Application of Analytic Solution in Relative Motion to Spacecraft Formation Flying in Elliptic Orbit

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Abstract

The current paper presents application of a new analytic solution in general relative motion to spacecraft formation flying in an elliptic orbit. The calculus of variations is used to analytically find optimal trajectories and controls for the given problem. The inverse of the fundamental matrix associated with the dynamic equations is not required for the solution in the current study. It is verified that the optimal thrust vector is a function of the fundamental matrix of the given state equations. The cost function and the state vector during the reconfiguration can be analytically obtained as well. The results predict the form of optimal solutions in advance without having to solve the problem. Numerical simulation shows the brevity and the accuracy of the general analytic solutions developed in the current paper.

1. Introduction

Satellite formation flying consists of a group of satellites working together to perform a unified space mission. Many future space missions will necessitate the use of formation flying technology for multiple satellites. Among the technologies associated with satellites in a formation, a fuel-optimal reconfiguration problem is one of important research fields. Most studies have utilized numerical methods (Park et al. 2008). Compared to analytic methods, numerical methods allow us to conduct more accurate and practical analyses. However, it is necessary for analytic solutions to be found because they provide insight into the feedback controller, and thus they are easily applied to formation flying, if they can be uncovered. For the actual on-board control system, it is preferable to have analytic solutions since they significantly reduce the computational loads. Palmer (2006) presented an elegant analytic solution for the problem by representing the continuous and variable thrust acceleration in a Fourier series with a period equal to the maneuver time. He used the Parseval's theorem (Boas 1983) to make the infinite sum into a closed form. However, this analytic solution is limited to formation flying in only a circular or near-circular orbit because the HCW equations are used. Using a similar approach, Cho et al. (2007) extended the previous result and obtained a solution to general-elliptic-orbit cases. In this study, the solution is still described in a Fourier series. Scott & Spencer (2007) chose the calculus of variations to obtain an analytic solution. They brought in the adjoint system to find an optimal thrust vector. This solution applies only

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to circular-reference-orbit case. Sharma et al. (2007) solved the problem for non-zero eccentricity, and also included the effects of nonlinear differential gravity. As shown by the results of the studies in these references, the solutions need the inverse of the fundamental matrix; consequently, it is very difficult to get an analytic solution. To overcome this difficulty, Cho & Park (2008) analytically solved the fuel-optimal reconfiguration problem in general relative motion using the calculus of variations approach. Unlike the previous studies, the analytic solution in the reference (Cho & Park 2008) does not require calculating the inverse of the fundamental matrix associated with the given state equations, but instead it shows that the optimal controls are the functions of this original fundamental matrix. The method proposed in the reference (Cho & Park 2008) significantly reduces calculations and readily applies to general linear cases.

It is very challenging to verify and apply the analytic solution in the reference (Cho & Park 2008) to spacecraft formation flying problems. Thus, in the current paper, the fuel-optimal reconfiguration problem is analytically solved using the approach proposed in the reference (Cho & Park 2008). The problem considered in the current paper is to fuel-optimal reconfiguration in an elliptic orbit. It is verified in the current study that once dynamics equations are analytically solved, the optimal control vector can be easily found by the method suggested in the reference (Cho & Park 2008). It is also shown that the state vector as well as the cost function for desired configuration also can be analytically obtained. The verification in the current paper demonstrates the brevity of the method and the accuracy of the approach presented in the reference (Cho & Park 2008).

2. Derivation of Analytic Solutions to Reconfigurations in Elliptic Orbits

In this section, analytic solutions are derived for the fuel-optimal reconfiguration problem in a general elliptic orbit. To obtain the analytic solutions, the Tschauner-Hempel equations (1965) are used and the technique of the calculus of variations is utilized. For this specific problem, it is verified that the general analytic solutions contain the fundamental matrix of the given system, and the solutions do not require the calculation of the inverse of the fundamental matrix. These insights on the analytic solutions prove a novel method to establish general solutions suggested by Cho & Park (2008) to reconfiguration maneuvers in relative motions.

Satellite formation flying is used when a group of satellites need to perform a unified space mission together. There are two kinds of satellites in the artificial formation flying. The main one is called the chief satellite, and the others are the deputy satellites. The deputy satellites surround the central chief satellite. The orbit of the chief is defined as the reference orbit, whereas the deputies revolve along an orbit that is relative to the reference orbit. The dynamics of relative motion should be known exactly in order to describe the relative motion and to perform reconfiguration maneuvers in formation flying. In an inertial coordinate, the relative motion between a chief and its deputies is considered rather than the separate motion of each satellite. For this reason, a reference orbit is needed to appropriately express the relative motion. The local vertical, local horizontal (LVLH) reference coordinate is used as one of the rotating reference coordinates. The origin of the LVLH coordinate is the position of the chief. The $x(t)$ axis lies in the radial direction, the $y(t)$ axis is in the along-track direction, and the $z(t)$ axis along the orbital angular momentum vector completes a right-handed coordinate system. Relative position vector $\mathbf{r} = [x \ y \ z]^T$ and relative velocity vector $\mathbf{v} = [\dot{x} \ \dot{y} \ \dot{z}]^T$ are defined as the difference of position and velocity between the chief and deputy, respectively. Linearized equations of relative motion have been introduced in many literatures.

To describe relative motion in elliptic orbits, the Tschauner-Hempel (TH) equations (1965) are

used:

$$
\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -2 \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} - \begin{bmatrix} -\dot{\theta}^2 & 0 & 0 \\ 0 & -\dot{\theta}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$
$$
- \begin{bmatrix} 0 & -\ddot{\theta} & 0 \\ \ddot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\rho(\theta)^3}{\Gamma^4} \begin{bmatrix} 2x \\ -y \\ -z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \tag{1}
$$

In this equation, $\theta(t)$ and e refer to the true anomaly and the eccentricity of the chief's orbit; respectively, $\rho(\theta) \triangleq 1 + e\cos\theta$ and $\Gamma \triangleq h^{3/2}/GM$ are defined, where h is the magnitude of the orbital angular momentum of the chief satellite, G is the universal gravitational constant, and M is the mass of the central body: Earth. The dot $(\cdot)$ represents the differentiation with respect to time (t), and it is assumed that the unconstrained thrust vector $[T_x(t) \; T_y(t) \; T_z(t)]^T$ can be continuously applied at the desired directions during the maneuver. For brevity, when changing the independent variable from time, t, to true anomaly, θ, the following transformation is considered:

$$
[\tilde{x} \; \tilde{y} \; \tilde{z}]^T = \dot{\theta}^{1/2}[x \; y \; z]^T \tag{2}
$$
$$
\tilde{\mathbf{u}} = [\tilde{u}_x \; \tilde{u}_y \; \tilde{u}_z]^T = [T_x \; T_y \; T_z]^T/\dot{\theta}^{3/2} \tag{3}
$$

The tildes are used to represent pseudo-values. Use of the same procedure as that derived by Humi (1993) makes eq. (1) very simple:

$$
\tilde{\boldsymbol{\xi}}' = \tilde{\mathbf{A}}(\theta)\tilde{\boldsymbol{\xi}} + \tilde{\mathbf{B}}(\theta)\tilde{\mathbf{u}} \tag{4}
$$

In this equation, primer $(')$ represents differentiation with respect to true anomaly, and

$$
\tilde{\boldsymbol{\xi}} = [\tilde{\mathbf{r}}^T \; \tilde{\mathbf{v}}^T]^T = [\tilde{x} \; \tilde{y} \; \tilde{z} \; \tilde{x}' \; \tilde{y}' \; \tilde{z}']^T,
$$
$$
\tilde{\mathbf{A}}(\theta) = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \\ \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_2 \end{bmatrix}, \quad \tilde{\mathbf{A}}_1 = \begin{bmatrix} 3/\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{5}
$$
$$
\tilde{\mathbf{A}}_2 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0}_{3\times3} \\ \mathbf{I}_{3\times3} \end{bmatrix}
$$

With eq. (2) and the relationship of $\dot{\theta}^{1/2} = \rho/\Gamma$, pseudo-positions $\tilde{\mathbf{r}} = [\tilde{x} \; \tilde{y} \; \tilde{z}]^T$ and pseudo-velocities $\tilde{\mathbf{v}} = \tilde{\mathbf{r}}' = [\tilde{x}' \; \tilde{y}' \; \tilde{z}']^T$ are transformed through the following relations from eq. (2):

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{\Gamma}{\rho} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}, \quad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \frac{e\sin\theta}{\Gamma} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} + \frac{\rho}{\Gamma} \begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{bmatrix} \tag{6}
$$

The actual thrust vector $\mathbf{T}$ is related with the pseudo-thrust vector $\tilde{\mathbf{u}}$ by $\mathbf{T} = (\rho^3/\Gamma^3)\tilde{\mathbf{u}}$.

The cost function is set as

$$
J = \frac{1}{2} \int_{\theta_0}^{\theta_f} \mathbf{T}^T\mathbf{T} d\varphi = \frac{1}{2\Gamma^6} \int_{\theta_0}^{\theta_f} \rho(\varphi)^6 \tilde{\mathbf{u}}^T \tilde{\mathbf{u}} d\varphi \tag{7}
$$

where θ_0 and θ_f are true anomalies of the chief satellite when thrusts start to fire and turn off, respectively. Here, true anomaly is used as the independent variable instead of time because it allows for easier calculations and naturally penalizes the control effort near perigee. The difference in the cost function caused by using different independent variables (true anomaly and time) is not negligible for high eccentricities, because nonlinear effects will be increased when true anomaly is used as an independent variable for high-eccentricity case (Sengupta et al. 2006).

The above equations can be adjoined via Lagrange multipliers to eq. (7) so that new augmented cost function J_{aug} is found (Bryson & Ho 1975):

$$J_{aug} = \int_{\theta_0}^{\theta_f} (H - \tilde{\boldsymbol{\lambda}}_r^T \tilde{\mathbf{r}}' - \tilde{\boldsymbol{\lambda}}_v^T \tilde{\mathbf{v}}') d\tau \tag{8}$$

where the Hamiltonian function is defined as

$$H \triangleq \frac{\rho^6}{2\Gamma^6} \tilde{\mathbf{u}}^T \tilde{\mathbf{u}} + \tilde{\boldsymbol{\lambda}}_r^T \tilde{\mathbf{v}} + \tilde{\boldsymbol{\lambda}}_v^T (\tilde{\mathbf{A}}_1 \tilde{\mathbf{r}} + \tilde{\mathbf{A}}_2 \tilde{\mathbf{v}} + \tilde{\mathbf{u}}) \tag{9}$$

If we define the adjoint vector by

$$\tilde{\boldsymbol{\lambda}} \triangleq \begin{bmatrix} \tilde{\boldsymbol{\lambda}}_r \\ \tilde{\boldsymbol{\lambda}}_v \end{bmatrix} \tag{10}$$

then the following equations must be satisfied for the optimality conditions (Bryson & Ho 1975):

$$\tilde{\boldsymbol{\lambda}}'^T = -\frac{\partial H}{\partial \tilde{\boldsymbol{\xi}}}, \quad \frac{\partial H}{\partial \tilde{\mathbf{u}}} = \mathbf{0} \tag{11}$$

For the system in this section, eq. (11) yields

$$\tilde{\boldsymbol{\lambda}}_r' = -\tilde{\mathbf{A}}_1^T \tilde{\boldsymbol{\lambda}}_v = -\tilde{\mathbf{A}}_1 \tilde{\boldsymbol{\lambda}}_v \tag{12}$$

$$\tilde{\boldsymbol{\lambda}}_v' = -\tilde{\mathbf{A}}_2^T \tilde{\boldsymbol{\lambda}}_v - \tilde{\boldsymbol{\lambda}}_r \tag{13}$$

$$\tilde{\mathbf{u}} = -\frac{\Gamma^6}{\rho^6} \tilde{\boldsymbol{\lambda}}_v \tag{14}$$

and the state and adjoint equations are

$$\begin{bmatrix} \tilde{\mathbf{r}}' \\ \tilde{\mathbf{v}}' \\ \tilde{\boldsymbol{\lambda}}_r' \\ \tilde{\boldsymbol{\lambda}}_v' \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_2 & \mathbf{0}_{3\times3} & -\frac{\Gamma^6}{\rho^6}\mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\tilde{\mathbf{A}}_1 \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\mathbf{I}_{3\times3} & -\tilde{\mathbf{A}}_2^T \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{v}} \\ \tilde{\boldsymbol{\lambda}}_r \\ \tilde{\boldsymbol{\lambda}}_v \end{bmatrix} \tag{15}$$

Yamanaka & Ankersen (2002) provide the fundamental matrix associated with the matrix $\tilde{\mathbf{A}}$:

$$\tilde{\boldsymbol{\Phi}} = \begin{bmatrix} 0 & -c & 0 & -s & 3es\Omega - 2 & 0 \\ 1 & s(1+1/\rho) & 0 & -c(1+1/\rho) & 3\rho^2\Omega & 0 \\ 0 & 0 & c/\rho & 0 & 0 & s/\rho \\ 0 & -c' & 0 & -s' & 3e(s'\Omega + s/\rho^2) & 0 \\ 0 & 2c-e & 0 & 2s & 3(1-2es\Omega) & 0 \\ 0 & 0 & -s/\rho & 0 & 0 & c/\rho \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_1 \\ \tilde{\boldsymbol{\Phi}}_2 \\ \tilde{\boldsymbol{\Phi}}_3 \\ \tilde{\boldsymbol{\Phi}}_4 \\ \tilde{\boldsymbol{\Phi}}_5 \\ \tilde{\boldsymbol{\Phi}}_6 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_A \\ \tilde{\boldsymbol{\Phi}}_A' \end{bmatrix} \tag{16}$$

In this equation, $s \triangleq \rho \sin \theta$ and $c \triangleq \rho \cos \theta$, and $\tilde{\boldsymbol{\Phi}}_1, \tilde{\boldsymbol{\Phi}}_2, \tilde{\boldsymbol{\Phi}}_3$, etc. are 1×6 row vectors and both $\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}$ and $\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}'$ are 3×6 matrices, and

$$\Omega \triangleq \frac{1}{\Gamma^2}(t - t_0) = \int_{\theta_0}^{\theta} \frac{1}{\rho(\varphi)^2} d\varphi \tag{17}$$

where t_0 and θ_0 are the time and true anomaly when thrusters start to fire.

Now, it will be shown that the inverse of $\tilde{\boldsymbol{\Phi}}$ is not needed, and this allows the calculation to be considerably reduced. It is noted that previous researches (Palmer 2006, Cho et al. 2007, Scott & Spencer 2007, Sharma et al. 2007) required the inverse of the fundamental matrix because the inverse is contained in a particular solution. If we set $\tilde{\boldsymbol{\lambda}}_{\mathrm{r}} = [\tilde{\lambda}_1 \quad \tilde{\lambda}_2 \quad \tilde{\lambda}_3]^T$ and $\tilde{\boldsymbol{\lambda}}_{\mathrm{v}} = [\tilde{\lambda}_4 \quad \tilde{\lambda}_5 \quad \tilde{\lambda}_6]^T$, then the result from eqs. (12,13) will be

$$\tilde{\lambda}_1' = -\frac{3}{\rho}\tilde{\lambda}_4, \quad \tilde{\lambda}_2' = 0, \quad \tilde{\lambda}_3' = \tilde{\lambda}_6,$$
$$\tilde{\lambda}_4' = -\tilde{\lambda}_1 + 2\tilde{\lambda}_5, \quad \tilde{\lambda}_5' = -\tilde{\lambda}_2 - 2\tilde{\lambda}_4, \quad \tilde{\lambda}_6' = -\tilde{\lambda}_3 \tag{18}$$

Then,

$$\tilde{\lambda}_4'' = \frac{3}{\rho}\tilde{\lambda}_4 + 2\tilde{\lambda}_5'$$
$$\tilde{\lambda}_5'' = -2\tilde{\lambda}_4' \tag{19}$$
$$\tilde{\lambda}_6'' = -\tilde{\lambda}_6'$$

or

$$\tilde{\boldsymbol{\lambda}}_{\mathrm{v}}'' = \tilde{\mathbf{A}}_1 \tilde{\boldsymbol{\lambda}}_{\mathrm{v}} + \tilde{\mathbf{A}}_2 \tilde{\boldsymbol{\lambda}}_{\mathrm{v}}' \tag{20}$$

where the matrices $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ are defined in eq. (5). Equation (20) is exactly the same as the homogeneous form of eq. (4). This means that without controls, the pseudo-position vector $\tilde{\mathbf{r}}$ satisfies the following relationship from eq. (5):

$$\tilde{\mathbf{r}}'' = \tilde{\mathbf{A}}_1 \tilde{\mathbf{r}} + \tilde{\mathbf{A}}_2 \tilde{\mathbf{r}}'$$

Therefore, $\tilde{\boldsymbol{\lambda}}_{\mathrm{v}}$ and $\tilde{\mathbf{r}}$ have the same form of solutions:

$$\tilde{\boldsymbol{\lambda}}_{\mathrm{v}} = \begin{bmatrix} \tilde{\lambda}_4 \\ \tilde{\lambda}_5 \\ \tilde{\lambda}_6 \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_1 \\ \tilde{\boldsymbol{\Phi}}_2 \\ \tilde{\boldsymbol{\Phi}}_3 \end{bmatrix} \tilde{\boldsymbol{\Lambda}}_0 = \tilde{\boldsymbol{\Phi}}_{\mathrm{A}} \tilde{\boldsymbol{\Lambda}}_0, \quad \tilde{\boldsymbol{\lambda}}_{\mathrm{v}}' = \tilde{\boldsymbol{\Phi}}_{\mathrm{A}}' \tilde{\boldsymbol{\Lambda}}_0 \tag{21}$$

where $\tilde{\boldsymbol{\Lambda}}_0$ is a 6×1 constant matrix determined from the boundary condition at $\theta = \theta_f$. Substituting eq. (21) into eq. (13) produces:

$$\tilde{\boldsymbol{\lambda}}_{\mathrm{r}} = \tilde{\mathbf{A}}_2 \tilde{\boldsymbol{\lambda}}_{\mathrm{v}} - \tilde{\boldsymbol{\lambda}}_{\mathrm{v}}' = (\tilde{\mathbf{A}}_2 \tilde{\boldsymbol{\Phi}}_{\mathrm{A}} - \tilde{\boldsymbol{\Phi}}_{\mathrm{A}}')\tilde{\boldsymbol{\Lambda}}_0$$

Let $\tilde{\boldsymbol{\Psi}}$ be the fundamental matrix associated with the adjoint system:

$$\tilde{\boldsymbol{\lambda}} \triangleq \begin{bmatrix} \tilde{\boldsymbol{\lambda}}_{\mathrm{r}} \\ \tilde{\boldsymbol{\lambda}}_{\mathrm{v}} \end{bmatrix} = \tilde{\boldsymbol{\Psi}} \tilde{\boldsymbol{\Lambda}}_0$$

where

$$\tilde{\boldsymbol{\Psi}} = \begin{bmatrix} -\tilde{\boldsymbol{\Phi}}_1' + 2\tilde{\boldsymbol{\Phi}}_2 \\ -2\tilde{\boldsymbol{\Phi}}_1 - \tilde{\boldsymbol{\Phi}}_2' \\ -\tilde{\boldsymbol{\Phi}}_3' \\ \tilde{\boldsymbol{\Phi}}_1 \\ \tilde{\boldsymbol{\Phi}}_2 \\ \tilde{\boldsymbol{\Phi}}_3 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_{\mathbf{B}} \\ \tilde{\boldsymbol{\Phi}}_{\mathbf{A}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3s/\rho & 0 & -e - 3c/\rho & 3\Omega(3\rho + e^2 - 1) - 3es/\rho^2 & 0 \\ 0 & e & 0 & 0 & 1 & 0 \\ 0 & 0 & s/\rho & 0 & 0 & -c/\rho \\ 0 & -c & 0 & -s & 3es\Omega - 2 & 0 \\ 1 & s(1 + 1/\rho) & 0 & -c(1 + 1/\rho) & 3\rho^2\Omega & 0 \\ 0 & 0 & c/\rho & 0 & 0 & s/\rho \end{bmatrix} \quad (22)$$

and where $\tilde{\boldsymbol{\Phi}}_{\mathbf{A}}$ and $\tilde{\boldsymbol{\Phi}}_{\mathbf{B}}$ are 3×6 matrices. It is noted that $\tilde{\boldsymbol{\Phi}}_{\mathbf{A}}$, the lower half of $\tilde{\boldsymbol{\Psi}}$, is exactly the same as the upper half of $\tilde{\boldsymbol{\Phi}}$. Then, $\tilde{\boldsymbol{\Phi}}$ is the fundamental matrix associated with the original system [eq. (4)] and $\tilde{\boldsymbol{\Psi}}$ is the fundamental matrix associated with the adjoint system [eqs. (12,13)], so $\tilde{\boldsymbol{\Psi}}^T \tilde{\boldsymbol{\Phi}} = \tilde{\mathbf{C}}$ is a hold where $\tilde{\mathbf{C}}$ is a constant matrix (Carter 1998). In the case in this section,

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & -e & 0 & 0 & -1 & 0 \\ e & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -e & 0 \\ 1 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\mathbf{C}}^{-1} = \frac{1}{1 - e^2} \begin{bmatrix} 0 & -e & 0 & 0 & 1 & 0 \\ e & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^2 - 1 \\ 0 & 1 & 0 & 0 & -e & 0 \\ -1 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & 1 - e^2 & 0 & 0 & 0 \end{bmatrix} \quad (23)$$

so that

$$\tilde{\boldsymbol{\Phi}}^{-1} = \tilde{\mathbf{C}}^{-1} \tilde{\boldsymbol{\Psi}}^T = \tilde{\mathbf{C}}^{-1}[\tilde{\boldsymbol{\Phi}}_{\mathbf{B}}^T \vdots \tilde{\boldsymbol{\Phi}}_{\mathbf{A}}^T] \quad (24)$$

Then, from eqs. (14,21), and the relationship of $\mathbf{T} = (\rho^3/\Gamma^3)\tilde{\mathbf{u}}$, the optimal thrust is

$$\tilde{\mathbf{u}}(\theta) = -\frac{\Gamma^6}{\rho(\theta)^6} \tilde{\boldsymbol{\Phi}}_{\mathbf{A}} \tilde{\boldsymbol{\Lambda}}_0, \quad \mathbf{T}(\theta) = -\frac{\Gamma^3}{\rho(\theta)^3} \tilde{\boldsymbol{\Phi}}_{\mathbf{A}} \tilde{\boldsymbol{\Lambda}}_0 \quad (25)$$

Equation (25) indicates that optimal thrusts are functions of $\tilde{\boldsymbol{\Phi}}_{\mathbf{A}}$, the upper half of $\tilde{\boldsymbol{\Phi}}$. Therefore, it is possible to find out the form of analytic solution beforehand, and the only requirement is to determine the unknown 6×1 constant matrix $\tilde{\boldsymbol{\Lambda}}_0$ from the boundary conditions at $\theta = \theta_f$. The solution of eq. (4) is (Kirk 2004):

$$\tilde{\boldsymbol{\xi}}(\theta) = \tilde{\boldsymbol{\Phi}}(\theta)\tilde{\boldsymbol{\Phi}}^{-1}(\theta_0)\tilde{\boldsymbol{\xi}}(\theta_0) + \tilde{\boldsymbol{\Phi}}(\theta) \int_{\theta_0}^{\theta} \tilde{\boldsymbol{\Phi}}^{-1}(\varphi)\tilde{\mathbf{B}}(\varphi)\tilde{\mathbf{u}}(\varphi)d\varphi \quad (26)$$

where $\tilde{\boldsymbol{\Phi}}(\theta)$ given in eq. (16) is the fundamental matrix associated with eq. (4), and $\tilde{\boldsymbol{\Phi}}(\theta)\tilde{\boldsymbol{\Phi}}^{-1}(\theta_0)$ is the state transition matrix associated with eq. (4). Considering eqs. (24,25), the function in the integration becomes:

$$\tilde{\boldsymbol{\Phi}}^{-1}\mathbf{B}\tilde{\mathbf{u}} = \tilde{\mathbf{C}}^{-1}[\tilde{\boldsymbol{\Phi}}_{\mathrm{B}}^{T} \quad \tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}]\begin{bmatrix} \mathbf{0}_{3\times 3} \\ \mathbf{I}_{3\times 3} \end{bmatrix}\left(-\frac{\Gamma^6}{\rho^6}\right)\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\tilde{\boldsymbol{\Lambda}}_0 = -\Gamma^6\tilde{\mathbf{C}}^{-1}\left(\frac{1}{\rho^6}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\right)\tilde{\boldsymbol{\Lambda}}_0 \qquad (27)$$

At $\theta = \theta_f$, the final state is given by eqs. (26,27):

$$\tilde{\boldsymbol{\xi}}(\theta_f) = \tilde{\boldsymbol{\Phi}}(\theta_f)\tilde{\boldsymbol{\Phi}}^{-1}(\theta_0)\tilde{\boldsymbol{\xi}}(\theta_0) - \Gamma^6\tilde{\boldsymbol{\Phi}}(\theta_f)\tilde{\mathbf{C}}^{-1}\left(\int_{\theta_0}^{\theta_f}\frac{1}{\rho^6}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\,d\varphi\right)\tilde{\boldsymbol{\Lambda}}_0$$

Then the 6×1 matrix $\tilde{\boldsymbol{\Lambda}}_0$ is calculated:

$$\tilde{\boldsymbol{\Lambda}}_0 = -\frac{1}{\Gamma^6}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}} \qquad (28)$$

where

$$\tilde{\mathbf{S}}(\theta) \triangleq \int_{\theta_0}^{\theta}\frac{1}{\rho^6}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\,d\varphi, \quad \tilde{\mathbf{S}}_{\mathrm{f}} = \tilde{\mathbf{S}}(\theta_f) \qquad (29)$$

and

$$\tilde{\mathbf{K}} = \tilde{\boldsymbol{\Phi}}_{\mathrm{f}}^{-1}\tilde{\boldsymbol{\xi}}_{\mathrm{f}} - \tilde{\boldsymbol{\Phi}}_0^{-1}\tilde{\boldsymbol{\xi}}_0 \qquad (30)$$

where $\tilde{\boldsymbol{\xi}}_{\mathrm{f}} \triangleq \tilde{\boldsymbol{\xi}}(\theta_f), \tilde{\boldsymbol{\xi}}_0 \triangleq \tilde{\boldsymbol{\xi}}(\theta_0), \tilde{\boldsymbol{\Phi}}_{\mathrm{f}} \triangleq \tilde{\boldsymbol{\Phi}}(\theta_f)$ and $\tilde{\boldsymbol{\Phi}}_0 \triangleq \tilde{\boldsymbol{\Phi}}(\theta_0)$. It is noted that the 6×6 matrix $\tilde{\mathbf{S}}$ is symmetric. Then, the cost function as shown in eq. (7) can be simplified, because $\tilde{\mathbf{K}}^{T}, \tilde{\mathbf{C}}^{T}$, and $\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}$ are constant:

$$J = \frac{1}{2}\int_{\theta_0}^{\theta_f}\mathbf{T}^{T}\mathbf{T}\,d\varphi = \frac{1}{2\Gamma^6}\int_{\theta_0}^{\theta_f}\frac{1}{\rho^6}\tilde{\mathbf{K}}^{T}\tilde{\mathbf{C}}^{T}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}}\,d\varphi = \frac{1}{2\Gamma^6}\tilde{\mathbf{K}}^{T}\tilde{\mathbf{C}}^{T}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}} \qquad (31)$$

In sum, the optimal thrust vector is a function of the upper half of the fundamental matrix [eq. (16)], and it is not needed to calculate its inverse. All results are succinctly represented in the following form:

$$\tilde{\mathbf{S}}(\theta) = \int_{\theta_0}^{\theta}\frac{1}{\rho^6}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}^{T}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\,d\varphi, \quad \tilde{\mathbf{K}} = \tilde{\boldsymbol{\Phi}}_{\mathrm{f}}^{-1}\tilde{\boldsymbol{\xi}}_{\mathrm{f}} - \tilde{\boldsymbol{\Phi}}_0^{-1}\tilde{\boldsymbol{\xi}}_0,$$

$$\mathbf{T}(\theta) = \frac{1}{\Gamma^3\rho(\theta)^3}\tilde{\boldsymbol{\Phi}}_{\mathrm{A}}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}}, \quad J = \frac{1}{2\Gamma^6}\tilde{\mathbf{K}}^{T}\tilde{\mathbf{C}}^{T}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}}, \qquad (32)$$

$$\tilde{\boldsymbol{\xi}} = \begin{bmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{v}} \end{bmatrix} = \tilde{\boldsymbol{\Phi}}(\tilde{\boldsymbol{\Phi}}_0^{-1}\tilde{\boldsymbol{\xi}}_0 + \tilde{\mathbf{C}}^{-1}\tilde{\mathbf{S}}\tilde{\mathbf{S}}_{\mathrm{f}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{K}}), \quad \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \frac{\Gamma}{\rho} & 0 \\ \frac{e\sin\theta}{\Gamma} & \frac{\rho}{\Gamma} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{v}} \end{bmatrix}$$

To solve the problem, the vector $\tilde{\mathbf{K}}$ must be firstly determined, which is straightforward because the states $\tilde{\boldsymbol{\xi}}_0$ and $\tilde{\boldsymbol{\xi}}_{\mathrm{f}}$ are given and $\tilde{\boldsymbol{\Phi}}_0^{-1}$ and $\tilde{\boldsymbol{\Phi}}_{\mathrm{f}}^{-1}$ are constant matrices. Since the fundamental matrix $(\tilde{\boldsymbol{\Phi}})$ has already been revealed, the symmetric matrix $\tilde{\mathbf{S}}$ is readily calculated, and the constant matrix $\tilde{\mathbf{C}}$ is given in eq. (23). Then, the cost function, J, the optimal thrust vector, $\mathbf{T}$, and the state variables during the reconfiguration, $\tilde{\boldsymbol{\xi}}$, can be solved in a completely analytic way. In previous research studies conducted (Scott & Spencer 2007, Sharma et al. 2007), $\tilde{\mathbf{S}}$ contains the inverse of the fundamental matrix associated with the original system, which makes the required computations much more complex. Furthermore, eq. (32) in the current study reveals the fact that the optimal thrust vector is an

explicit function of the fundamental matrix so the form of solutions can be established without time-consuming calculations, while previous results (Sharma et al. 2007) do not present explicit forms of the solutions due to the complexity of the inverse of the fundamental matrix. Equation (32) yields exactly the same results obtained by Cho et al. (2007), which will be shown in Section 3.

3. Application of the Solutions for Spacecraft Formation Flying in an Ellptic Orbit

In this section, the general solution derived in Section 2 is applied to fuel-optimal reconfiguration in an elliptic orbit. This example illustrates that the general solutions obtained in the current study can be readily applied to spacecraft formation flying in an elliptic orbit. The cost function is set as eq. (7) and the fundamental matrix ($\tilde{\mathbf{\Phi}}$) is already given in eq. (16). Then, the general solutions are described in eq. (32). The constant matrix $\tilde{\mathbf{C}}$ is given by eq. (23), and the symmetric matrix $\tilde{\mathbf{S}}$ is:

$$
\tilde{\mathbf{S}} = \begin{bmatrix}
\tilde{S}_{11} & \tilde{S}_{12} & 0 & \tilde{S}_{14} & \tilde{S}_{15} & 0 \\
\tilde{S}_{21} & \tilde{S}_{22} & 0 & \tilde{S}_{24} & \tilde{S}_{25} & 0 \\
0 & 0 & \tilde{S}_{33} & 0 & 0 & \tilde{S}_{36} \\
\tilde{S}_{41} & \tilde{S}_{42} & 0 & \tilde{S}_{44} & \tilde{S}_{45} & 0 \\
\tilde{S}_{51} & \tilde{S}_{52} & 0 & \tilde{S}_{54} & \tilde{S}_{55} & 0 \\
0 & 0 & \tilde{S}_{63} & 0 & 0 & \tilde{S}_{66}
\end{bmatrix}
\tag{33}
$$

The integration may seem to be difficult, but evaluating in terms of the eccentric anomaly E mitigates this problem; when this is done, each element is:

$$
\tilde{S}_{11} = A_5, \ \ \tilde{S}_{12} = \tilde{S}_{21} = A_4 + B_3, \ \ \tilde{S}_{14} = \tilde{S}_{41} = -A_6 - B_2, \ \ \tilde{S}_{15} = \tilde{S}_{51} = 3C_5,
$$
$$
\tilde{S}_{22} = A_3 + 2B_1 + C_1, \ \ \tilde{S}_{24} = \tilde{S}_{42} = -A_2 - 2B_4, \ \ \tilde{S}_{25} = \tilde{S}_{52} = 2B_2 - 3eC_2 + 3C_3 + 3D_1,
$$
$$
\tilde{S}_{33} = A_1, \ \ \tilde{S}_{36} = \tilde{S}_{63} = A_2, \ \ \tilde{S}_{44} = A_1 + 2B_6 + C_1,
$$
$$
\tilde{S}_{45} = \tilde{S}_{54} = 2B_3 - 3eC_6 - 3C_7 - 3D_2, \ \ \tilde{S}_{55} = 4A_5 - 12eB_5 + 9e^2C_4 + 9E_1, \ \ \tilde{S}_{66} = A_3
$$

where A_1, A_2, A_3, etc. are given by Cho et al. (2007).

For a numerical verification, the same example as in the reference (Cho et al. 2007) is demonstrated. Cho at al. (2007) obtained analytical solutions by a method using Fourier series which is different method from that in the current study. The semi-major axis and the eccentricity of the chief satellite's orbit are assumed to be 7.78×10^6 (m) and 0.1, respectively. The deputy satellite performs reconfiguration while the chief satellite revolves one orbit. For a direct comparison with the results derived in the reference (Cho et al. 2007), the same initial and final conditions for formation resizing are used:

$$
\boldsymbol{\xi}_0 = [250(\mathrm{m}) \ \ 0(\mathrm{m}) \ \ 500(\mathrm{m}) \ \ 0(\mathrm{m/s}) \ \ -0.539385(\mathrm{m/s}) \ \ 0(\mathrm{m/s})]^T
\tag{34a}
$$
$$
\boldsymbol{\xi}_\mathrm{f} = [500(\mathrm{m}) \ \ 0(\mathrm{m}) \ \ 1000(\mathrm{m}) \ \ 0(\mathrm{m/s}) \ \ -1.07876(\mathrm{m/s}) \ \ 0(\mathrm{m/s})]^T
\tag{34b}
$$

From eqs. (23,32,33), the matrices $\tilde{\mathbf{C}}$, $\tilde{\mathbf{K}}$, and $\tilde{\mathbf{S}}$ are determined, and the cost function, J, has a value of $3.61816 \times 10^{-8} \mathrm{m}^2/\mathrm{s}^4$ from eq. (32). The optimal thrust acceleration vector, $\mathbf{T}$, an explicit function of the upper half of the fundamental matrix ($\tilde{\mathbf{\Phi}}_\mathrm{A}$), is immediately calculated from eq. (32)
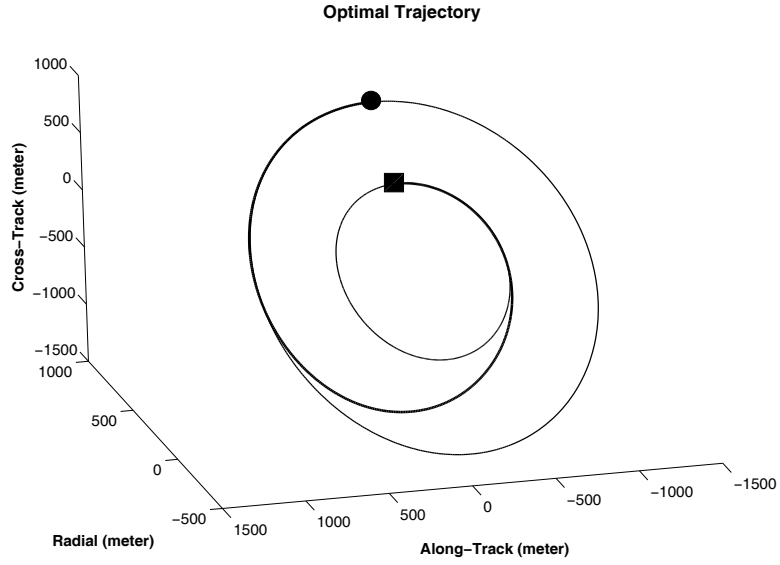
Optimal Trajectory

Figure 1. Three-dimensional optimal reconfiguration trajectory for an elliptic reference orbit.

or eq. (25):

$$
\begin{aligned}
\mathbf{T}(\theta) &= \frac{1}{\Gamma^3 \rho^3} \tilde{\boldsymbol{\Phi}}_A \tilde{\mathbf{S}}_f^{-1} \mathbf{CK} = -\frac{\Gamma^3}{\rho^3} \tilde{\boldsymbol{\Phi}}_A \tilde{\boldsymbol{\Lambda}}_0 \\
&= -\frac{\Gamma^3}{\rho^3} \left[\begin{array}{c}
-\tilde{\lambda}_2 \rho \cos\theta - \tilde{\lambda}_4 \rho \sin\theta + \tilde{\lambda}_5 (3e\rho \sin\theta\,\Omega - 2) \\
\tilde{\lambda}_1 + \tilde{\lambda}_2 \sin\theta(1+\rho) - \tilde{\lambda}_4 \cos\theta(1+\rho) + 3\tilde{\lambda}_5 \rho^2 \Omega \\
\tilde{\lambda}_3 \cos\theta + \tilde{\lambda}_6 \sin\theta
\end{array} \right]
\end{aligned}
$$

where $\tilde{\boldsymbol{\Lambda}}_0 = [\tilde{\lambda}_1 \ \ \tilde{\lambda}_2 \ \ \tilde{\lambda}_3 \ \ \tilde{\lambda}_4 \ \ \tilde{\lambda}_5 \ \ \tilde{\lambda}_6]^T$, and $\tilde{\lambda}_1 = 1.13538 \times 10^{-10}$, $\tilde{\lambda}_2 = -5.72925 \times 10^{-14}$, $\tilde{\lambda}_3 = 0$, $\tilde{\lambda}_4 = -3.84234 \times 10^{-10}$, $\tilde{\lambda}_5 = -2.19623 \times 10^{-14}$, and $\tilde{\lambda}_6 = 4.13561 \times 10^{-9}$. The previous analytic solutions derived by Cho et al. (2007) are eqs. (29,41a,b) in the paper as following:

$$
T_x(\theta) = \frac{\Gamma^3}{2\rho^3}[-\lambda_2 \rho \cos\theta + \lambda_3(3e\rho \sin\theta\,\Omega - 2) + \lambda_5 \rho \sin\theta]
$$

$$
T_y(\theta) = \frac{\Gamma^3}{2\rho^3}[\lambda_2 \sin\theta(1+\rho) + 3\lambda_3 \rho^2 \Omega - \lambda_4 + \lambda_5 \cos\theta(1+\rho)]
$$

$$
T_z(\theta) = \frac{\Gamma^3}{2\rho^3}[\lambda_0 \cos\theta + \lambda_1 \sin\theta]
$$

It is directly observed that the two solutions above have exactly the same form except for the arrangement of coefficients. Numerical simulations are shown graphically in Figure 1, Figure 2, and Figure 3. Figure 1 shows the three-dimensional optimal trajectory ($\tilde{\boldsymbol{\xi}}$) obtained from eq. (32) during the reconfiguration in the LVLH frame.
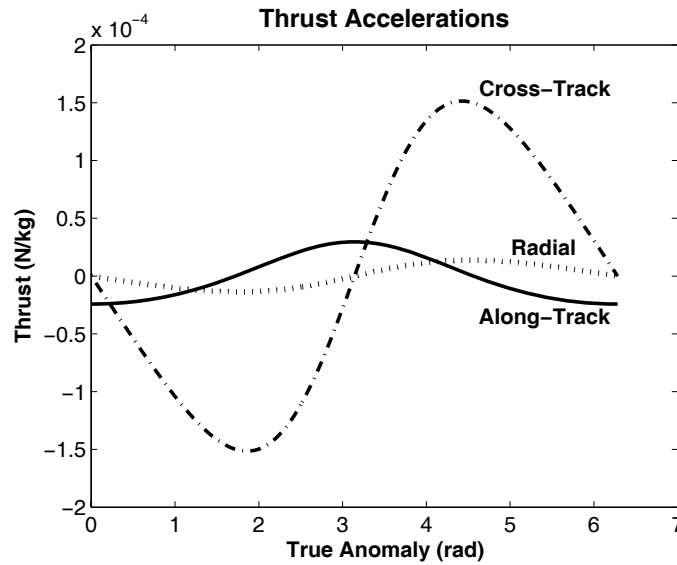
Figure 2. Thrust profiles for each of the three thrusters for an elliptic reference orbit.

The narrow lines are trajectories with initial and final formations. The solid line represents the reconfiguration trajectory. The square and the circle represent initial and final positions, respectively. Figure 2 demonstrates the thrust functions for each of the three thrusters. Figures 1 and 2 are exactly the same as those in Cho et al. (2007). In Figure 3 the differences between the optimal thrust profiles obtained from Cho et al. (2007) and the current paper are shown. It is obvious that the differences are negligible.

5. Conclusions

A new analytic method for solving reconfiguration problems in relative motion is verified, which does not need the inverse of the fundamental matrix. The analytical solutions are derived for the problem of spacecraft formation flying in an elliptic orbit. It is shown that the optimal control functions contain the components of the fundamental matrix of the given dynamic equations. For the interested problem, the fundamental matrix is found, and the optimal thrust accelerations, the cost function, and the states during the maneuver are also analytically calculated. The results obtained in the current paper explain and confirm the explicit form of the optimal solutions suggested by Cho & Park (2008).
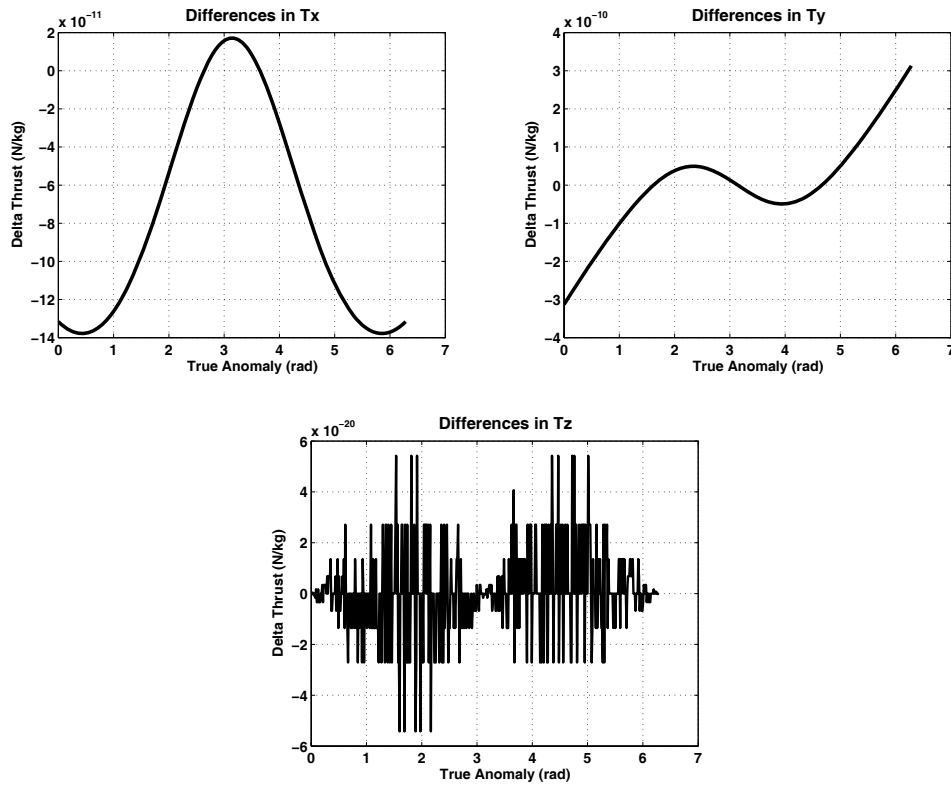
Figure 3. Differences between the optimal thrust profiles obtained from Cho et al. (2007) and the current paper for an elliptic reference orbit.

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