Recent Applications of Variational Principles to Finite Element Methods

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Most of the development of finite element procedures in the numerical analysis of problems in Continuum Mechanics has been centered around its kinematical aspects. It is the main purpose of this lecture to show that extremely useful results can also be obtained by focusing attention primarily on the statical aspects. In a general sense kinematics and statics are complementary and intimately related through virtual work or energy considerations. Whenever a problem in continuum Mechanics is "conservative", the statical properties can be generated as the variational equations of extremization of an appropriate functional of its kinematical description. The converse is also true in the linear cases and a general method for obtaining such conjugate functionals is due to O.C. Friedrichs [1].

Such variational principles are extremely useful for the purpose of obtaining approximate solutions in terms of a finite number of parameters or unknown functions of a single variable. In the former case the problem is reduced to one of algebraic type, in the second to that of solving a set of ordinary differential equations. Both can presently be handled efficiently by electronic computers.
The Rayleigh-Ritz method, for instance, applies to functionals of the kinematical type and the finite element method using displacement models is but an extension of it, required to handle efficiently problems of complicated topology or non-uniform constitutive properties.
A converse of the Rayleigh-Ritz method was originally proposed by E. Trefftz [2], it is however only a particular case of the dual method based on the conjugate functional. We now examine in more detail certain fields of application.

1. Linear elasticity theory.

The principle of minimum of total energy and the principle of minimum complementary energy are the conjugate principle of kinematics and statics. When related by Friedrichs's method they also generate two so-called two-field principles, one of which is known as the Hellinger-Reissner principle [3].

Displacement models of finite elements satisfy rigorously the kinematical properties when they are "conforming", that is when they satisfy the required continuity properties of displacements across interfaces. Hence both the individual and the global equilibrium conditions can be obtained by application of the principle of minimum of total potential.

Equilibrium models should satisfy rigorously the statical properties. To this purpose they are constructed from parametric stress fields in equilibrium with the given body forces and they must be "diffusive", that is transmit rigorously the surface tractions across interfaces. This is sometimes the difficult property to incorporate. Furthermore, as the associated strain field is not, as a rule, integrable, equilibrium models do not provide local displacements but only functionals of the unknown displacement field. The approximate compatibility conditions are obtained by application of the pinciple of minimum complementary energy. Hybrid models of a wide variety of type can be constructed using the two-field principles. In this case both equilibrium and compatibility are approximated.

A closer study of rigorously kinematical or rigorously statical approximations [4], [3], reveals an interesting property of strain energy bounding. If the kinematical boundary conditions are homogeneous, the approximate and exact values of the strain energy under a given set of loads are related as follows:

Kinematical Approximation $\leq$ Exact $\leq$ Statical Approximation

(conforming displacement models) (diffusive equilibrium models)

Thus a dual approximate analysis will provide upper and lower bounds to the energy and a quantitative estimate of the state of energy convergence. Applications of this principle have established is practical value [5]. We illustrate it here on the example of torsion of a box-beam, and bending and torsion of a swept-back aircraft wing [6].

Dual analysis of plate bending problems, when transverse shearing deformation is neglected, poses the difficult problem of obtaining conformity under the Kirchhoff assumption. This problem was solved in 1963 by using a bi-cubic interpolation scheme and more detailed varsion of the generation of the stiffness matrix published in 1968 [7]. It was also recognized that the diffculty was of exactly the same nature as that of removing kinematical deformation modes in eqilibrium models of plate stretching [9] and this was further clarified by the static-geometric analogies between displacements and stress-functions [10]. Thus there is also complete analogy between the triangular displacement model of second degree for stretching and the equilibrium plate bending model published in [8].
A comparison of performance of the plate bending models of both types is presented.

2. Steady state heat conduction.

The use of conjugate functionals in heat conduction problems is also beneficial. Starting from the dissipation functional in terms of a temperature field, the Friedrichs method leads to the conjugate functional in terms of a field of heat fluxes in thermal equilibrium with heat sources or sinks. The corresponding finite element models and some numerical applications are described in [11]. Very close bounds can be obtained for the global dissipation function.

3. Elastodynamics.

Hamilton's principle is again a suitable variational principle for a kinematical approximation to dynamic problems. In addition to the stiffness matrix, it furnishes by computation of the kinetic energy a coherent mass matrix for the conforming finite element.
In 1952 Toupin presented a conjugate principle [12] that is also derivable from Hamilton's principle through the Friedrichs technique [13].
Géradin [14] showed how equilibrium models can satisfy this conjugate principle with coherent inverse-mass matrices and succeeded in bringing the formulation back to the standard eigenvalue form. Dual elasto-dynamic approximations by finite elements generally furnish upper and lower bounds to natural frequencies.
This will be illustrated on cantilever plate bending vibration problems.
While displacement models usually require "reduction" techniques in inertial degrees of freedom to obtain economical evaluation of the low frequency spectrum of natural vibrations, equilibrium models provide an automatic reduction because inertial degrees of freedom are considerably less than the number of global stress parameters.

4. Geometrical non-linearities.

Finite displacements of thin-walled structures, remaining in the linear elastic range of material properties, provide another instance of conservative problems. In this case there are several choices possible for the strain tensor, although they are practically equivalent when the elastic material range allows only very small strains to take place.
In contrast to this the different possible choices of stress tensors are not equivalent. Using the Green tensor for strains and the Kirchhoff stress tensor, that is the energy conjugate, Washizu [15] transformed the minimum total potential principle into a two-field principle that is in fact the corresponding Friedrichs Canonical form.

He concluded to the impossibility of obtaining a conjugate principle involving the stresses only. However by a different choice of stresses and divorcing the rotational equilibrium conditions, the translational equilibrium conditions can be satisfied by stress-function method for instance, and a conjugate principle obtained [16].

It is hoped that by this new principle the range of application of equilibrium type finite elements can be extended into the geometrically non linear range and provide new methods for analyzing elastic stability.

5. Fluid mechanics.

Isoenergetic flows can be handled by variational principles. However the version of Hamilton's principle that is required is not very practical and Eulerian type principles are usually preferred, such as those given by Bateman. A logical derivation of Bateman's principles from Hamilton's principle is of theoretical interest as it requires careful distinction between lagrangian and Eulerian variations [17].

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