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H E A T C O N D U C T I O N

- I. DUAL MINIMUM THEOREMS IN HEAT CONDUCTION PROBLEMS AND
FINITE ELEMENT SOLUTION BASED ON COMPATIBLE TEMPERATURE FIELDS.

2. FINITE ELEMENT SOLUTION OF HEAT CONDUCTION PROBLEMS BASED ON
BALANCED HEAT FLOW FIELDS. BOUNDS TO THE DISSIPATION FUNCTIONAL.

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SUMMARY

The steady-state heat conduction problem is formulated in terms of the basic physical quantity attached to this kind of boundary value problem : a dissipation functional.

Dual minimum principles are derived from which respectively upper and lower bounds to the dissipation functional can be obtained from corresponding models of finite elements.

The temperature model is derived and tested from the viewpoint of numerical effectiveness.

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1. INTRODUCTION

Finite element methods are best known from their applications to linear elasticity theory. Dual single-field variational principles, like the principle of minimum total energy and the principle of minimum complementary energy, were shown to be advantageous in the construction of mathematical models of finite elements and in the numerical estimation of the accuracy of the approximations (1, 2, 3).

Non structural continuum problems have received less attention and only recently (4, 5) has the dual formulation been presented for other field problems where variational principles are available.

In the present paper attention is focused on the steady state heat conduction problem, although other boundary value problems like steady seepage through porous media, electrostatic and electromagnetic fields, isoenergetic flows of ideal fluids can benefit from similar treatment.

The basic dual theorems can be derived either from variational manipulations⁵ or from inequalities based on thermal behaviour of the continuum. The latter approach starts from an integrated thermal balance of the continuum and is analogous to the virtual work principle of elasticity. It stresses the fact that the variational character of the theorems is not essential for their use in conjunction with the finite element method. The basic physical quantity of steady state heat conduction, i.e. the dissipation functional, is introduced and the two minimum theorems are derived from its positive definite character. Special emphasis is given on the nature of the transition conditions required when the domain is subdivided into finite subdomains. The dual character of the theorems extends to the benefit of the dissipation functional a numerical estimate of the convergence by upper bounds (temperature approximations) and lower bounds (heat flow approximations) when no heat sinks are prescribed. When the body is in contact with a uniform temperature bath, taken as zero level, the role of bounds is reversed. The general case of a mixed boundary value problem where both sinks and non-uniform external temperature prevail can be treated by superposition if convergence estimates are required.

The following sections are devoted to the construction of mathematical models of finite elements of the compatible temperature type based on the first minimum theorem⁶. The particular concepts of shaping and bubble functions are introduced for the temperature field.

An original treatment of the convection boundary of the continuum is presented which is necessary to obtain correct bounds in a dual analysis. Some numerical examples show practical use of 2 D - and axisym - temperature finite elements in solving heat conduction problems.

2. FIELD EQUATIONS AND BOUNDARY CONDITIONS

The heat conducting body occupies a volume D bounded by a surface ∂D , the outward normal of which has direction cosines n_i ($i = 1, 2, 3$). Let $T(x_i)$ be the steady but non uniform temperature distribution referred to cartesian coordinates.

The temperature gradient is denoted by

$$\partial_i T = \frac{\partial T}{\partial x_i} \quad (1)$$

and heat flow q_i is governed by Fourier's law

$$q_j = -k_{ji} \partial_i T \quad (2)$$

where k_{ij} is a symmetrical heat conductivity tensor. When the principal axes of this tensor have constant orientation from point to point and the cartesian axes are taken parallel to them, the simpler law prevails

$$q_1 = -k_1 \partial_1 T \quad q_2 = -k_2 \partial_2 T \quad q_3 = -k_3 \partial_3 T \quad (3)$$

If Q denotes a heat sink distribution per unit volume, the heat flow balance requires

$$\partial_j q_j + Q = 0 \quad (4)$$

Boundary conditions of various type are considered. If the body is in contact with a heat bath, whose temperature can be maintained at a prescribed level T_e , the body's surface temperature will depend on the outgoing flux $n_j q_j$

$$T = T_e + \frac{1}{h} n_j q_j \quad (5)$$

If the convection exchange coefficient h tends to infinity, this reduces to a prescribed temperature condition. If it tends to zero, we have an adiabatic wall condition.

Should we write (5) in the equivalent form

$$n_j q_j = h (T - T_e) = q_e \quad (6)$$

the right hand side is an equivalent surface heat sink. We can then think of the rate of heat absorption q_e to be a prescribed quantity; in which case the bath temperature T_e must be adjusted to conform with the temperature reached by the body's surface.

The concept of a heat bath temperature can be extended, as a useful artifact, to interface transition conditions. On each interface A between two heat conducting bodies we can distinguish between two faces, conventionally denoted by A_+ and A_- (fig.1.).

The conditions generalizing (5) can then be written as

$$\begin{aligned} T_+ &= T_e + \frac{1}{h_+} n_{j+} q_{j+} \\ T_- &= T_e + \frac{1}{h_-} n_{j-} q_{j-} \end{aligned} \quad (5')$$

If the interface temperature T_e is prescribed, each condition is independent of the other and there is, properly speaking, no transition condition. There is an interface heat sink

$$n_{j+} q_{j+} + n_{j-} q_{j-} = h_+ (T_+ - T_e) + h_- (T_- - T_e) = q_e \quad (6')$$

whose intensity has to be regulated to comply with the surface temperatures T_+ and T_- .

Should we prescribe q_e instead of T_e , this last value must be adjusted to

$$T_e = \frac{h_+ T_+ + h_- T_- - q_e}{h_+ + h_-}$$

In this case, eliminating T_e between the two conditions (5'), the conditions can be written

$$T_+ - \frac{1}{h_+} n_{j+} q_{j+} = T_- - \frac{1}{h_-} n_{j-} q_{j-} \quad (7)$$

$$n_{j+} q_{j+} + n_{j-} q_{j-} = q_e \text{ prescribed.}$$

Those are real transition conditions. Introducing an equivalent transitional convection coefficient h , defined by

$$\frac{1}{h} = \frac{1}{h_+} + \frac{1}{h_-} \quad (8)$$

they can be put into the equivalent form

$$\begin{aligned} n_{j+} q_{j+} &= h (T_+ - T_-) + \frac{h}{h_-} q_e \\ n_{j-} q_{j-} &= h (T_- - T_+) + \frac{h}{h_+} q_e \end{aligned} \quad (7')$$

Usually the prescribed heat sink value is zero; there is heat flow continuity and only the transitional convection coefficient h need be known.

If the interface is an artificial one, introduced for the purpose of numerical analysis by finite elements in a homogeneous body, we can let h_+ and h_- tend to infinity in the first of equations (7) and obtain the transitional requirement of temperature continuity.

3. VIRTUAL POWER AND HEAT SINK POTENTIAL

A vector field

$$e_i = \partial_i T \quad (9)$$

satisfies integrability conditions for the existence of T :

$$\partial_2 e_3 - \partial_3 e_2 = 0 \quad \partial_3 e_1 - \partial_1 e_3 = 0 \quad \partial_1 e_2 - \partial_2 e_1 = 0 \quad (10)$$

analogous to the compatibility equations a strain field must satisfy in order to integrate a displacement field.

Inasmuch as non-compatible strain fields do occur for certain types of

approximation in elasticity theory, we are led to introduce the concept of "generalized temperature gradient field".

Such a field is composed of e_i fields, that do not necessarily satisfy (9) in each subdomain, and surface temperature discontinuities, either at interfaces or at the outer boundary; it will be symbolized by e .

Let q symbolize an independent heat flow field, piecewise differentiable in each subdomain. We then construct a "virtual power functional"

$$(e, q) = - \int_D e_i q_i dD + \int_S n_i q_i (T - T_e) dS \quad (11)$$

analogous to the structure of a virtual work in elasticity theory. S denotes the outer boundary plus the set of both faces of each interface and we recall that T_e has the same value on both faces of an interface.

If e is such that the vector field e_i obeys to (10) in each simply connected subdomain and a temperature field can be constructed, whose surface values coincide with those introduced in (11), it is called "compatible" and denoted by the symbol ∂T . Then, by simple integration by parts, we find

$$(\partial T, q) = \int_D T \partial_i q_i dD - \int_S n_i q_i T_e dS$$

Let us now denote by Q and q_e the heat sinks required to balance the flow field q , that is

$$\partial_i q_i = -Q \quad \text{in } D$$

$$n_i q_i = q_e \quad \text{on } \partial D$$

$$(n_i q_i)_+ + (n_i q_i)_- = q_e \quad \text{on } A$$

We then obtain an identity between virtual power and a heat sink potential :

$$(\partial T, q) = - \int_D T Q dD - \int_{A + \partial D} T_e q_e dS \quad (12)$$

In the heat sink potential, each heat sink intensity is multiplied by its corresponding temperature. This result is analogous to the identity between the internal form of virtual work and its external form (product of external forces by the conjugate displacements) when the strain field is compatible.

4. SCALAR PRODUCT AND DISSIPATION FUNCTIONAL

The result of section 3 makes no use of Fourier's law nor of the nature of the boundary conditions; it relies only on the concepts of compatibility for temperature gradients and balance, or equilibrium, for heat flow.

To any flow field we can now associate a generalized temperature gradient field, through Fourier's law

$$q_i = -k_{ij} e_j \quad (13)$$

and the heat convection equations

$$n_i q_i = h (T - T_e) \quad (14)$$

applied on the totality of S . Substitution of this into (11) would create confusion by having the same symbol for the generalized temperature gradient field already present in (11) and the new field "conjugate" to q . For this reason we denote the latter one by \hat{e} , and obtain

$$(e, \hat{e}) = \int_D k_{ij} e_i \hat{e}_j dD + \int_S h(T - T_e)(\hat{T} - T_e) dS \quad (15)$$

This expression is clearly in the nature of a scalar product between two generalized temperature gradient fields; in particular

$$(e, \hat{e}) = (\hat{e}, e) \quad (16)$$

Conversely, inverting Fourier's law

$$e_j = -r_{jk} q_k \quad (17)$$

where r_{jk} is a heat resistivity tensor, reciprocal to the heat conduc-

tivity one,

$$k_{ij} r_{jk} = \delta_{ik} \quad (18)$$

and using (14) in reverse, we are able to express the same scalar product in a dual form as

$$(q, \bar{q}) = \int_D r_{ij} q_i \bar{q}_j dD + \int_S \frac{(n_i q_i)(n_j \bar{q}_j)}{h} dS \quad (19)$$

In fact, since there is a one-one correspondance between a generalized temperature gradient field and its conjugate flow field, those fields represent the same element in function space. Thus, when we use the symbol f , it will represent either one of the conjugate fields. In this case (13) and (14) are analogous to the stress-strain relations.

5. THE DUAL VARIATIONAL PRINCIPLES

The definition of a scalar product induces a Hilbert space structure in the generalized temperature gradient fields (or their associated flow fields) provided

$$(f, f) = (e, e) = (q, q)$$

be positive definite. This is however a valid assumption, based on physical considerations related to the second law of thermodynamics. If θ denotes the absolute value of the reference temperature, from which the (small) temperature excursion T are measured, then $(f, f)/\theta^2$ is the (linearized) rate of increase of entropy of the body and its surroundings. We shall call

$$\frac{1}{2} (f, f) \quad \text{the dissipation functional.}$$

It will prove convenient to use the property of a Hilbert space to be a metric space. It furnishes a concept of distance between two fields, the square of which is defined by

$$(f - \hat{f}, f - \hat{f}) \geq 0$$

and the positive definite property ensures that if the distance vanishes then $f = \hat{f}$.

Consider now a heat conduction problem where the volume heat sinks are specified

$$Q = \bar{Q} \quad \text{in } D \quad (20)$$

and the boundary conditions are mixed. More precisely, we separate S in two subsets

$$S_1 \cup S_2 = S$$

and $A + \partial D$ in two corresponding subsets

$$B_1 \cup B_2 = A + \partial D$$

The set B_1 is the set S_1 where both faces belonging to the same interface have been merged. The heat sink temperatures are prescribed on B_1

$$T_e = \bar{T}_e \quad \text{on } B_1 \quad (21)$$

The set B_2 corresponds similarly to S_2 and the heat sink flow rate is specified on it

$$q_e = \bar{q}_e \quad \text{on } B_2 \quad (22)$$

Let ∂T_0 denote a fixed temperature gradient field with an integrated temperature distribution $T_0(x)$ in each subdomain and surface temperature discontinuities $T_0 - \bar{T}_e$ on B_1 .

Let $\partial \hat{T}$ denote an adjustable (containing unknown parameters) temperature gradient field, with integrated temperature distribution $\hat{T}(x)$ in each subdomain and surface temperature discontinuities $\hat{T} - 0$.

Then, by the principle of superposition, $\partial(\hat{T} + T_0)$ will be an adjustable temperature gradient field, with integrated temperature distribution $\hat{T} + T_0$ and surface temperature discontinuities $\hat{T} + T_0 - \bar{T}_e$ on B_1 .

Similarly introduce a fixed heat flow field, denoted by q_0 , which balances the heat sink rates \bar{Q} in D and \bar{q}_e on B_2 . Introduce also an adjustable heat flow field \hat{q} which balances zero heat sink rates in D and on B_2 . Then, by superposition, $\hat{q} + q_0$ will be an adjustable heat flow field in equilibrium with the prescribed heat sink rates.

If the squared distance

$$(\hat{T} + \partial T_0 - \hat{q} - q_0, \hat{T} + \partial T_0 - \hat{q} - q_0) \geq 0$$

happens to be zero, we have

$$\hat{T} + \partial T_0 = \hat{q} + q_0$$

and both fields become the conjugate of one another and represent an exact solution to the problem. This solution is unique. Indeed if s_1 and s_2 are two solutions, $s_2 - s_1$ must be a temperature gradient field with $\bar{T}_e = 0$ on B_1 , that is of type \hat{T} .

Again $s_2 - s_1$ must also be a heat flow field in equilibrium with $\bar{Q} = 0$ in D and $\bar{q}_e = 0$ on B_2 , that is of type \hat{q} . Thus

$$(s_2 - s_1, s_2 - s_1) = (\hat{T}, \hat{q}) = 0$$

in view of equation (12). This establishes that $s_2 = s_1$.

In numerical analysis the number of adjustable parameters is generally insufficient to produce an exact solution. A useful form of best approximation is then furnished by trying to minimize the squared distance between a temperature gradient formulation and a heat flow formulation:

$$(\hat{T} + \partial T_0 - \hat{q} - q_0, \hat{T} + \partial T_0 - \hat{q} - q_0) \text{ minimum}$$

In view of the result

$$(\hat{T}, \hat{q}) = 0 \tag{23}$$

this is equivalent to

$$\begin{aligned}
 & (\partial \hat{T} + \partial T_0 - q_0, \partial \hat{T} + \partial T_0 - q_0) + (\hat{q} + q_0 - \partial T_0, \hat{q} + q_0 - \partial T_0) \\
 & - (q_0 - \partial T_0, q_0 - \partial T_0) \text{ minimum} \quad (24)
 \end{aligned}$$

The first term is non-negative and depends only on the adjustable parameters in $\partial \hat{T}$, the second is also non-negative and depends only on the adjustable parameters in \hat{q} , the last one being fixed can be discarded. Consequently we may minimize separately the first and the second term, which procedure will furnish immediately the two most important variational principles governing steady heat conduction.

5.1. THE PRINCIPLE OF VARIATION OF TEMPERATURES

Minimization of the first term in (24) is equivalent to

$$\frac{1}{2} (\partial T, \partial T) - (\partial T, q_0) \text{ minimum} \quad (T = \hat{T} + T_0) \quad (25)$$

Explicitly, using the definition of the scalar product and (12) :

$$\begin{aligned}
 & \frac{1}{2} \int_D k_{ij} \partial_i T \partial_j T \, dD + \frac{1}{2} \int_{S_1} h(T - \bar{T}_e)^2 \, dS + \frac{1}{2} \int_{S_2} h(T - T_e)^2 \, dS \\
 & + \int_D T \bar{Q} \, dD + \int_{B_2} \bar{q}_e T_e \, dS + \int_{B_1} q_{e0} \bar{T}_e \, dS \text{ min.} \quad (26)
 \end{aligned}$$

The last term, that contains the fixed heat sink rates q_{e0} balancing the particular flow field q_0 on B_1 , can be discarded as constant. The Euler equations and boundary conditions of this principle of variation of temperatures are found to be

$$- \partial_i (k_{ij} \partial_j T) + \bar{Q} = 0 \quad \text{in } D \quad (27)$$

$$n_i k_{ij} \partial_j T + h(T - \bar{T}_e) = 0 \quad \text{on } S_1 \quad (28)$$

$$n_i k_{ij} \partial_j T + h(T - T_e) = 0 \quad \text{on } S_2 \quad (29)$$

stemming from volume and surface variations of the temperature field T . It must be understood that, according to whether parts of S_1 or S_2 belong to A_+ or A_- , h and T must be replaced by h_+ , T_+ or h_- , T_- respectively.

An additional result is obtained from the arbitrary variation of the unspecified heat sink temperature T_e on S_2 :

$$-h_+(T_+ - T_e) - h_-(T_- - T_e) + \bar{q}_e = 0 \quad (30)$$

$$\text{or} \quad -h(T - T_e) + \bar{q}_e = 0 \quad (31)$$

according to whether \bar{q}_e is at an interface or at the outer boundary. The correctness of equations (27) to (31) can be verified from the considerations developed in section 2.

An analysis of the limiting case, when a convection coefficient goes to infinity, shows that the corresponding contribution of the integrals on S_1 or S_2 in (26) must disappear; in compensation the corresponding face temperature must be set equal to \bar{T}_e if it belongs to S_1 , to T_e if it belongs to S_2 .

5.2. THE PRINCIPLE OF VARIATION OF HEAT FLOW

Minimization of the second term in (24) is equivalent to

$$\frac{1}{2} (q, q) - (\partial T_o, q) \text{ minimum} \quad (q = \hat{q} + q_o) \quad (32)$$

Explicitly, and since for T_o the heat sink temperature on B_1 is \bar{T}_e ,

$$\begin{aligned} & \frac{1}{2} \int_D \tau_{ij} q_i q_j dD + \frac{1}{2} \int_S \frac{(n_i q_i)^2}{h} dS + \int_D \bar{q} T_o dD \\ & + \int_{B_1} q_e \bar{T}_e dS + \int_{B_2} \bar{q}_e T_{eo} dS \quad \min \end{aligned} \quad (33)$$

The third and fifth terms, which are fixed constants, can be dropped.

In this "principle of variation of heat flow", analogous to the complementary energy principle of elasticity theory, the heat flow vector may not be varied unrestrictedly. It is constrained a priori to verify the heat balance equations :

$$\partial_j q_j + \bar{Q} = 0 \quad \text{in } D \quad (34)$$

$$(n_j q_j)_+ + (n_j q_j)_- = \bar{q}_e \quad \text{on } B_2 \quad (35)$$

or $n_j q_j = \bar{q}_e$

Similarly, in the B_1 integral of (33) q_e must be understood to mean $(n_j q_j)_+ + (n_j q_j)_-$ or $n_j q_j$, according to whether the part of B_1 considered belongs to A or to ∂D .

This complementary variational principle can also be obtained from the temperature principle through a classical FRIEDRICHS transformation⁵. In so doing one also obtains useful two-field variational principles as intermediate steps. Those will not be considered here.

6. BOUNDING OF THE DISSIPATION FUNCTIONAL

To obtain a guaranteed upper or lower bound to the dissipation functional by approximations based on the preceding variational principles, it is necessary to split the general heat conduction problem in two parts.

In part 1 we keep the given data \bar{Q} in D and \bar{q}_e on B_2 but set $\bar{T}_e = 0$ on B_1 .

In part 2 we do the reverse, set $\bar{Q} = 0$ in D and $\bar{q}_e = 0$ on B_2 but introduce the data \bar{T}_e on B_1 .

By the principle of superposition, the general solution is then obtained by adding the solutions of parts 1 and 2.

6.1. BOUNDING OF THE DISSIPATION FUNCTIONAL IN PROBLEM 1

Setting $\bar{T}_e = 0$ on B_1 means that we can dispense with the particular field T_0 and approximate the solution of problem 1 through either one

of the variational principles

$$\frac{1}{2} (\partial \hat{T}, \partial \hat{T}) - (\partial \hat{T}, q_0) \text{ minimum} \quad (36)$$

$$\frac{1}{2} (q, q) \text{ minimum} \quad (37)$$

Take (36) first and assume a solution

$$\hat{T} = \alpha_i \hat{T}_{(i)} \quad (38)$$

where the summation is finite and the $\hat{T}_{(i)}$ temperature fields have assumed distributions. Then (36) goes into the problem of minimizing a quadratic form in the unknown coefficients α_i

$$\frac{1}{2} \alpha_i \alpha_j (\partial \hat{T}_{(i)}, \partial \hat{T}_{(j)}) - \alpha_i (\partial \hat{T}_{(i)}, q_0)$$

The coefficients are determined by solving the linear system

$$\alpha_j (\partial \hat{T}_{(i)}, \partial \hat{T}_{(j)}) = (\partial \hat{T}_{(i)}, q_0) \quad (i = 1, 2, \dots, n) \quad (39)$$

Denoting by $\hat{\theta}$ the best approximate temperature field obtained by substituting those coefficients into (38), we see that (39) can be written

$$(\partial \hat{T}_{(i)}, \partial \hat{\theta}) = (\partial \hat{T}_{(i)}, q_0) \quad (i = 1, 2, \dots, n)$$

Multiplying each of these equations by the corresponding coefficient and adding

$$(\partial \hat{\theta}, \partial \hat{\theta}) = (\partial \hat{\theta}, q_0) \quad (40)$$

This property is analogous to the Clapeyron theorem of elasticity theory. It is obvious that it holds true also for the exact solution

$$(s, s) = (s, q_0) \quad (41)$$

We recall that the symbol s for the exact solution, represents either the exact flow field or its conjugate temperature gradient field.

Then the property

$$(s, s - q_0) = 0$$

equivalent to (41) follows directly from (23), because s , as a temperature gradient field, is of type $\partial\hat{T}$ and $s - q_0$, as a flow field is of type \hat{q} (balancing zero heat sink rates in D and on B_2).

Since the approximate solution will not, in general, allow the minimum of (36) to be reached, we have

$$\frac{1}{2} (\partial\hat{\theta}, \partial\hat{\theta}) - (\partial\hat{\theta}, q_0) \geq \frac{1}{2} (s, s) - (s, q_0)$$

Then, in view of (40) and (41), we obtain the boundings

$$(s, s) \geq (\partial\hat{\theta}, \partial\hat{\theta}) \quad (s, q_0) \geq (\partial\hat{\theta}, q_0) \quad (42)$$

Turn next to (37). Since again any best approximate solution p obtained from this variational principle, will not as a rule reach the absolute minimum :

$$(p, p) \geq (s, s) \quad (43)$$

Thus

$$(\partial\hat{\theta}, \partial\hat{\theta}) \leq (s, s) \leq (p, p) \quad (44)$$

This bracketing of the exact value of the dissipation functional through dual approximate analyses is very useful to appreciate the accuracy of the numerical solutions.

6.2. BOUNDING OF THE DISSIPATION FUNCTIONAL IN PROBLEM 2

Setting $\bar{q} = 0$ in D and $\bar{q}_e = 0$ on B_2 means that we can dispense of the particular flow field q_0 and approximate the solution of problem 2 through either one of the variational principles

$$\frac{1}{2} (\partial T, \partial T) \quad \text{minimum} \quad (45)$$

$$\frac{1}{2} (\hat{q}, \hat{q}) - (\partial T_0, \hat{q}) \quad \text{minimum} \quad (46)$$

Denoting by θ the best approximate temperature field obtained through (45), it is clear that

$$(\partial\theta, \partial\theta) \geq (s, s)$$

Again, implementing (46) by a finite expansion

$$\hat{q} = \beta_1 \hat{q}_{(1)}$$

we find that the best approximation $\hat{q} = \hat{p}$ has the property

$$(\hat{p}, \hat{p}) = (\partial T_0, \hat{p}) \quad (47)$$

analogous to the Pasternak reduction theorem in structural analysis. This property is shared by the exact solution of problem 2

$$(s, s) = (\partial T_0, s) \quad (48)$$

because s , as a flow field, is of type \hat{q} and $s - \partial T_0$, as a temperature gradient field, is of type ∂T .

Furthermore

$$\frac{1}{2} (\hat{p}, \hat{p}) - (\partial T_0, \hat{p}) \geq \frac{1}{2} (s, s) - (\partial T_0, s)$$

Thus, in view of (47) and (48), we obtain

$$(s, s) \geq (\hat{p}, \hat{p}) \quad (\partial T_0, s) \geq (\partial T_0, \hat{p})$$

The bracketing of the approximate analyses is now seen to be reversed

$$(\hat{p}, \hat{p}) \leq (s, s) \leq (\partial\theta, \partial\theta) \quad (49)$$

which explains the necessity of splitting the general problem when it is wished to make use of this type of convergence criterion.

7. MATHEMATICAL MODELS OF TEMPERATURE FINITE ELEMENTS

These two minimum theorems may now be applied to the construction of mathematical models of finite elements either of the compatible temperature type or of the balanced heat flow type.

In this paper we shall restrain the discretization of the continuum problem to the most widely used way ⁽⁶⁾, i.e. discretization of the sole temperature field, continuous and piecewise differentiable.

The corresponding discretization of the temperature gradient follows through rigorous application of (1) and the whole weight of the approximation falls on the balance equation (4) which is only averaged as in a Galerkin process on each finite element domain by the shape functions chosen for the temperature field. The same is true for heat flux boundary specifications (22) which are translated into equivalent generalized fluxes and there are discontinuities in heat flow transmission between adjacent elements, the heat flux being solely constrained (6,7). On the other hand temperature boundary specifications (21) can be accounted for exactly if expressible in terms of the dito shaping functions. The procedure is then to minimize the functional (26) (dissipation functional plus heat sink potential of prescribed thermal loads) with respect to the finite temperature degrees of freedom.

7.1. TEMPERATURE ELEMENTS GENERATION

Hereafter matrix formulation ⁺ will be used for brevity and simplicity. The temperature field is discretized by assuming

$$T(x) = m^T(x) a = a^T m(x) \quad (50)$$

within the domain E of the finite element and on its boundary ∂E .

The boundary consists of several parts $\partial_\sigma E$ ($\sigma = 1, \dots, s$) corresponding to adjacent elements or to parts of the external boundary ∂D .

In (50) a and m are column vectors collecting respectively the parameters α_i to be determined by the minimum condition (26) (so that heat balance be approximated at best) and the assumed independent spatial modes $M_i(x)$ - usually polynomials - of the temperature field. Both the α_i - and M_i 's are in finite number n .

⁺ Unless otherwise specified, a small letter denotes a column vector, a capital letter denotes a matrix and superscript T denotes the transpose operation.

Temperature gradients are readily

$$e = \partial T = D(x) a \quad (51)$$

where ∂ is the matrix differential operator

$$\partial^T = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

and D a $(3 \times n)$ matrix of modes derivatives.

7.2. LOCAL BOUNDARY TEMPERATURES AND SHAPING FUNCTIONS

Continuity requirements for the temperature are still to be satisfied between elements. For this sake the behavior of $T(x)$ is analyzed on each face $\partial_\sigma E$ of the boundary ∂E : this furnishes the number of degrees of freedom of $T(x)$ on each $\partial_\sigma E$ and a local temperature coordinate is attached to each such degree of freedom.

Local coordinates pertaining to $\partial_\sigma E$ are collected in the column matrix t_σ which determines completely $T(x)$ on it. Transition conditions are then satisfied along $\partial_\sigma E$ provided the same boundary modes exist for the adjacent element and the associated t_σ take the same value across the interface. Local temperatures here upon are related to the parameters of the field by

$$t_\sigma = R_\sigma a \quad (52)$$

Identification of temperature distribution along $\partial_\sigma E$ furnishes

$$m^T(x) a = p_\sigma^T(x) t_\sigma \quad \text{for } x \in \partial_\sigma E$$

where the elements of $p_\sigma(x)$ are the boundary temperature modes on $\partial_\sigma E$. This relation is valid for any a so that

$$m(x) = R_\sigma^T p_\sigma(x) \quad (53)$$

The set of relationships (52) for all the parts $\partial_\sigma E$ of the boundary results in the global matrix relation

$$t = R a \quad (54)$$

where t collects the n_b boundary local temperatures (some of them are readily common to several faces in order to insure uniqueness for temperature at vertices).

This states that the knowledge of a determines completely the temperature on the boundary ∂E of the element but the question is now : does some value of a always correspond to an arbitrarily given t ?

The simplest case is that of a non singular matrix R implying that the number n of parameters be equal to the number n_b of boundary local temperatures. Inverting (54) and substituting into (50) there comes a translated discretization of the temperature field

$$T(x) = p^T(x) t \quad (55)$$

$$\text{where } p(x) = (R^T)^{-1} m(x) \quad (56)$$

are the shaping functions of the temperature field. In other words the independent boundary temperatures are sufficient to determine the temperature within E . This is the case when the elements M_i of the spatial modes matrix $m(x)$ are the complete set of coordinates polynomials of degree $N \leq 2$, for finite elements of the 2D-triangle or 3D-tetrahedron type.

7.3. LOCAL INTERNAL TEMPERATURES AND BUBBLE FUNCTIONS

The case where R is a non square matrix of rank n_b is still simple. In other words there are more field parameters a_i than local boundary temperatures ($n > n_b$) and the way to by-pass this difficulty is to choose ($n - n_b = n_i$) local internal temperatures v to complete the set of n_b relations (54) by

$$v = S a \quad (57)$$

where the row i of S corresponds to the row vector $m^T(x_i)$.
The system

$$s = \begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} R \\ S \end{pmatrix} a = N a \quad (58)$$

can now be solved to give the general solution since the $(n \times n)$ matrix N is non singular. This furnishes

$$a = P s = U t + V v \quad (59)$$

and after substitution into (50) there comes

$$T(x) = p^T(x) t + q^T(x) v \quad (60)$$

where $p(x) = U^T m(x) \quad (61)$

$$q(x) = V^T m(x) \quad (62)$$

are respectively once more the shaping functions and the bubble functions. The latter represent indeed internal temperature fields which vanish on the boundary of the element.

7.4. GENERALIZED HEAT FLUXES

At this point all the required conditions but temperature boundary specifications are satisfied to invoke the minimum principle (26). In this appears the heat sink potential which introduces natural definitions for the generalized fluxes. The part due to internal heat generation gives in view of (60)

$$\int_E \bar{Q} T(x) dE = t^T f_t + v^T f_v \quad (63)$$

where $f_t = \int_E \bar{Q} p(x) dE$ $f_v = \int_E \bar{Q} q(x) dE \quad (64)$

are defined as generalized heat fluxes conjugated respectively to the local temperature sets t and v . They appear as linear functionals where the actual prescribed heat generation is weighted by the shaping and bubble functions. Similarly for the (normal) heat flux at the boundary

$$\int_{\partial E} q_e T(x) d\partial E = t^T g_t \quad (65)$$

gives the definition of the generalized fluxes conjugated to the sole local boundary temperatures

$$S_t = \int_{\partial E} q_e p(x) d\partial E \quad (66)$$

The approximate solution to the temperature distribution using principle (6) will be shown depend only of the values of the generalized fluxes (64), (66) and many actual heat generation distributions and heat flux specifications can produce the same approximate answer. This is the announced characteristic feature of temperature finite elements models where the knowledge of the generalized heat fluxes corresponds to only a weak knowledge of the actual heat sink distribution.

7.5. ELEMENT CONDUCTIVITY MATRIX

Turning now to the dissipation term of principle (26) it can be computed directly for the element in terms of the parameters from the initial discretization (50) as

$$F_i = \frac{1}{2} \int_E e^T C e dE = \frac{1}{2} a^T K_{aa} a \quad (67)$$

where F_i denotes the internal part of the dissipation functional and

$$K_{aa} = \int_E D^T(x) C D(x) dE, \quad C = (k_{ij})$$

and thereafter transformed through (59) in a quadratic form in the local temperatures

$$F_i = \frac{1}{2} t^T K_{tt} t + v^T K_{vt} t + \frac{1}{2} v^T K_{vv} v \quad (68)$$

with $K_{tt} = U^T K_{aa} U$

$$K_{vt} = V^T K_{aa} U$$

$$K_{vv} = V^T K_{aa} V$$

It can also be obtained directly in the form (68) by performing integrations on the basis of the expansion (60) of T in shaping and bubble functions; in this case the generalized conductivity matrices are

$$K_{tt} = \int_E [\partial p^T(x)]^T C [\partial p^T(x)] dE$$

$$K_{vt} = \int_E [\partial q^T(x)]^T C [\partial p^T(x)] dE$$

$$K_{vv} = \int_E [\partial q^T(x)]^T C [\partial q^T(x)] dE$$

This last conductivity matrix is never singular because a bubble temperature function necessarily involves non zero temperature gradients within E; this allows the associated local internal temperatures to be eliminated at the element level; indeed the minimization to be performed from principle (26) is finally in view of (63), (65) and (68) - i.e. without convection effect.

$$\frac{1}{2} t^T K_{tt} t + v^T K_{vt} t + \frac{1}{2} v^T K_{vv} v + t^T (f_t + g_t) + v^T f_v$$

minimum

The minimum conditions are

$$K_{tt} t + K_{tv} v = -f_t - g_t \quad (69)$$

$$K_{vt} t + K_{vv} v = -f_v \quad (70)$$

Solving the last with respect to the internal temperatures produces

$$v = -K_{vv}^{-1} (f_v + K_{vt} t)$$

and the conductivity relation

$$K_E t_E = f_E + g_E \quad (71)$$

with $K_E = K_{tt} - K_{tv} K_{vv}^{-1} K_{vt}$ (72)

$$f_E = -f_t + K_{tv} K_{vv}^{-1} f_v \quad (73)$$

$$g_E = -g_t \quad (74)$$

Relation (71) describes the discretized thermal properties of each element with respect to boundary temperatures $t_E = t$ but the case of the boundary part F_c of the dissipation functional need still be treated.

7.6. SPECIFIC CONVECTION BOUNDARY ELEMENTS

This case is no longer treated by generalized fluxes (6) but results in essentially different elements as follows : an external degree of freedom is allocated to the boundary and a fictitious element is considered out of the actual body (fig. 2). The local temperatures sequence for this element comprises the set of temperatures pertaining to the boundary itself plus the external one T_e .

$$t_c^T = (t^T \quad T_e) \quad (75)$$

The external part F_c of the dissipation functional becomes then a quadratic form in these local temperatures

$$F_c = \frac{1}{2} \int_{\partial_1 E} h(T - T_e)^2 d\partial E = \frac{1}{2} t_c^T K_{cc} t_c \quad (76)$$

where $\partial_1 E$ is the part of the convection boundary S_1 allocated to the element. The minimization to be performed from principle (26) is in the present case

$$\frac{1}{2} t_c^T K_{cc} t_c + t_c^T g_c \quad \text{minimum}$$

where g_c is defined in a similar manner to (66) on $\partial_1 E$.

The discretized thermal properties of those specific convection boundary elements are then summarized in

$$K_{cc} t_c = - g_c \quad (77)$$

7.7. THERMAL PROPERTIES OF THE ASSEMBLED ELEMENTS

The principle for assembling the elements is well known and fully described elsewhere (3). Let us simply recall that it consists in stating which boundary temperatures must have common values at interfaces to implement the continuity requirements. The way to gather generalized fluxes follows from thermal balance at the interface. At this step it should be noted that, while f_E and g_E (73, 74) are boundary generalized fluxes for the element, the g_E 's are internal thermal loads for the whole body: this is clearly seen from the definition (64) of the f_E 's where the generalized fluxes are computed from the known heat generation distribution while the boundary heat fluxes defining the g_E 's by (66) are unknown.

Elementar conductivity matrices are also assembled in a master conductivity matrix and the thermal way of life of the whole body reads

$$K_s t_s = g_s \quad (78)$$

which can then be solved for the set of local temperatures t_s after due account is taken of the prescribed ones (boundary and external).

8. NUMERICAL EXAMPLES

The general method to derive temperature finite element models has been successfully applied ^{6, 7} to several types of geometry (2 D - triangle and quadrangle, AXISYM-triangle and 2 D - tetraedron). The detailed matrices will not be presented here as long as the procedure to be followed is the standard one previously described.

Preference is given to some numerical illustrations which stress the efficiency of the method in solving steady state heat conduction problems.

8.1. HUMAN BODY THERMAL MODEL ⁸

The thermal behavior of the human body has been modeled to simulate the major aspects of the heat transfer within the living tissues. The purpose is to develop thermoregulatory systems for protective suits used in current extravehicular activity space units. These systems must remove the metabolic heat generated by the body which was achieved in the most recent Apollo space suits by a network of water-cooled tubes in direct contact with the skin.

The modeled human body is a set of rectangular strips (fig. 3A) whose width is the half distance between cooling tubes. The depth is divided into three layers : an outer layer of skin, a skeletal muscle and a constant temperature inner core.

The tissues were considered to be isotropic and the thermal properties were taken as constant. The human body, in the vicinity of the parallel cooling tubes in contact with the skin, will be modeled as a parallelepiped with rectangular cross section. The thermal gradient parallel to the tubes will be considered zero so that the problem becomes two-dimensional in rectangular coordinates. The thermal field perpendicular to the tubes will be taken as symmetrical and the lines of symmetry which are located at the center of the cooling tubes and halfway between the tubes, are adiabatic boundaries. Since all the heat developed inside the body is assumed to be removed through the part β of the skin surface (prescribed heat flux), the remaining uncontacted part is also considered adiabatic. The inner boundary (interface core-skeletal muscle) is given the essentially constant temperature of the human body core (99,7°F).

Such simplifications are necessary to reach the limited data on the parameters involved and it is clear that the following numerical results are not immediately reliable for biological purposes.

Nevertheless we performed such a simplified numerical work through a 2 D - finite element mesh of 82 parabolic T-field elements (fig. 3.B). The temperature distribution is mapped for a case of high total metabolic rate (2600 Btu/hr = 760 watts). From a physiological viewpoint it is noteworthy that the required minimum skin temperature for the removal of 2600 Btu/hr is 38,1°F which is very low from a comfort standpoint.

8.2. PRESSURE VESSEL TEMPERATURE DISTRIBUTION

As a second example, an axisymmetrical type of geometry is involved in computing the temperature distribution in the pressure vessel of a nuclear reactor (fig. 4.A.). The temperature map due to critical conditions is required for further thermal stresses analysis. Fig. 4.A. presents an axial section of the hollow cylindrical type pressure vessel. The inner part noted I is occupied by a hot fluid and the corresponding boundary suffers convective heat transfer. The situation is analogous for part II of the boundary with a colder fluid. The remainder of the outer surface is considered adiabatic and the vessel isotropic and homogeneous,

but for a bottom ring of different material.

Fig. 4.B. presents the temperature map resulting from a treatment using a 202 finite element mesh of the parabolic temperature-type. In addition 38 specific convection boundary elements were needed to idealize the heat convection boundary.

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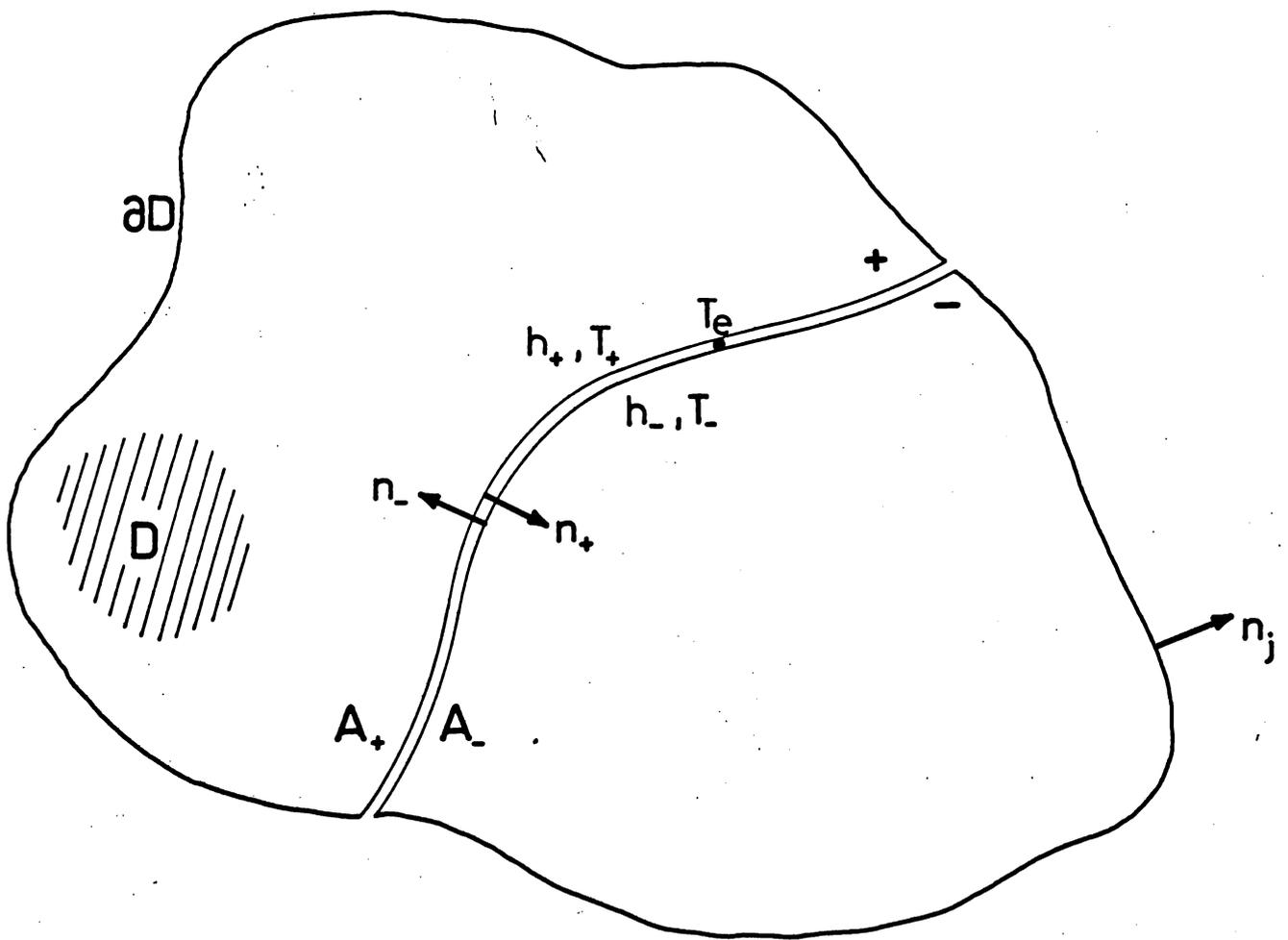


FIGURE 1 INTERFACE TRANSITION CONDITIONS

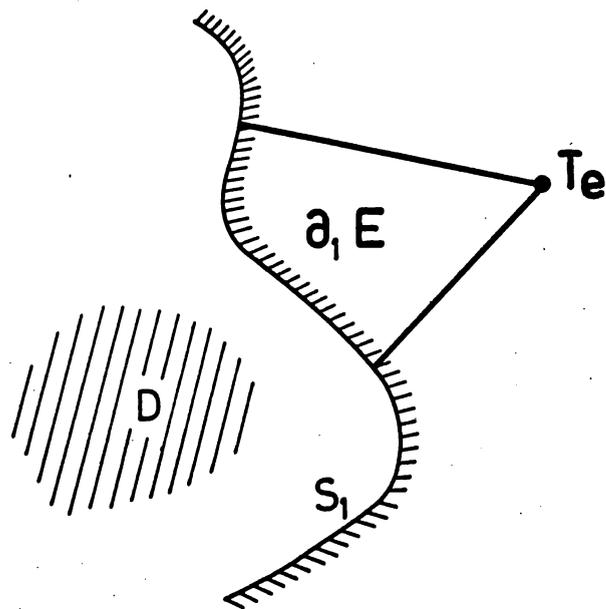


FIGURE 2 SPECIFIC CONVECTION BOUNDARY ELEMENT

FIGURE 3

HUMAN BODY THERMAL MODEL

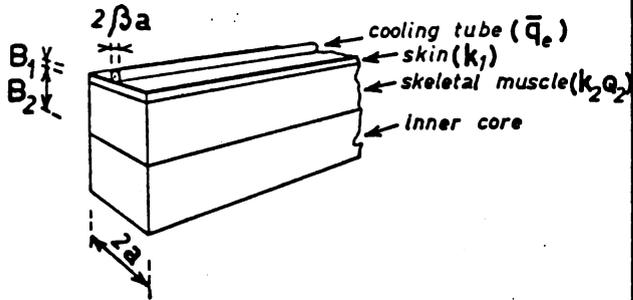


FIGURE 3a
REPRESENTATIVE STRIP

CHARACTERISTIC DATA	
$B_1 = 0.008$	FT.
$B_2 = 0.065$	FT.
$A = 0.032$	FT.
$K_1 = 0.242$	BTU / HR - FT - °F
$K_2 = 0.311$	BTU / HR - FT - °F
$Q_2 = 1398.$	BTU / HR - FT ³
$\bar{q}_e = 1696.$	BTU / HR - FT ²

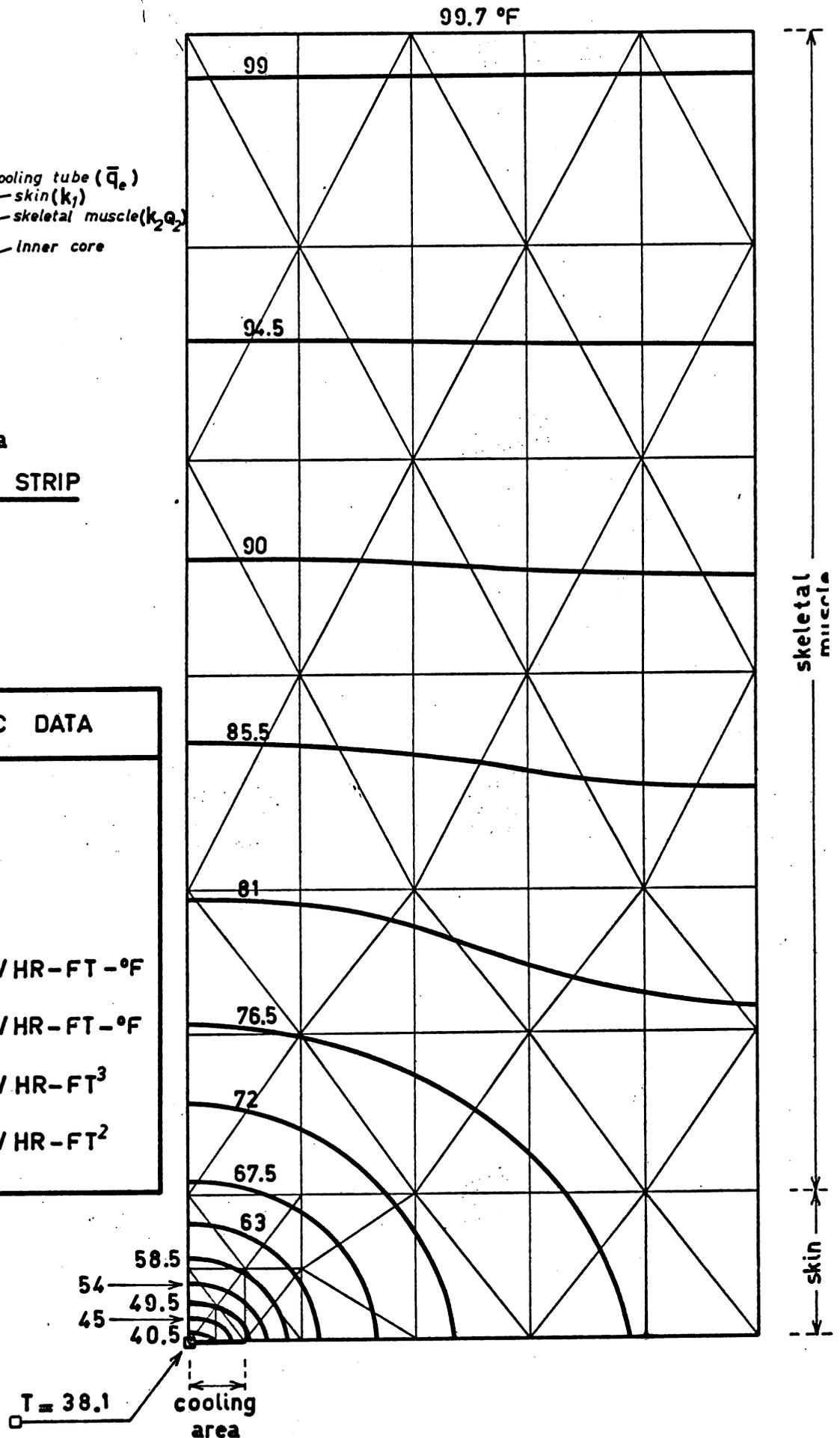


FIGURE 3b TEMPERATURE DISTRIBUTION $\beta = .1$

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SUMMARY

Heat flow models are derived from the second minimum principle of steady-state heat conduction. Either the heat flow field itself is discretized, or this is done through a heat stream function. Numerical bounds are presented for the dissipation functional of certain problems where both the temperature and the heat flow models were applied.

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1. INTRODUCTION

A dual formulation of the boundary value problem of heat conduction was presented (1,2) and was shown to provide a numerical estimate of the accuracy of approximate solutions.

In this paper two ways of constructing mathematical models of finite elements based on the second single-field variational principle, i.e. principle of variation of heat flow, are described.

The more direct way is to discretize balanced heat flow fields. Another way is to introduce a heat stream function, whereby the heat flow continuity at interfaces is obtained in a manner that is completely analogous to temperature continuity in temperature models.

In both cases the knowledge obtained for the temperature field is weak since it will be based only on values of a finite number of functionals of that field. This does not appear to be a severe limitation since those weighted values are sufficient to give a realistic picture of the temperature distribution.

Heat flow models enable numerical estimates to be made of the dissipation functional that bound the exact value from the opposite side to the estimates obtained from temperature models. This important evaluation of convergence is illustrated on several examples.

2. SUMMARY OF THE BASIC PRINCIPLE

The basic variational principle was obtained previously as ⁽¹⁾

$$\frac{1}{2} \int_D r_{ij} q_i q_j dD + \frac{1}{2} \int_S \frac{(n_i q_i)^2}{h} dS + \int_{B_1} q_e \bar{T}_e dS \text{ min.} \quad (1)$$

$$\left. \begin{array}{l} \text{with } q_e = (n_j q_j)_+ + (n_j q_j)_- \\ \text{or simply } q_e = n_j q_j \end{array} \right\} \text{ on } B_1 \quad (2)$$

We recall that B_1 is the set of interfaces or boundary surfaces where the heat sink temperature is prescribed.

The heat flow vector in (1) must be constrained to satisfy "a priori"

$$\partial_j q_j + \bar{Q} = 0 \quad \text{in } D \quad (3)$$

$$\left. \begin{array}{l} (n_j q_j)_+ + (n_j q_j)_- = \bar{q}_e \\ \text{or simply } n_j q_j = \bar{q}_e \end{array} \right\} \text{ on } B_2 \quad (4)$$

where B_2 is the set of interfaces or boundary surfaces where the heat sink intensity is prescribed.

Because of (3) and (4) the admissible heat flows are balanced heat flow fields.

Recuperation of the temperature field occurs in principle through the generalized temperature gradient field

$$e_j = -r_{jk} q_k \quad \text{in } D \quad (5)$$

that can be obtained by inverting Fourier's law. However it does not in general comply with the integrability conditions

$$\partial_2 e_3 - \partial_3 e_2 = 0 \quad \partial_3 e_1 - \partial_1 e_3 = 0 \quad \partial_1 e_2 - \partial_2 e_1 = 0 \quad (6)$$

required to construct a pointwise defined temperature field T , such that

$$e_j = \partial_j T \quad (7)$$

The boundary conditions for T are

$$\left. \begin{aligned} T_+ &= \bar{T}_e + \left(\frac{n_j q_j}{h}\right)_+ \quad \text{and} \quad T_- = \bar{T}_e + \left(\frac{n_j q_j}{h}\right)_- \\ \text{or simply} \quad T &= \bar{T}_e + \frac{n_j q_j}{h} \end{aligned} \right\} \text{on } B_1 \quad (8)$$

and similar conditions on B_2 , except that there \bar{T}_e is not prescribed but unknown.

3. MATHEMATICAL MODELS OF HEAT FLOW FINITE ELEMENTS

3.1. HEAT FLOW ELEMENTS GENERATION

Once again matrix formulation is used hence forward.

The heat flow is discretized in the form

$$q(x) = S(x) b + G(x) c \quad (9)$$

within the domain E of the finite element. b and c are column matrices of heat flow parameters to be determined by the minimum condition (1). The heat sink distribution in equilibrium with (9) must be

$$\partial^T q(x) = \partial^T S(x) b + \partial^T G(x) c \quad (10)$$

where ∂^T is the matrix differential operator

$$\partial^T = \left(\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \right).$$

The elements of $G(x)$ and $S(x)$ are usually taken to be polynomials of low degree. We have then distinguished those which generate heat flows in the absence of heat sinks, because

$$\partial^T S(x) = 0 \quad (11)$$

and those which correspond to certain modal distributions of heat sinks within the element

$$\bar{Q}(x) = \partial^T G(x) c = m^T(x) c \quad (12)$$

3.2. GENERALIZED HEAT FLUXES

The boundary of the finite element consists of several parts $\partial_\sigma E$ ($\sigma = 1, \dots, s$). Assumption (9) generates along part $\partial_\sigma E$ of the boundary a set of independent heat flux modes to each of which a generalized heat flux coordinate is attached.

The set of generalized flux coordinates for $\partial_\sigma E$ constitutes a column matrix g_σ and the following identity must hold for arbitrary b and c

$$n_\sigma^T S(x) b + n_\sigma^T G(x) c = f_\sigma^T(x) g_\sigma \quad x \in \partial_\sigma E \quad (13)$$

where $f_\sigma^T = [f_1(x) \dots f_{k_\sigma}(x)] \quad x \in \partial_\sigma E$

$$n_\sigma^T = [n_1(x) \quad n_2(x) \quad n_3(x)] \quad x \in \partial_\sigma E$$

are respectively the set of k_σ independent heat flux modes and the outward normal components along $\partial_\sigma E$.

In fact definition (13) of a generalized heat flux as intensity factor of a heat flux mode produces its dependence on the values of the parameters b and c ; in other words we are provided with a relationship of the form

$$g_\sigma = A_\sigma b + B_\sigma c \quad (14)$$

and identity (13) corresponds to

$$n_\sigma^T S(x) = f_\sigma^T A_\sigma \quad n_\sigma^T G(x) = f_\sigma^T B \quad x \in \partial_\sigma E \quad (15)$$

The knowledge of g_σ determines the complete heat flux distribution along $\partial_\sigma E$ so that continuity of heat fluxes at interfaces can be secured simply by reciprocity of corresponding generalized flux coordinates.

Any predetermined set of boundary heat flux modes may be replaced by a new set of k_σ linearly independent combinations of the elements of the old set. Advantage is taken of this freedom to give simple physical significance to the generalized flux coordinates, either as local heat flux values or moments of various order.

The set of relations (14) for all parts of the boundary ∂E results in a global relation

$$g_b = A b + B c \quad (16)$$

where the boundary flux coordinates g_σ are listed in g_b in some conventional order and A and B correspondingly partitioned from the $(k_\sigma \times n)$ and $(k_\sigma \times t)$ matrices A_σ and B_σ .

3.3. GENERALIZED TEMPERATURES

The heat sink potential part of functional (1) is next calculated by assuming the external temperature T_e to be defined along the whole boundary ∂E ; it then becomes formally

$$\begin{aligned} P &= \sum_{\sigma} \int_{\partial_{\sigma} E} T_e n_{\sigma}^T q(x) d\partial E \\ &= \sum_{\sigma} g_{\sigma}^T \int_{\partial_{\sigma} E} T_e f_{\sigma}(x) d\partial E \end{aligned} \quad x \in \partial_{\sigma} E \quad (17)$$

We now define generalized temperature coordinates t_σ along each boundary by

$$P = \sum_{\sigma} g_{\sigma}^T t_{\sigma} \quad x \in \partial_{\sigma} E \quad (18)$$

and consequently the generalized temperatures sequence along $\partial_\sigma E$ collects, in the same order as in g_σ , linear functionals of the unknown temperature such as

$$t_j = \int_{\partial_\sigma E} T_e f_j(x) d\partial E \quad x \in \partial_\sigma E \quad (19)$$

$$j = 1, \dots, k_\sigma$$

where the boundary heat flux modes play the role of weighting functions. Again if the coordinates t_σ of each boundary are collected in a single column matrix t_b , in the same order as the g_σ 's in g_b , the heat sink potential (18) takes the form

$$P = g_b^T t_b \quad (20)$$

which summarizes in a concise way the conjugate character of generalized fluxes and generalized temperatures.

Besides, if internal heat sink modes (12) are considered, conjugate generalized temperatures can be defined by analyzing the remaining heat sink potential

$$\int_E T Q dE = \left[\int_E T m^T(x) dE \right] c = t_1^T c \quad (21)$$

To each internal heat sink coordinate in c a conjugate generalized temperature in t_1 is thus attached, such that

$$t_1 = \int_E T m(x) dE \quad (22)$$

Our knowledge of the temperature field will remain the "weak" one provided by the values taken by the linear functionals (19) and (22). This is expected since the discretized temperature gradient field (5)

$$e(x) = -C^{-1} q(x) \quad C^{-1} = (r_{ij})$$

will in general not be integrable.

3.4. GENERALIZED CONDUCTIVITY RELATIONS

As suggested hereupon we wish to extend the principle of variations of heat flow to the situation where internal heat sink distribution instead of being fixed, is variable and depends on adjustable parameters c .

Their variations are then constrained by the heat balance equation

$$\partial^T \delta q(x) + m^T(x) \delta c = 0 \quad (23)$$

This condition is removed by a Lagrangian multiplier λ an added to the variation of the internal part of the dissipation functional

$$\delta F_i + \int_E [e^T \delta q(x) + \lambda \partial^T \delta q(x) + \lambda m^T(x) \delta c] dE = 0$$

Partial integration on the second term gives

$$\begin{aligned} \delta F_i + \int_E [(e^T - \partial^T \lambda) \delta q(x) + \lambda m^T(x) \delta c] dE \\ + \int_{\partial E} \lambda n^T \delta q(x) d\partial E = 0 \end{aligned}$$

and introduces in the principle of variation of heat flow the additional term

$$\left[\int_E T m^T(x) dE \right] \delta c = t_i^T \delta c \quad (24)$$

if λ is taken as the integrated gradient field e and $\partial E \leftarrow S_1$.

This is now applied to discretization (9). The internal dissipation functional is a quadratic positive definite form

$$\begin{aligned} F_i = \frac{1}{2} \int_E q^T(x) C^{-1} q(x) dE = \frac{1}{2} b^T R_{bb}^1 b + b^T R_{bc}^1 c \\ + \frac{1}{2} c^T R_{cc}^1 c \end{aligned} \quad (25)$$

with generalized resistivity matrices

$$R_{bb}^1 = \int_E S^T C^{-1} S \, dE$$

$$R_{bc}^1 = \int_E S^T C^{-1} G \, dE \quad (26)$$

$$R_{cc}^1 = \int_E G^T C^{-1} G \, dE$$

The boundary part of the dissipation functional leads also to a quadratic form through the heat flux

$$n_\sigma^T(x) q(x) = b^T A_\sigma^T f_\sigma + c^T B_\sigma^T f_\sigma \quad (27)$$

as

$$F_b = \frac{1}{2} \sum_\sigma \frac{1}{h} \int_{\partial_\sigma E} (n_\sigma^T q)^2 \, d\partial E$$

$$= \frac{1}{2} b^T R_{bb}^2 b + b^T R_{bc}^2 c + \frac{1}{2} c^T R_{cc}^2 c \quad (28)$$

with additional generalized resistivity matrices

$$R_{bb}^2 = \sum_\sigma A_\sigma^T H_\sigma A_\sigma$$

$$R_{bc}^2 = \sum_\sigma A_\sigma^T H_\sigma B_\sigma \quad (29)$$

$$R_{cc}^2 = \sum_\sigma B_\sigma^T H_\sigma B_\sigma$$

where

$$H_\sigma = \frac{1}{h} \int_{\partial_\sigma E} f_\sigma f_\sigma^T \, d\partial E \quad (30)$$

We recall here that advantage is taken of the subdivision of the structure in elements to incorporate eventual heat convection losses at interfaces and not only along the external boundary ∂D .

The global dissipation functional reads now

$$\frac{1}{2} b^T R_{bb} b + b^T R_{bc} c + \frac{1}{2} c^T R_{cc} c \quad (31)$$

if the resistivity matrices collect the separate ones as for instance

$$R_{bb} = R_{bb}^1 + R_{bb}^2 \quad (32)$$

The global heat sink potential yields from (20) and (21)

$$t_b^T g_b + t_i^T c = t_b^T A b + (t_b^T B + t_i^T) c$$

Substitution of this and (31) into principle of variation of heat flow and identification of the coefficients of the variations δb and δc produces

$$\begin{aligned} R_{bb} b + R_{bc} c &= -A^T t_b \\ R_{cb} b + R_{cc} c &= -B^T t_b - t_i \end{aligned} \quad (33)$$

The first equation may be solved for b and substituted into relation (16) as long as the parameters b are independent and each of them represent a non zero contribution to the dissipation functional; this furnishes

$$g_b + (A R_{bb}^{-1} R_{bc} - B) c = -A R_{bb}^{-1} A^T t_b \quad (34)$$

which is a conductivity relation completely analogous to that of a temperature-type element. The internal heat sink distribution is supposedly given, i.e. c is known, and the way it acts on the discretized behavior of the element is through internal generalized fluxes given by

$$g_i = (A R_{bb}^{-1} R_{bc} - B) c \quad (35)$$

The generalized conductivity matrix

$$K_b = -A R_{bb}^{-1} A^T \quad (36)$$

involves only part A of the heat flux connection matrix.

If an estimation of the generalized temperatures t_i conjugate to the generalized internal heat fluxes is desired, we invoke the second relation of (33) to produce

$$t_i = (R_{cb} R_{bb}^{-1} A^T - B^T) t_b + (R_{cb} R_{bb}^{-1} R_{bc} - R_{cc}) c \quad (37)$$

In summary the discretized thermal relations of the element are

$$K_b t_b = g_b + g_i \quad (38)$$

The way to collect elements together is then the same as for the temperature models (1).

3.5. REMARKS ON HEAT FLOW FINITE ELEMENTS

Generalized conductivity matrices are always singular because the homogeneous system

$$K_b t_b = 0 \quad (39)$$

has amongst its non trivial solutions at least the uniform reference temperature of the element. These solutions are identical to those of

$$A^T t_b = 0 \quad (40)$$

since R_{bb}^{-1} is positive definite, and let us show that an uniform temperature field T_0 is solution of (40). Indeed the corresponding boundary and internal generalized temperatures are obtained directly by inserting this field into the definitions (19) and (22). Then, in view of the balance equation satisfied by any heat flow field

$$\partial^T q(x) + m^T(x) c = 0 \quad (41)$$

integrating over the finite element domain E with T_0 as weighting function, we obtain

$$\begin{aligned} & - \int_E q^T(x) \partial T_0 \, dE + \int_{\partial E} T_0 n^T(x) q(x) \, d\partial E \\ & + \left(\int_E T_0 m^T(x) \, dE \right) c = 0 \end{aligned}$$

where an integration by parts has been performed on the first term of (41). Now the reference temperature field produces no temperature gradient field and expressions (20) and (21) of the associated heat sink potential furnish

$$g_b^T t_b^o + c^T t_i^o = 0 \quad (42)$$

which expresses the vanishing of the total outgoing heat flux and is only normal because of the detailed equilibrium equations (3) and (4) : it is the analogue of the overall equilibrium conditions between generalized forces in elasticity.

Substituting decomposition (16) of the g_b 's and noting that the relation holds for arbitrary b and c , there follows

$$A^T t_b^o = 0 \quad (43)$$

$$B^T t_b^o + t_i^o = 0 \quad (44)$$

This completes the proof for the uniform reference temperature field. Usually there are no other non zero solutions to (43) and the element is a "sound" one. If the case may be, an internal temperature field is taken from (44)

$$t_i = - B^T t_b$$

and relations (33) lead to the vanishing of b and c . In other words a temperature field exists that produces no heat flow. This is however not the case for the family of 2-D triangles or 3-D tetrahedra elements provided the approximate functions in (10) are a complete set of polynomials.

4. EXAMPLES AND NUMERICAL EFFECTIVENESS OF HEAT FLOW ELEMENTS

The general method to derive heat flow finite elements has been successfully applied ^(2,3) to several degrees of polynomial approximation for the heat flow. The simplest 2-D elements will be briefly described and attention focused on an example with internal heat sink distribution.

A numerical example shows that the generalized temperatures associated with heat flow models give a good idea of local distribution even in case of non integrable gradient fields.

4.1. CONSTANT HEAT FLOW MODEL (fig.1A)

The simplest case when no internal heat sinks are considered and when the polynomial degree for the heat flow is the lowest is very easy to handle. Heat flux at interface is constant and a single generalized heat flux is sufficient to insure perfect transmission to the adjacent element. Consequently there is only one generalized temperature by interface, which is the mean of the temperature distribution along the interface and is located at mid-edge.

4.2. LINEAR HEAT FLOW MODEL (fig.1B)

The heat flow components are written in a form implementing 'a priori' the homogeneous heat balance equation and depend so far on five independent parameters. This time the heat flux at interfaces is linearly varying and requests two generalized coordinates to enable identification of the whole heat flux between adjacent elements. The conjugate generalized temperatures are once more weighting averages of the temperature distribution along the interface located to each third of the edge.

4.3. HEAT FLOW MODEL WITH INTERNAL HEAT SINKS

We will derive now a general but still simple model according to the standard procedure developed in section 3. It is a 2D-triangle (fig. 1.c) similar to the preceding ones but we want to take into account a possible internal heat sink distribution.

The following discretization is assumed

$$\begin{aligned} q_x &= a_1 - \frac{Q}{2} x \\ q_y &= a_2 - \frac{Q}{2} y \end{aligned} \tag{45}$$

which can be identified to (9) provided

$$\begin{aligned}
 q^T &= (q_x \ q_y) ; \quad b^T = (\alpha_1 \ \alpha_2) ; \quad c = -\frac{Q}{2} \\
 S &= E ; \quad G^T = (x \ y) ; \quad m^T = (1 \ 1)
 \end{aligned}
 \tag{46}$$

The generalized fluxes are next calculated; assumption (45) generates only a constant heat flux distribution along the sides of the triangle as can be easily verified from (13); in this case $k_\sigma = 1$ and $f_\sigma = \frac{1}{L_{ij}}$ and relation (16) becomes

$$\begin{vmatrix} s_1 \\ s_2 \\ s_3 \end{vmatrix} = \begin{vmatrix} y_{21} & x_{12} \\ y_{32} & x_{23} \\ y_{13} & x_{31} \end{vmatrix} b = \begin{vmatrix} x_1 y_{21} + y_1 x_{12} \\ x_2 y_{32} + y_2 x_{23} \\ x_3 y_{13} + y_3 x_{31} \end{vmatrix} \frac{Q}{2}$$

where $x_{ij} = x_i - x_j$ $y_{ij} = y_i - y_j$ $i, j =$ vertices of the triangle

The conjugate generalized boundary temperature are then the means

$$\begin{aligned}
 t_1 &= \frac{1}{L_{12}} \int_{\partial_\sigma E=12} T_e \, d\partial E ; \quad t_2 = \frac{1}{L_{23}} \int_{\partial_\sigma E=23} T_e \, d\partial E ; \\
 t_3 &= \frac{1}{L_{31}} \int_{\partial_\sigma E=31} T_e \, d\partial E
 \end{aligned}$$

as can be achieved from (19) and are identical to those of the element described in 4.1. The additional heat sink coordinate c introduces a fourth generalized temperature which is following (22)

$$t_4 = \int_E T \, dE$$

We possess at this point all the elements to perform the detailed resistivity matrices of the element as from (26) and definitions (46)

$$R_{bb}^1 = E \begin{vmatrix} \frac{1}{k_x} & 0 \\ 0 & \frac{1}{k_y} \end{vmatrix} ; \quad R_{bc}^1 = \begin{vmatrix} \frac{S_x}{k_x} \\ \frac{S_y}{k_y} \end{vmatrix} ; \quad R_{cc}^1 = \left(\frac{I_x}{k_x} + \frac{I_y}{k_y} \right)$$

where k_x , k_y are the orthotropic conductivity coefficients and

$$\begin{aligned} E &= \int_E dE & S_x &= \int_E x dE & S_y &= \int_E y dE \\ I_x &= \int_E x^2 dE & I_y &= \int_E y^2 dE \end{aligned}$$

The conductivity relation is then detailed following (34), (35), (36).

4.4. NUMERICAL EXAMPLE

Figure 2A illustrates a conduction-cooled turbine blade having its root maintained at a uniform temperature T_0 and operating in an effective gas with temperature T_f . The tip is considered to be insulated, the body exchanges heat with the gas by convection and the disk acts as a heat sink to cool the blade.

A finite element analysis (fig. 2B) has been performed on the basis of a 24 triangular mesh of successively constant and linear heat flow models. The temperature plotting shows a remarkable accuracy for the corresponding generalized temperatures versus the unidimensional analytical solution. This attempts to demonstrate that the abstract concept of generalized temperature gives a good picture of the detailed local field even in case of non integrable gradient fields, provided some physical sense guides the choice and interpretation of those generalized quantities.

5. CONSTRUCTION OF HEAT FLOW MODELS BY MEANS OF HEAT STREAM FUNCTIONS

Recourse to the concept of weak generalized temperatures to assemble heat flow models may be responsible for some lack of understanding in the real usefulness and value of the model. As a matter of fact, if informations on temperature distributions are desired, it is the most natural way to develop the heat flow element. However, if only heat flow information is requested, temperature concepts can seemingly be discarded by using heat stream functions.

The introduction of heat stream functions allows automatic satisfaction of the heat flow balance in the absence of volume heat sinks and this will be assumed to be the nature of the problem. Two cases will be considered :

- two-dimensional flow where a scalar stream function is sufficient to express balance;
- three-dimensional flow where a vector-type of stream function is necessary.

5.1. TWO-DIMENSIONAL HEAT FLOW MODELS BASED ON A STREAM FUNCTION

The homogeneous form of the balance equation

$$\partial_j q_j = 0 \quad (47)$$

is satisfied by using a stream function ψ such that

$$q_1 = \partial_2 \psi \quad q_2 = -\partial_1 \psi \quad (48)$$

or
$$q_j = \epsilon_{jm} \partial_m \psi \quad (49)$$

where ϵ_{jm} is the two-dimensional alternator symbol.

The flux through a boundary of element described in anti-clockwise sense (fig. 3.A) is then given by

$$n_j q_j = \epsilon_{jm} n_j \partial_m \psi = n_1 \partial_2 \psi - n_2 \partial_1 \psi = \frac{\partial \psi}{\partial s} \quad (50)$$

We know that in the heat flow variational principle, q_j must not only satisfy the homogeneous balance equation (47), where we assumed restriction to $\bar{Q} = 0$, but also

$$(n_j q_j)_+ + (n_j q_j)_- = \bar{q}_0 \quad \text{on } B_2$$

or $n_j q_j = \bar{q}_0$

For simplicity we take $\bar{q}_0 = 0$ at all interfaces, consequently discarding all artificial impositions of temperatures \bar{T}_0 at interfaces.

Thus we consider only an boundary and interface conditions

$$(n_j q_j)_+ + (n_j q_j)_- = 0 \quad \text{on } \Lambda \text{ set of all interfaces} \quad (51)$$

$$n_j q_j = \bar{q}_0 \quad \text{on } \partial_2 D \quad (52)$$

$$T_0 = \bar{T}_0 \quad \text{on } \partial_1 D \quad (53)$$

Considering (50), (51) can be rewritten in the form

$$\left(\frac{\partial \psi}{\partial n}\right)_+ + \left(\frac{\partial \psi}{\partial n}\right)_- = 0 \quad \text{on } \Lambda \quad (54)$$

Noting that ds_+ and ds_- are described in opposite senses, this may be satisfied by simple continuity of the stream function ψ across the interfaces. The condition, if sufficient, is however not quite necessary since (54) obviously allows a constant difference between the two face values of ψ . This jump does in fact occur for multiply-connected domains D , as is well known and will be briefly recalled.

Take first a simply connected domain, like that of each finite element itself. Then we have everywhere inside satisfaction of (47) and

$$\int_D n_j q_j dD = 0$$

This is, by partial integration, equivalent to

$$\int_{\partial D} n_j q_j ds = \int_{\partial D} \frac{\partial \psi}{\partial s} ds = \int_{\partial D} d\psi = 0$$

Thus ψ is single valued throughout the domain.

Consider now a domain with an internal cavity (fig. 3.B) of contour c_0 . Here

$$\int_{c_0} d\psi = \int_{c_0} n_j q_j ds = g_0 \quad (55)$$

where g_0 is the total heat flux penetrating into the cavity, a quantity that needs not be necessarily zero. Thus ψ has a cyclic constant when the contour of the cavity is described. The same cyclic constant is valid for any closed contour c , described in the domain and circling once around this cavity. For, if we join c_0 and c by a two-faced cut 1234, the integral

$$\int_{12341} d\psi = \int_c d\psi + \int_2^3 d\psi - \int_{c_0} d\psi + \int_4^1 d\psi = 0$$

because the inner domain of the contour 12341 is simply connected. However, as there is reciprocity of heat flux across the cut, the second and last integrals cancel, leaving

$$\int_c d\psi = \int_{c_0} d\psi = g_0 \quad (56)$$

For this reason it will be necessary in the discretization process by finite elements to isolate a sequence of interfaces providing a barrier extending from the inner cavity to the external contour, across which a constant jump in the value of ψ must be left open.

The generalization to several internal cavities is immediate.

In terms of the heat stream function the heat flow variational principle (1) reads

$$\frac{1}{2} \int_D r_{ij} c_{im} c_{jn} \partial_m \psi \partial_n \psi dD + \frac{1}{2} \int_S \frac{1}{h} \left(\frac{\partial \psi}{\partial s} \right)^2 ds + \int_{\partial_1 D} \frac{\partial \psi}{\partial s} \bar{T}_e ds \text{ minimum} \quad (57)$$

The continuity of ψ holds on all interfaces except the barriers

$b_1 \dots b_m$ whose separations make D simply connected. The only "a-priori" condition remaining on ψ is (52) in its new form

$$\frac{\partial \psi}{\partial s} = \bar{q}_s \quad \text{on } \partial_2 D \quad (58)$$

Before discretizing let us examine the Euler equations and natural boundary conditions of this new principle.

Variation of the first integral

$$\frac{1}{2} \int_D r_{ij} \epsilon_{im} \epsilon_{jn} (\partial_m \psi \partial_n \delta \psi + \partial_n \psi \partial_m \delta \psi) dD$$

is first reduced to

$$\int_D (r_{ij} \epsilon_{im} \epsilon_{jn} \partial_n \psi \partial_m \delta \psi) dD$$

by exchanging in the first term the subscripts m and n and also i and j and noting that $r_{ji} = r_{ij}$. In view of

$$e_i = -r_{ij} q_j = -r_{ij} \epsilon_{jn} \partial_n \psi$$

this is also

$$- \int_D \epsilon_{im} e_i \partial_m \delta \psi dD = - \int_S \epsilon_{im} n_m e_i \delta \psi dS + \int_D \epsilon_{im} \partial_m e_i \delta \psi dD \quad (59)$$

The second term in (59) will be set separately equal to zero, since it is the only one containing the arbitrary variation $\delta \psi$ within each finite element. The Euler equation is thus

$$\epsilon_{im} \partial_m e_i = \partial_2 e_1 - \partial_1 e_2 = 0 \quad (60)$$

It is the integrability condition for the temperature field; inside each element then this temperature field for which

$$e_i = \partial_1 T \quad (61)$$

will be determined except for an additive constant. Considering further that

$$\epsilon_{im} n_m e_i = \epsilon_{im} n_m \partial_1 T = n_2 \partial_1 T - n_1 \partial_2 T = -\frac{\partial T}{\partial s}$$

the remaining contribution of (59) will be

$$\int_S \frac{\partial T}{\partial s} \delta \psi \, dS$$

and adding to it the variations of the second and third terms in (57)

$$\int_S \left(\frac{\partial T}{\partial s} \delta \psi + \frac{1}{h} \frac{\partial \psi}{\partial s} \delta \frac{\partial \psi}{\partial s} \right) ds + \int_{\partial_1 D} \bar{T}_e \delta \frac{\partial \psi}{\partial s} ds = 0$$

The integrations by parts on S are obtained by circulating anti-clockwise around each finite element, which covers all the faces of S . For each element we can replace

$$\oint \frac{\partial T}{\partial s} \delta \psi \, ds = - \oint T \delta \frac{\partial \psi}{\partial s} \, ds$$

and obtain

$$\int_S \left(\frac{1}{h} \frac{\partial \psi}{\partial s} - T \right) \delta \frac{\partial \psi}{\partial s} \, ds + \int_{\partial_1 D} \bar{T}_e \delta \frac{\partial \psi}{\partial s} \, ds = 0 \quad (61)$$

On each interface

$$\left(\frac{\partial \psi}{\partial s} \right)_+ = - \left(\frac{\partial \psi}{\partial s} \right)_-$$

and the same holds for their variations, hence

$$\left(-T + \frac{1}{h} \frac{\partial \psi}{\partial s} \right)_+ = \left(-T + \frac{1}{h} \frac{\partial \psi}{\partial s} \right)_- \quad \text{on interfaces} \quad (62)$$

which is equivalent to formula (7) of reference 1. The variation of $\partial \psi / \partial s$ along $\partial_1 D$ yields

$$T = \bar{T}_e \quad \text{on } \partial_1 D \quad (63)$$

Finally the variation of $\partial\psi/\partial s$ on $\partial_2 D$ vanishes by virtue of (58).

The discretization of the scalar ψ follows similar rules to that of T in the temperature models, identification of nodal ψ values implying complete continuity of ψ across interfaces and consequently heat flow diffusivity. The possible \bar{q}_e distributions along $\partial_2 D$ are of course limited by the shaping functions introduced to represent ψ . By contrast any distribution \bar{T}_e along $\partial_1 D$ will be translated into generalized temperatures, linear functionals of \bar{T}_e resulting from the discretization of the last integral in (57).

An example of translation of boundary requirements in terms of the heat stream function is given figure 3.C.

5.2. THREE-DIMENSIONAL HEAT FLOW MODELS BASED ON A VECTOR STREAM FUNCTION

The vector stream function \vec{a} such that

$$q_j = \epsilon_{jmn} \partial_m a_n \quad (64)$$

where ϵ_{jmn} is the three-dimensional alternating symbol, allows to satisfy "a priori" the homogeneous heat balance equation

$$\partial_j q_j = 0$$

(64) is equivalent to

$$\vec{q} = \text{rot } \vec{a}$$

and since $\text{rot grad } \phi \equiv 0$

there is an indeterminacy in \vec{a} constituted by the addition of the gradient of an arbitrary function.

It is easily seen that continuity of \vec{a} is sufficient for heat flux diffusivity across interfaces. If a face portion is parallel to axes 2 and 3, the heat flux density is $q_1 = \partial_2 a_3 - \partial_3 a_2$ and it depends only on derivatives in directions tangent to the face portion. More generally

$$n_j q_j = \epsilon_{jmn} n_j \partial_m a_n$$

depends on derivatives in face tangent directions, since $\epsilon_{jmn} n_j \partial_m$ is the vector product of the normal and the gradient operator.

Again in a domain without internal cavities \vec{a} may be taken to be continuous throughout. In the presence of internal cavities with non zero penetrating heat flux, surface discontinuities must be organized and the method loses much of its interest.

Figure 4 presents some of the simplest heat flow models with their two types of discretization; it can be easily verified that the heat stream function approach leads to finite elements topologically identical to the corresponding temperature ones for 2-D approximations. The case of 3-D models is slightly different : their topological analogues are the class of the simplest 3-D conforming displacement models of elasticity.

6. BOUNDS TO THE DISSIPATION FUNCTIONAL

The upper and lower bound character to the dissipation functional of approximations based on either the principle of variations of temperatures or the principle of variation of heat flow has been fully demonstrated in preceding developments (1,2).

To obtain guaranteed bounds it is necessary to split the general problem in two parts, one where the prescribed temperature of the effective heat sink distribution is set equal to zero (problem 1) and one where all the prescribed heat sink rates vanish (problem 2).

In the following two examples of type 2 will be briefly described.

The awaited bounds for this category are an upper value for temperature models and the reverse for heat flow models.

6.1. CONDUCTION-COOLED TURBINE BLADE

A dual analysis of the blade described in section 4.4 (fig. 2) has been performed and yields correct upper and lower bounds.

The mesh of figure 2 was used for each case of analysis and the different models together with their associated dissipation value are listed in the table hereafter.

FINITE ELEMENT MODEL	NB. DEGREES OF FREEDOM	$10^{-4} D$
Linear temperature field	23	20.91
Parabolic temperature field	67	20.53
Linear heat flow field	88	20.47
Constant heat flow field	44	19.55

6.2. LAMINATED SLAB (fig. 5)

A laminated slab with orthotropic conductivities k_1 and k_2 is analyzed. The lateral surface exchanges heat with an isothermal fluid bath at T_f except the lower edge which is partly adiabatic and partly in contact with an external heat source at T_o .

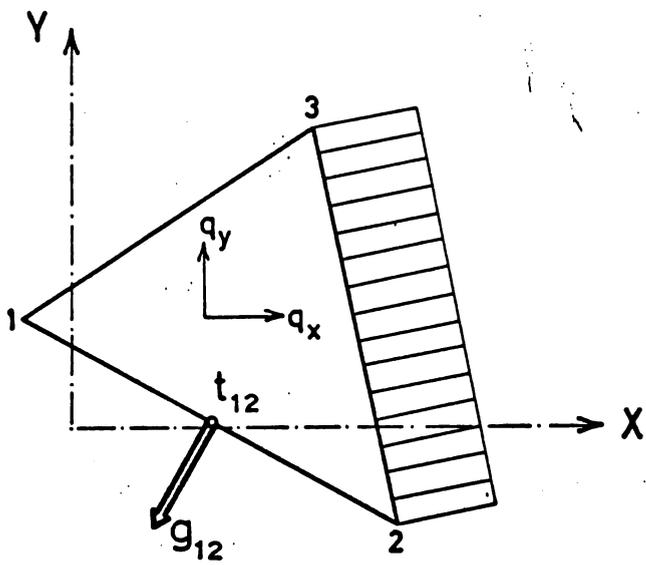
The detailed data are

$$\begin{array}{llll} k_1 = .8 & h_1 = .9 & T_o = 1,000 & a = 10 \\ \text{w/cm}^\circ\text{c} ; & \text{w/cm}^2 \cdot \text{c} ; & \cdot\text{c} ; & \text{cm.} \\ k_2 = .4 & h_2 = .5 & T_f = 100 & b = 6 \end{array}$$

Figure 5 presents the temperature maps for several models of approximations and the associated bounds are summarized figure 6. The cubic temperature model-treatment is nearly exact with respect to bounds bracketing. Heat flow models tend to converge faster than temperature ones but they need more degrees of freedom for a given grid an equal degrees in the polynomial discretization of the element since they involve interface unknowns solely.

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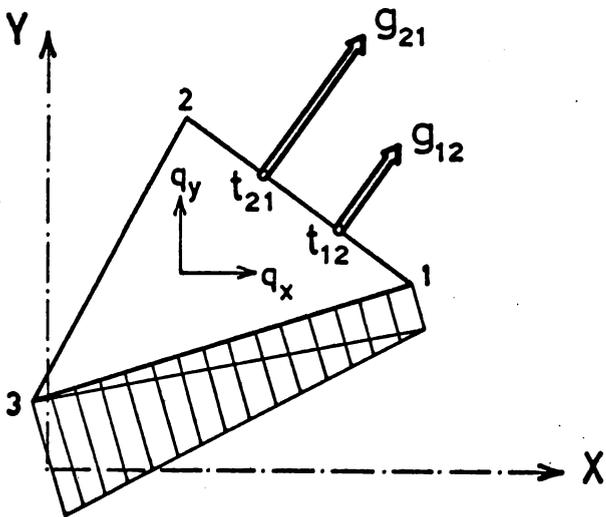


$$q_x = \alpha_1$$

$$q_y = \alpha_2$$

$$t_{ij} = \frac{1}{L_{ij}} \int_{ij} T_e ds$$

FIGURE 1a CONSTANT HEAT FLOW MODEL



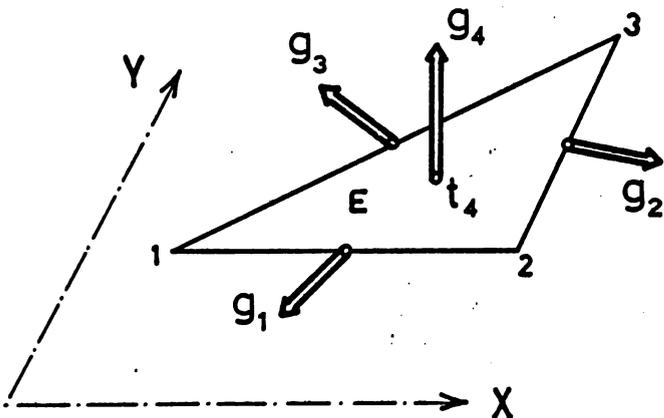
$$q_x = \alpha_1 + \alpha_2 X + \alpha_3 Y$$

$$q_y = \alpha_4 + \alpha_5 X - \alpha_2 Y$$

$$t_{ij} = \frac{2}{L_{ij}^2} \int_{ij} T_e (L_{ij} - s) ds$$

$$t_{ji} = \frac{2}{L_{ij}^2} \int_{ij} T_e s ds$$

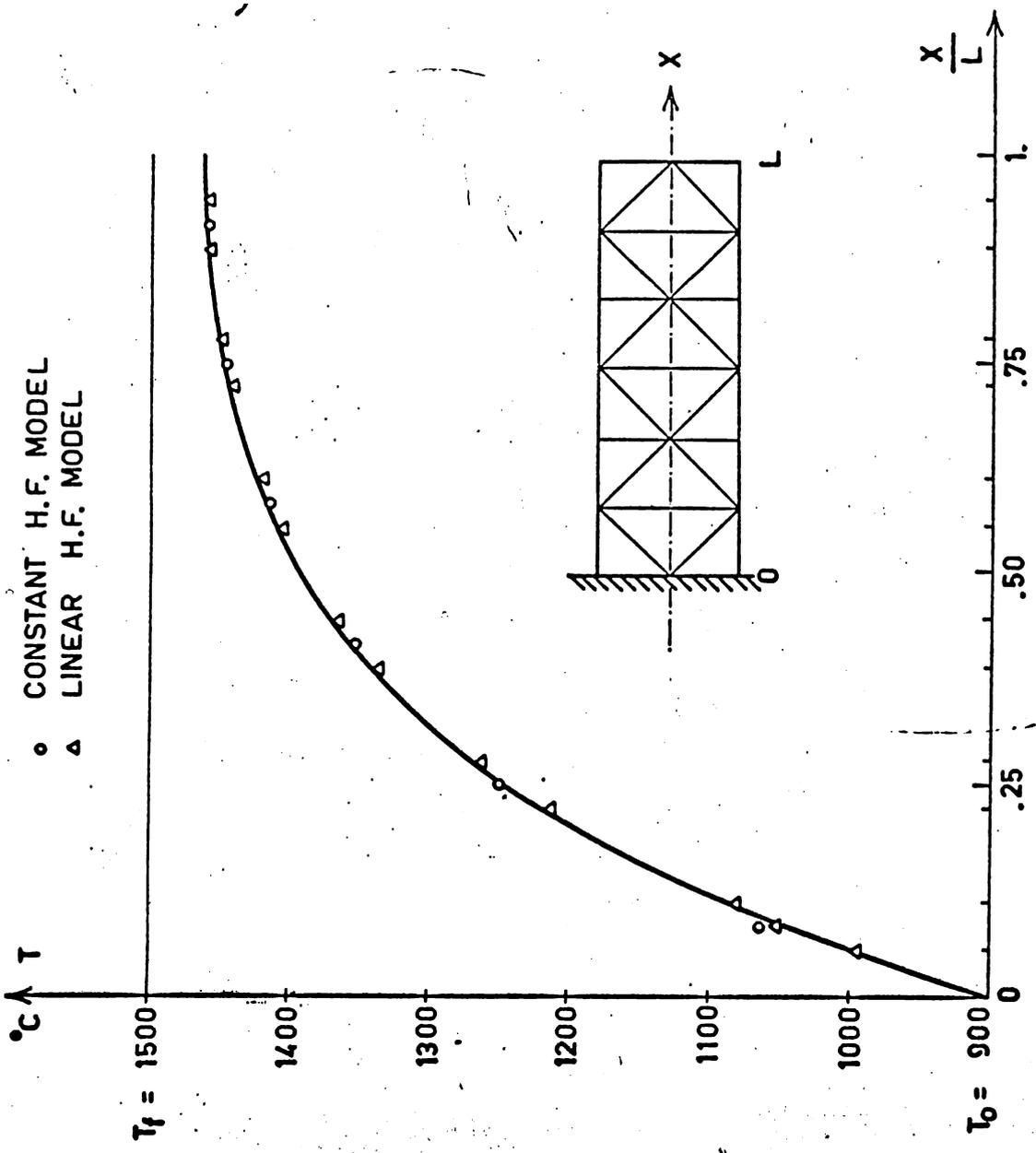
FIGURE 1b LINEAR HEAT FLOW MODEL



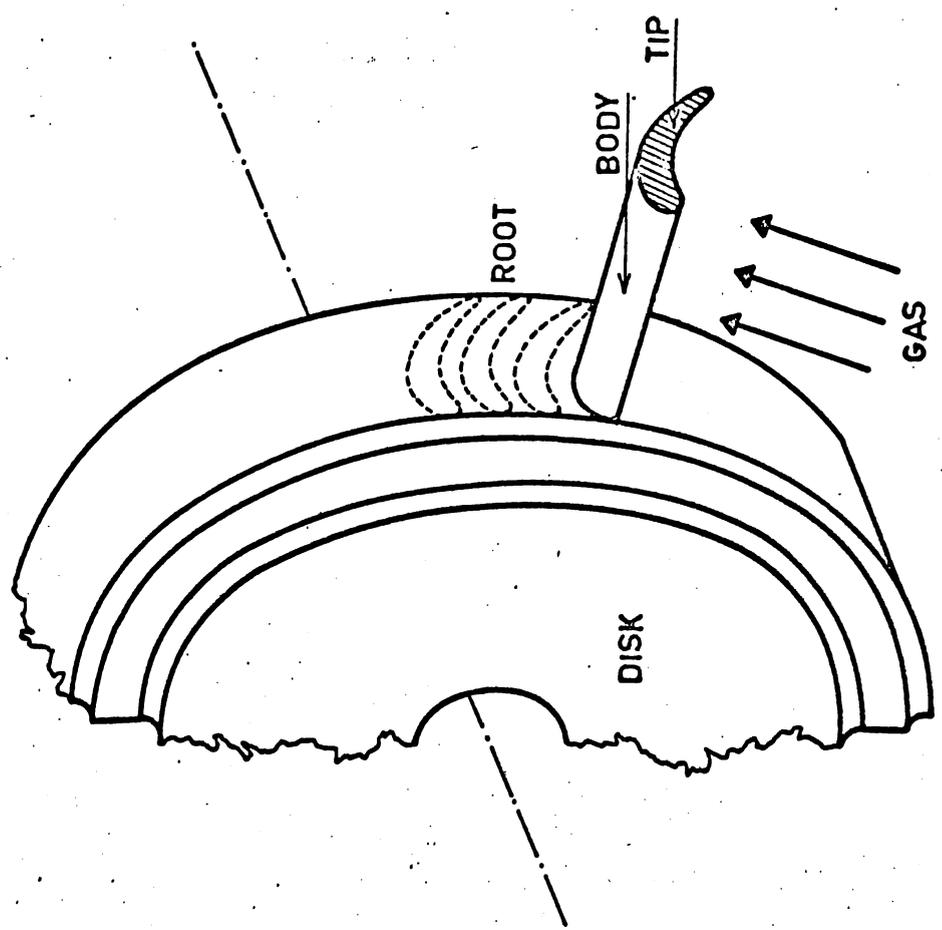
$$t_4 = \int_E T dE$$

FIGURE 1c CONSTANT HEAT FLOW MODEL

WITH INTERNAL HEAT GENERATION



(a)



(b)

FIG.2 CONDUCTION COOLED TURBINE BLADE ANALYSIS BY HEAT FLOW ELEMENTS

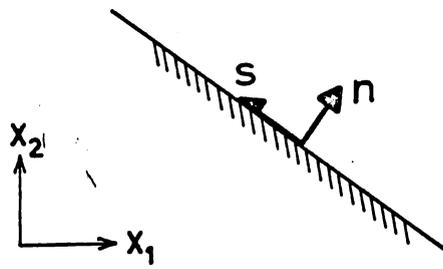


FIGURE 3.A

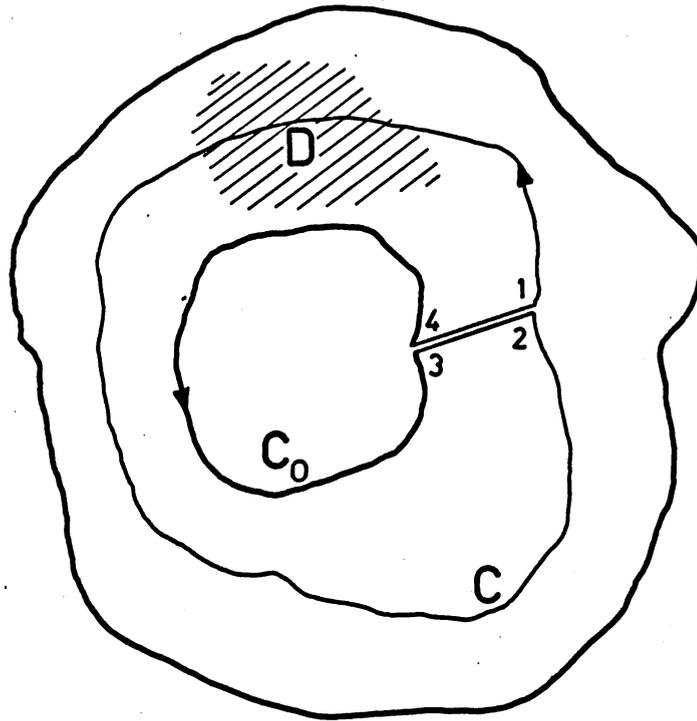


FIGURE 3.B

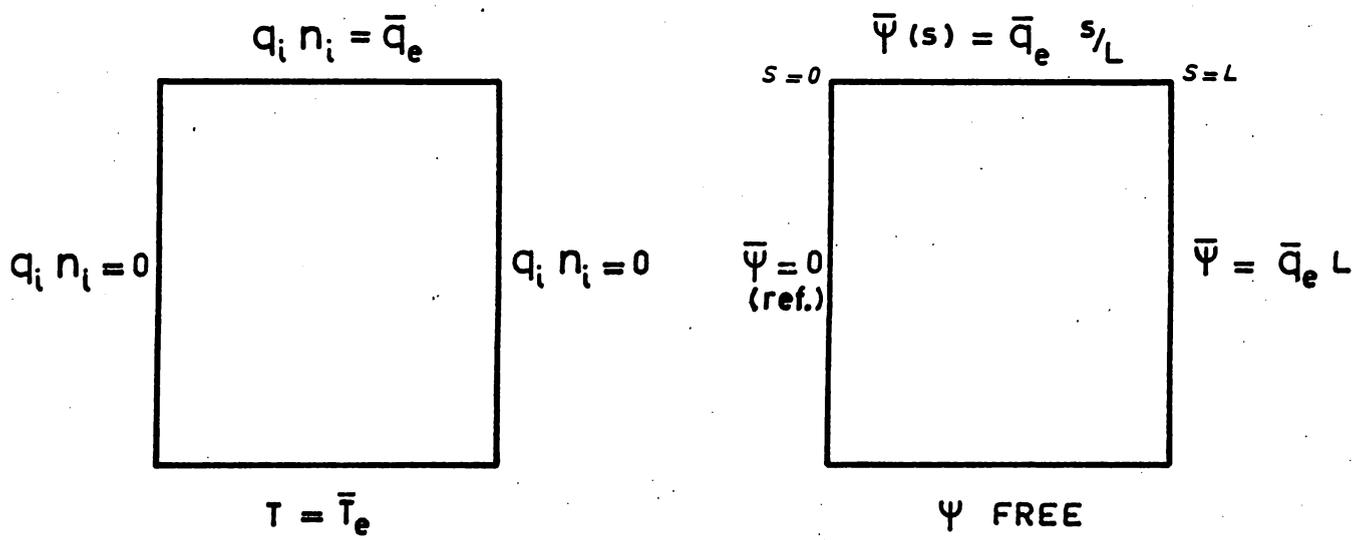
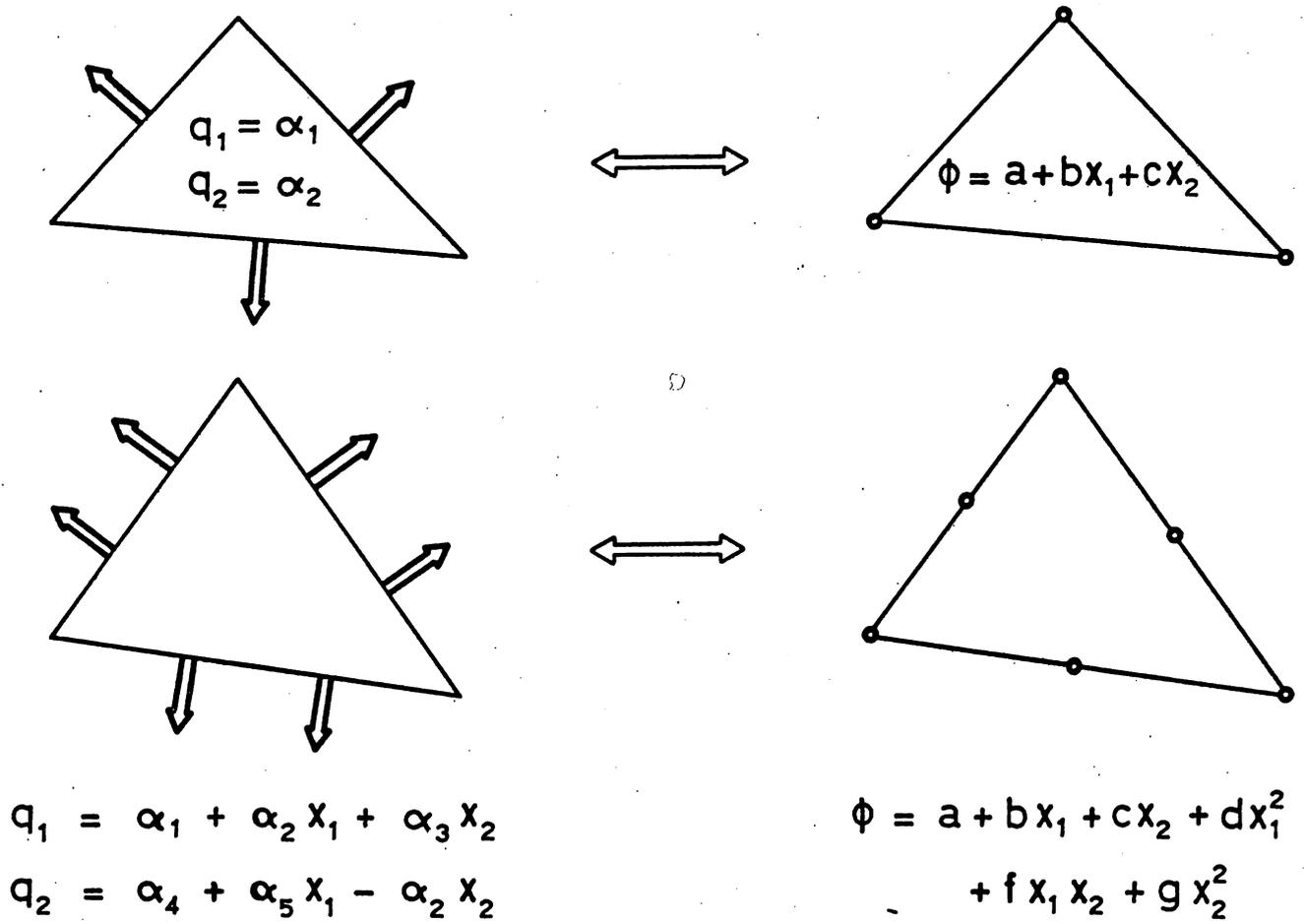


FIGURE 3.C

2 - D HEAT FLOW ELEMENTS



3 - D HEAT FLOW ELEMENT

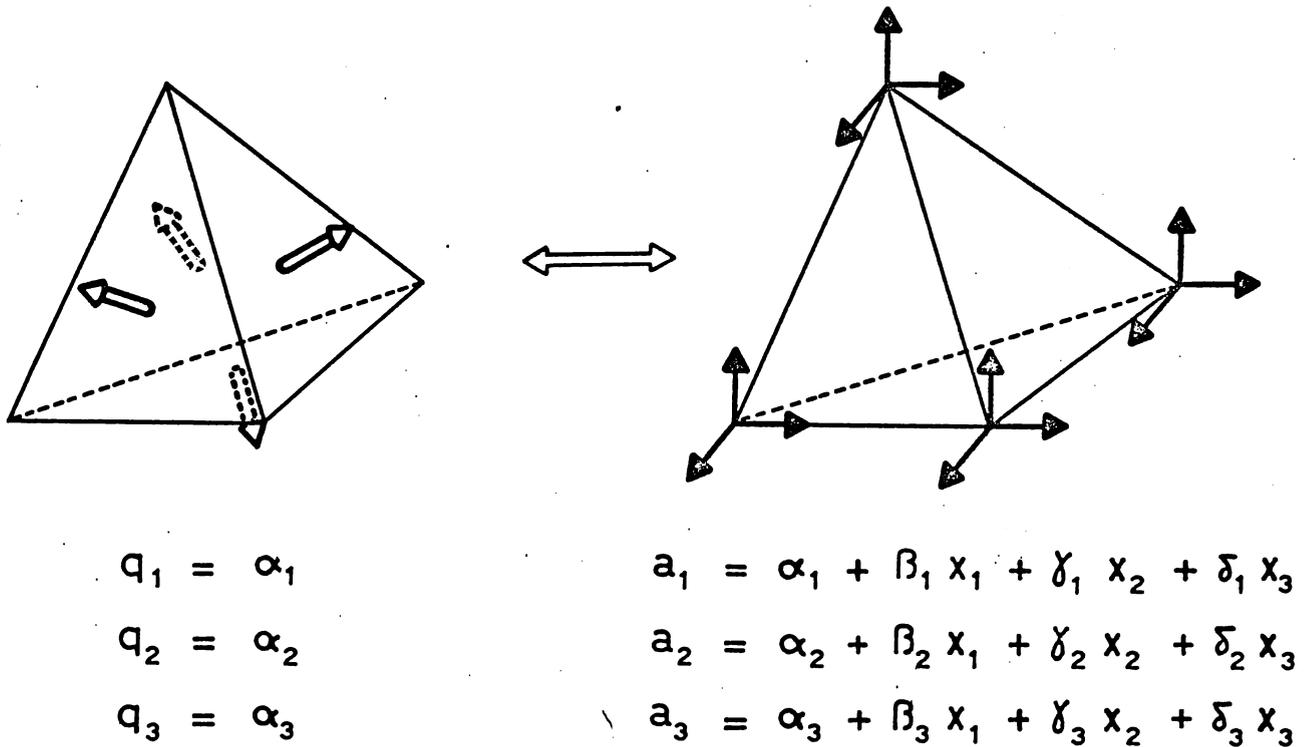
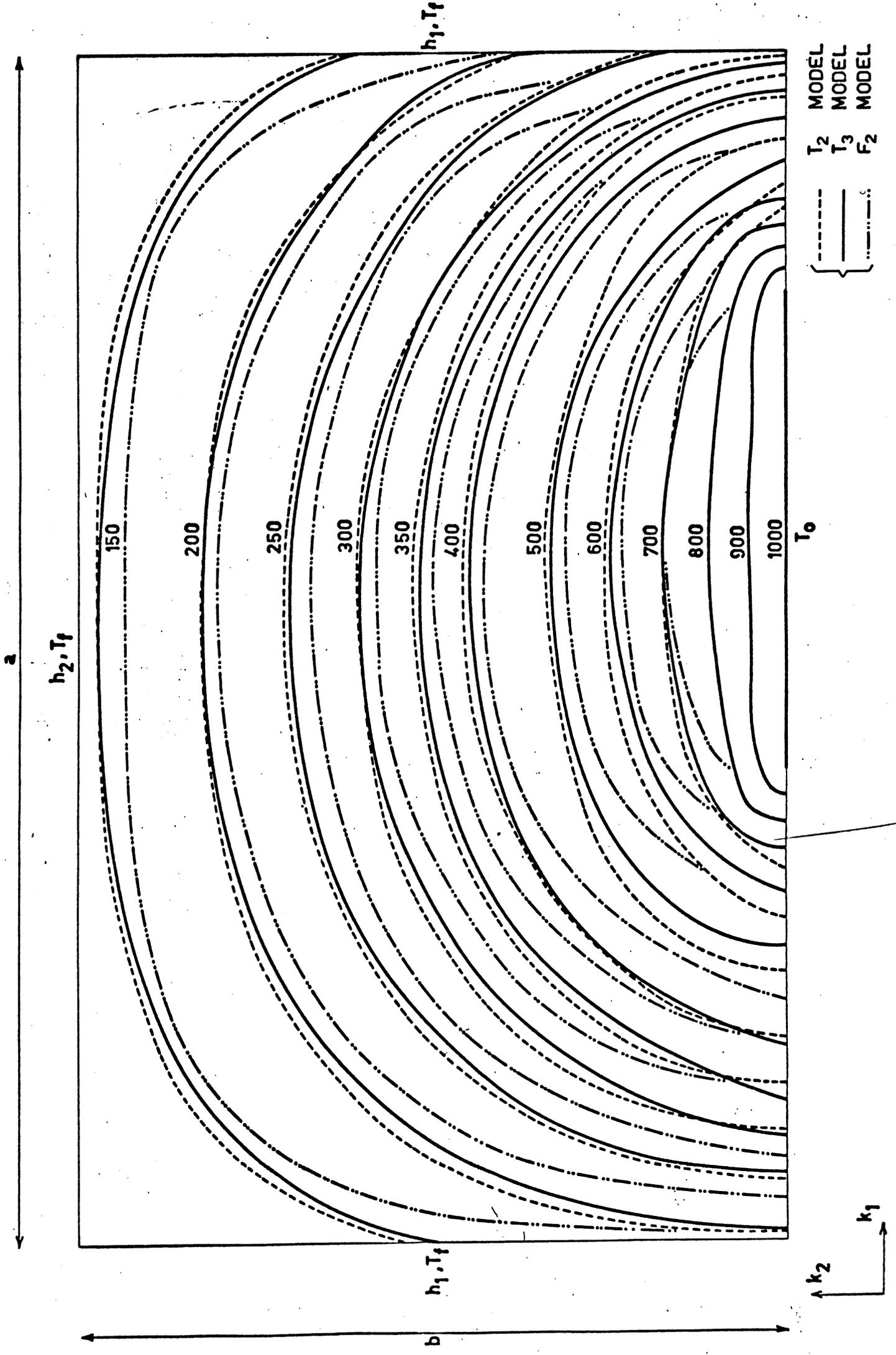


FIGURE 4 DISCRETIZATION OF HEAT FLOW MODELS



TEMPERATURE DISTRIBUTIONS FOR THE SLAB

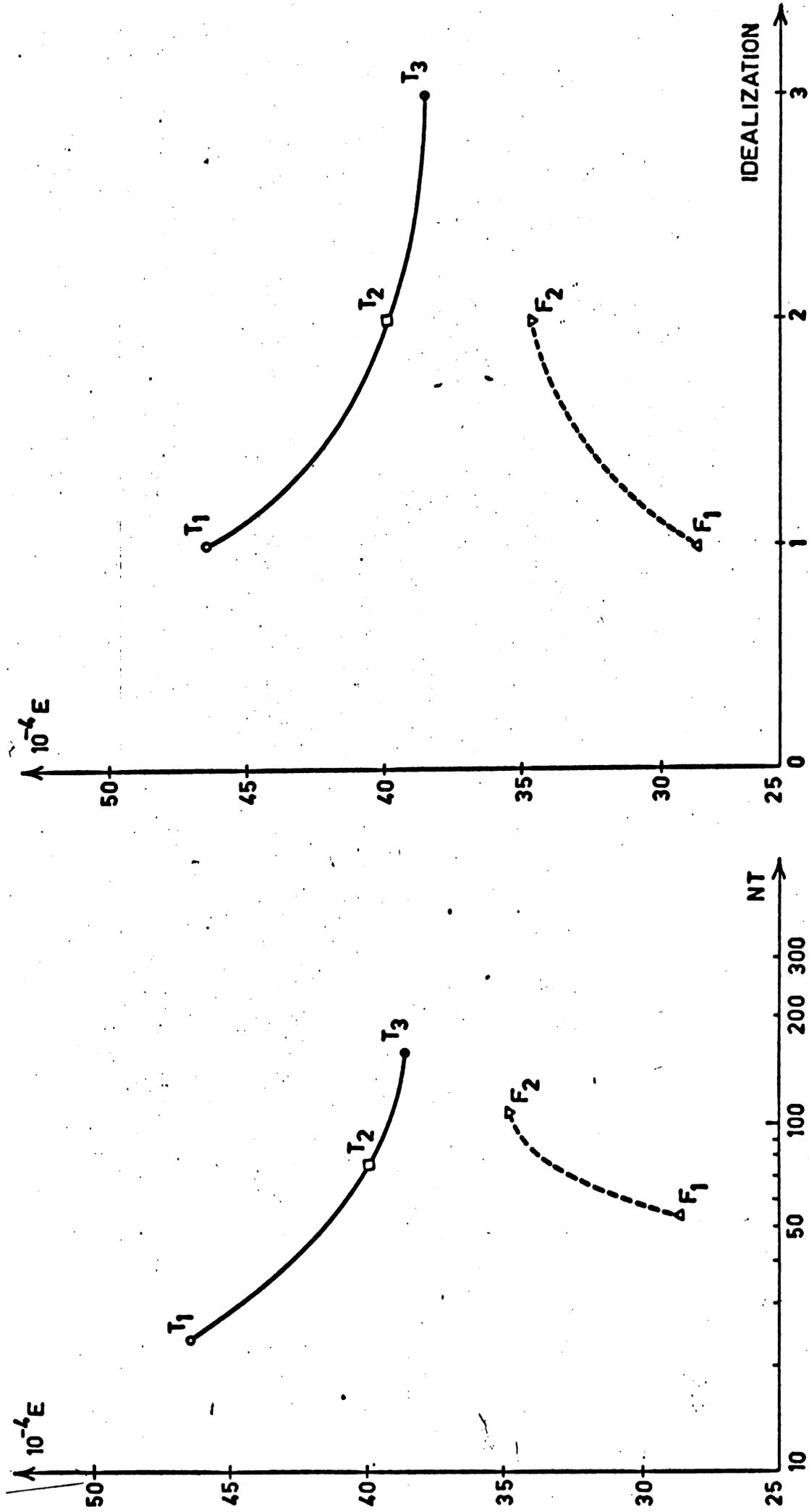


FIGURE 6 BOUNDS TO THE DISSIPATION FUNCTIONAL