.5) are respectively the transpose of those in (A.3) and (A.4), as befits the fact that, from Clapeyron's theorem, the strain energy density is numerically equal to either of the forms

$$W = \frac{1}{2}\,(\sigma_x\ \tau_{xy}\ \sigma_y)'(\varepsilon_x\ \gamma_{xy}\ \varepsilon_y)$$

$$= \frac{1}{2}\,(\sigma_p\ \tau_{mp}\ \sigma_m)'(\varepsilon_p\ \gamma_{mp}\ \varepsilon_m)$$

Contour integrals for the invariants of the strain tensor.

By analogy with the formula

$$\vec{u} = u^p \vec{g}_p + u^m \vec{g}_m$$

we define contravariant base vectors $\vec{g}^p$ and $\vec{g}^m$ by

$$\vec{u} = u_p \vec{g}^p + u_m \vec{g}^m \qquad \text{(B}$$

The scalar product of this with $\vec{g}_p$ and $\vec{g}_m$ yields in view of the defini: of the covariant components u_p and u_m

$$\vec{g}^p \cdot \vec{g}_p = \vec{g}^m \cdot \vec{g}_m = 1 \qquad \vec{g}^p \cdot \vec{g}_m = \vec{g}^m \cdot \vec{g}_p = o$$

Hence

$\vec{g}^p$ has cartesian components 1 and o

$\vec{g}^m$ $-\frac{m}{p}$ and $\frac{1}{p}$ (B.

The following consequences will be found useful later :

$$\frac{\partial \vec{g}^p}{\partial p} = o \qquad \frac{\partial \vec{g}^p}{\partial m} = o$$

$$\frac{\partial \vec{g}^m}{\partial p} = -\frac{1}{p}\vec{g}^m \qquad \frac{\partial \vec{g}^m}{\partial m} = -\frac{1}{p}\vec{g}^p \qquad \text{(B.:}$$

We now establish, first in cartesian coordinates, a formula for calculating first invariant $\theta_1 = \varepsilon_x + \varepsilon_y$ of the strain tensor by contour integration (fig. B)

$$\iint \theta_1 dxdy = \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = \oint (\ell u + mv)ds$$

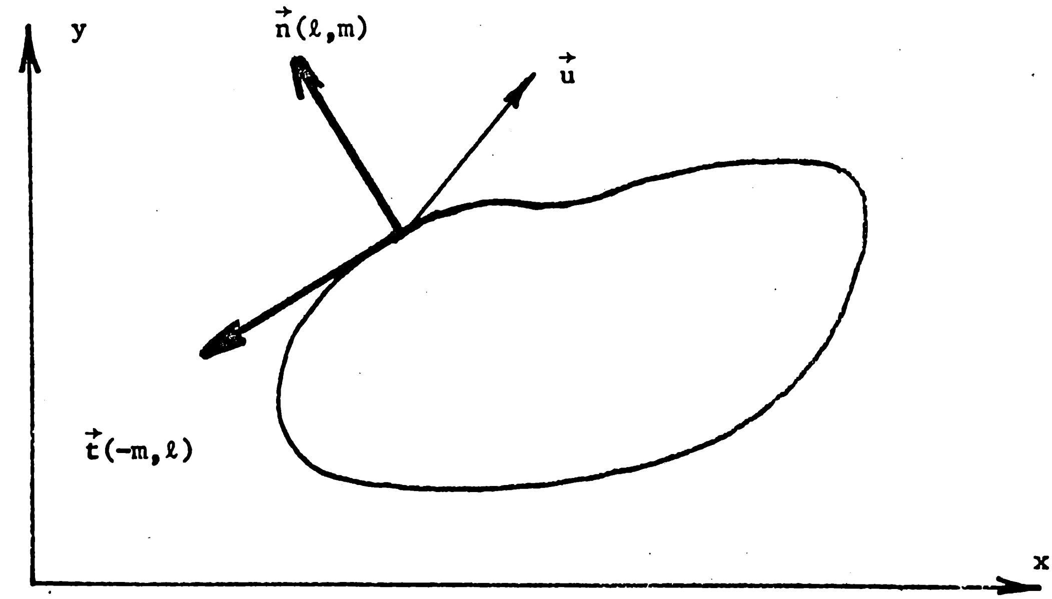


Figure B

Hence, if A denotes the area over which the double integral is extended, we obtain the intrinsic definition

$$\theta_1 = \lim_{A\to o} \frac{1}{A} \iint \theta_1 \, dA = \lim_{A\to o} \frac{1}{A} \oint (\vec{u} \cdot \vec{n}) ds \tag{B.4}$$

which allows the direct evaluation of θ_1 in natural coordinates.
For the second invariant of the deviator

$$\theta_2 = (\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2$$

$$\theta_2 = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2 + 4 \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

Hence

$$\theta_2 = \theta_1^2 + \omega^2 + 2\,\alpha \qquad (B.5)$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\alpha = 2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

θ_1 has already been reduced to a contour integral; for ω (twice the material rotation) we have

$$\iint \omega \, dxdy = \oint (\ell v - mu)ds$$

and, consequently, the intrinsic definition

$$\omega = \lim_{A \to o} \frac{1}{A} \oint (\vec{u} \cdot \vec{t})ds \qquad (B.6)$$

Finally

$$\iint \alpha \, dxdy = \oint \left\{ u \left(m \frac{\partial v}{\partial x} - \ell \frac{\partial v}{\partial y}\right) + v \left(\ell \frac{\partial u}{\partial y} - m \frac{\partial u}{\partial x}\right) \right\} ds$$

$$= \oint \left(v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t}\right) ds$$

and the intrinsic definition

$$\alpha = \lim_{A \to o} \frac{1}{A} \oint \left(\frac{\partial \vec{u}}{\partial t} \times \vec{u}\right)_z \, ds \qquad (B.7)$$

In this last formula, the z direction on which the vector product is projected, has a unit vector $\vec{1}_x \times \vec{1}_y$.

We now apply formulas (B.4), (B.6) and (B.7) to our trapezoidal coordinates. Along AB of figure 2, the outward normal is $-\vec{g}^m / |\vec{g}^m|$ and the projection of $\vec{u}$ on it

$$-\frac{1}{|\vec{g}^m|} \vec{u} \cdot \vec{g}^m = -\frac{1}{|\vec{g}^m|} (u_p \, \vec{g}^p \cdot \vec{g}^m + u_m \, \vec{g}^m \cdot \vec{g}^m)$$

$$= -\frac{p}{\sqrt{1+m^2}} \left(- u_p \frac{m}{p} + \frac{1+m^2}{p^2} u_m \right)$$

The flux of $\vec{u}$ along AB is finally

$$-\left(- m\, u_p + \frac{1+m^2}{p}\, u_m\right) dp$$

Adding the contribution of CD

$$\frac{\partial}{\partial m}\left(\frac{1+m^2}{p}\, u_m - m\, u_p\right) dmdp$$

Along AC the outward normal is $-\vec{g}^p / |\vec{g}^p| = -\vec{g}^p$, the projection of $\vec{u}$ $-u_p\, \vec{g}^p \cdot \vec{g}^p - u_m\, \vec{g}^m \cdot \vec{g}^p = -u_p + \frac{m}{p} u_m$ the flux of $\vec{u}$ along AC $(-p\, u_p + m\, u_m) dm$.

Adding the contribution of BD

$$\frac{\partial}{\partial p}(p\, u_p - m\, u_m) dpdm$$

Dividing the total flux by the area pdpdm, we find, according to the contour integral definition of θ_1

$$\theta_1 = \frac{1}{p}\frac{\partial}{\partial p}(p\, u_p - m\, u_m) + \frac{1}{p}\frac{\partial}{\partial m}\left(\frac{1+m^2}{p}\, u_m - m\, u_p\right)$$

$$= \varepsilon_p + \frac{1+m^2}{p^2}\,\varepsilon_m - \frac{m}{p}\,\gamma_{mp} \qquad \text{(B.8)}$$

In a similar manner we calculate the circulation of $\vec{u}$ along the contour which is

$$u\,\sqrt{1+m^2}\; dp = u_p\, dp \qquad \text{along AB}$$

$$-v\, p\, dm = -u_m\, dm \qquad \text{along AC}$$

and finally for the whole contour

$$\left(\frac{\partial u_p}{\partial m} - \frac{\partial u_m}{\partial p}\right) dpdm$$

Whence

$$\omega = \frac{1}{p}\left(\frac{\partial u_p}{\partial m} - \frac{\partial u_m}{\partial p}\right) \qquad \text{(B.9)}$$

The contour integral for α requires :

along AB
$$\frac{\partial \vec{u}}{\partial t} = \frac{1}{\sqrt{1+m^2}} \frac{\partial}{\partial p} (u_p \vec{g}^p + u_m \vec{g}^m)$$

$$= \frac{1}{\sqrt{1+m^2}} \left(\frac{\partial u_p}{\partial p} \vec{g}^p + \left(\frac{\partial u_m}{\partial p} - \frac{u_m}{p}\right) \vec{g}^m \right)$$

along AC
$$\frac{\partial \vec{u}}{\partial t} = - \frac{1}{p} \frac{\partial}{\partial m} (u_p \vec{g}^p + u_m \vec{g}^m)$$

$$= - \frac{1}{p} \left(\left(\frac{\partial u_p}{\partial m} - \frac{u_m}{p}\right) \vec{g}^p + \frac{\partial u_m}{\partial m} \vec{g}^m \right)$$

Noting the following values of the vector products

$$\vec{g}^p \times \vec{g}^p = \vec{g}^m \times \vec{g}^m = 0$$

$$(\vec{g}^p \times \vec{g}^m)_z = - (\vec{g}^m \times \vec{g}^p)_z = \begin{vmatrix} 1 & 1 \\ -\frac{m}{p} & \frac{1}{p} \end{vmatrix} = \frac{1}{p}$$

we obtain

along AB
$$\left(\frac{\partial \vec{u}}{\partial t} \times \vec{u}\right)_z ds = \frac{dp}{p} \left(u_m \frac{\partial u_p}{\partial p} - u_p \frac{\partial u_m}{\partial p} + \frac{u_m u_p}{p}\right)$$

along AC
$$= - \frac{dm}{p} \left(u_m \frac{\partial u_p}{\partial m} - u_p \frac{\partial u_m}{\partial m} - \frac{u_m u_p}{p}\right)$$

and finally

$$\alpha = - \frac{1}{p^2} \frac{\partial}{\partial m} \left(u_m \varepsilon_p - u_p \frac{\partial u_m}{\partial p} + \frac{u_m u_p}{p}\right) + \frac{1}{p} \frac{\partial}{\partial p} \left(\frac{u_m}{p} \frac{\partial u_p}{\partial m} - \frac{u_m^2}{p^2} - \frac{u_p}{p} \varepsilon_m\right)$$

After development and reorganization of terms, we find

$$\omega^2 + 2 \alpha = \frac{1}{p^2} (\gamma_{mp}^2 - 4 \varepsilon_m \varepsilon_p) \qquad \text{(B.IO)}$$

and this gives for the invariant θ_2

$$\theta_2 = \theta_1^2 + \frac{1}{p^2} (\gamma_{mp}^2 - 4\ \varepsilon_m\ \varepsilon_p)$$

$$= (\varepsilon_p + \frac{m^2-1}{p^2}\ \varepsilon_m - \frac{m}{p}\ \gamma_{mp})^2 + \frac{1}{p^2} (\gamma_{mp} - \frac{2\ m}{p}\ \varepsilon_m)^2 \qquad \text{(B.II)}$$

$$B_1^{**} = \begin{vmatrix} 0 & & & & & \\ 0 & 0 & & \text{SYMMETRY} & & \\ 0 & 0 & 0 & & & \\ 0 & -2\,m_o H & 0 & \frac{f(\beta_o)}{\lambda} & & \\ 0 & 0 & -\frac{4}{3}\,m_o H & 0 & \frac{g(\beta_o)}{\lambda} & \\ 0 & -\frac{2}{3}\,m_o H & 0 & \frac{g(\beta_o)}{\lambda} & -\frac{2}{3}\,m_o(m_o I-\alpha H) & \frac{h(\beta_o)}{\lambda} \end{vmatrix}$$

where :

$$f(\beta_o) = \lambda \int_{-1}^{1} \frac{dX}{Z} = \frac{1}{\beta_o}\,\ell n\,\frac{1+\beta_o}{1-\beta_o}$$

$$g(\beta_o) = -\frac{\lambda}{\beta_o} \int_{-1}^{1} \frac{XdX}{Z} = \lambda\int_{-1}^{1} \frac{X^2 dX}{Z}$$

$$= \frac{1}{\beta_o}\left(\frac{1}{\beta_o}\,\ell n\,\frac{1+\beta_o}{1-\beta_o} - 2\right)$$

$$h(\beta_o) = -\frac{\lambda}{\beta_o} \int_{-1}^{1} \frac{X^3 dX}{Z} = \lambda \int_{-1}^{1} \frac{X^4 dX}{Z}$$

$$= \frac{1}{\beta_o}\left(-\frac{2}{3} - \frac{2}{\beta_o^2} + \frac{f(\beta_o)}{\beta_o^2}\right)$$

$$H(m_o) = \frac{1}{m_o} \int_{-1}^{1} \frac{mdY}{1+m^2} = \frac{1}{2\,m_o^2}\,\ell n\,\frac{r(m_o)}{s(m_o)}$$

$$r(m_o) = 1 + m_o^2\,(\alpha_o+1)^2$$

$$I(m_o) = \frac{1}{m_o^2} \int_{-1}^{1} \frac{m^2 dY}{1+m^2} = \frac{1}{m_o^3}\left(2m_o - \arctan m_o(\alpha_o+1) + \arctan m_o(\alpha_o-1)\right)$$

$$s(m_o) = 1 + m_o^2(\alpha_o-1)^2$$

<u>Figure 5</u>

$$
B_1^* = \begin{vmatrix}
0 & & & & & \\
0 & 2\lambda N(m_o) & & & \text{SYMMETRY} & \\
0 & \frac{4}{3} m_o N(m_o) & \frac{8}{3}\lambda N(m_o) & & & \\
0 & 0 & 0 & 0 & & \\
0 & 2\lambda L(m_o) & \frac{4}{3} m_o L(m_o) & 0 & 2\lambda P(m_o) & \\
0 & \frac{4}{3} m_o L(m_o) & \frac{4}{3}\lambda L(m_o) & 0 & \frac{4}{3} m_o M(m_o) & \frac{8}{3}\lambda L(m_o)
\end{vmatrix}
$$

where $L(m_o) = (1+\nu)(C - \alpha_o A) + 2\, m_o^2(E - \alpha_o B)$ $\qquad N(m_o) = (1+\nu)\, A + 2\, G$

$M(m_o) = (1+\nu)(F + 2\, G) - \frac{4}{3} m_o^2(m_o I - \alpha_o H)$ $\qquad P(m_o) = (1+\nu)\, F + 2\, G$

2

$$
\left|\begin{array}{cccccccc}
0 & & & & & & & \\
0 & -m_o^2H^2\lambda & & & & & & \\
0 & -\frac{2}{3}m_o^3H & -\frac{4}{3}m_o^2H^2\lambda & & & & & \\
0 & m_oH & 0 & -1 & & & & \\
0 & -m_oHK\lambda & -\frac{2}{3}m_oH(Km_o-1) & K+\frac{1}{2\lambda}\beta_o g(\beta_o) & -\left(K^2\lambda+\frac{g(\beta_o)}{2\lambda}\right) & & & \\
0 & -\frac{m_oH}{3}(Km_o-1) & -\frac{4m_oHK\lambda}{3} & -\frac{g(\beta_o)}{2\lambda} & +\frac{K}{3}(1-2Km_o) & -\frac{4}{3}K^2\lambda-\frac{1}{2\lambda}h(\beta_o) & & \\
0 & -m_oH & 0 & 1 & -K-\frac{1}{2\lambda}\beta_o g(\beta_o) & \frac{1}{2}\frac{g(\beta_o)}{\lambda} & \frac{1}{2}\frac{f(\beta_o)}{\lambda} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

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13. ABSTRACT

This paper presents a tool to study the tapered structures by the direct stiffness method. It appears that a natural coordinate system is convenient to simplify the boundary conditions and in consequence the analytic calculations. The development of a particular element is taken as example and illustrated in details: the tapered spar. By means of the special coordinate system the conformity with the already developed cover sheets (quadratic elements, linear elements) is easily realized.

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