DUALITY IN STRUCTURAL ANALYSIS BY FINITE ELEMENTS

B. Fraeijs de Veubeke

INTRODUCTION

In linear elasticity theory the displacement field, of cartesian components u_i ($i = 1, 2, 3$), is characterized by gradients of negligible magnitude before unity

$$|D_j u_i| \ll 1 \qquad D_j = \partial/\partial x_j \tag{1}$$

The coordinates x_j of a material point are, for specificity, considered to be those of the initial or "reference" configuration of the elastic body, although the inequalities (1) allow to forget the distinction between such lagrangian coordinates and the coordinates of the same point in the final configuration.

The elements of the strain tensor can then be reduced to

$$\epsilon_{ij} = \frac{1}{2}(D_i u_j + D_j u_i) = \epsilon_{ji} \tag{2}$$

and those of the rotation vector to

$$\omega_i = \frac{1}{2} e_{ijk} D_j u_k \tag{3}$$

where e_{ijk} denotes the permutation symbol. It obviously follows that

$$|\epsilon_{ij}| << 1 \qquad \text{and} \qquad |\omega_i| << 1 \tag{4}$$

are restrictions fully equivalent to (1). In any simply connected domain, the conservation of energy can be stated in the form

$$\delta \int_R W dR = \int_R X_j \delta u_j dR + \int_{\partial R} t_j \delta u_j dS \tag{5}$$

where the integrations are performed in the domain R and on its bounding surface ∂R in the reference configuration. The δu_j denote arbitrary perturbations of the displacement field; x_j are external force components per unit initial volume; t_j surface traction force components. The right-hand side is then the elementary work performed by the external forces in the perturbation. It is equated to an increase in strain energy. From a thermodynamical standpoint this energy is the

internal energy if there is no heat exchange between particles (adiabatic transformation); it is the free energy if the transformation is kept isothermal. Introducing the stress tensor τ_{ij} equilibrium at the boundary implies

$$t_j = n_i \tau_{ij} \tag{6}$$

where the n_i are the direction cosines of the outward normal along ∂R. The surface integral can then be transformed in a volume integral

$$\int_{\partial R} n_i \tau_{ij} \delta u_j dS = \int_R D_i (\tau_{ij} \delta u_j) dR$$

and the statement of energy conservation written down, for a unit volume, as

$$\delta W = X_j \delta u_j + D_i (\tau_{ij} \delta u_j)$$

an expanded form of which is

$$\delta W = (X_j + D_i \tau_{ij}) \delta u_j + \tau_{ij} D_i \delta u_j \tag{7}$$

In the case each δu_j is an arbitrary constant, the volume element is merely translated and there can be no energy increase, while $D_i \delta u_j = 0$; hence

$$X_j + D_i \tau_{ij} = 0 \tag{8}$$

Those are the translational equilibrium equations satisfied by the stresses; the conservation energy reduces now to

$$\delta W = \tau_{ij} D_i \delta u_j \tag{9}$$

If the displacement perturbation field is a small rigid body rotation

$$\delta u_j = e_{jmn} x_n \delta \omega_m \tag{10}$$

where $\delta \omega_m$ are arbitrary constants. This expression neglects the fact that the coordinates of the particles are actually $x_n + u_n$; however, this approximation is logical in view of the assumptions (1) of geometrical linearity as is easily seen by comparison with an exact calculation. The energy increase must again vanish; consequently, after substitution of (10) into (9), one can equate separately to zero the coefficients of the $\delta \omega_m$ and obtain the rotational equilibrium conditions:

$$e_{jmn} \tau_{ij} \delta_{in} = e_{jmn} \tau_{nj} = 0$$

which are equivalent to a statement of symmetry for the stress tensor

$$\tau_{ij} = \tau_{ji} \tag{11}$$

In view of this last result (9) can be written as

$$\delta W = \tau_{ij} D_i \delta u_j + \tau_{ij} D_j \delta u_i = \tau_{ij} \delta (D_i u_j + D_j u_i)$$

or, finally,

$$\delta W = \tau_{ij} \delta \epsilon_{ij} \tag{12}$$

In writing this statement in the equivalent form of general stress-strain relations

$$\tau_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \tag{13}$$

we consider the energy density W to be a function of the elements of a symmetrical strain tensor but keep the formal distinction between ϵ_{ij} and ϵ_{ji} in case $i \neq j$. Usually W is known explicitly in terms of the direct strains (ϵ_{11}, ϵ_{22}, ϵ_{33}) and shearing strains (γ_{12}, γ_{23}, γ_{31}) defined as

$$\gamma_{ij} = D_i u_j + D_j u_i = \epsilon_{ij} + \epsilon_{ji} \qquad (i \neq j) \tag{14}$$

Thus, a shearing stress is given as a result of (12) by

$$\tau_{12} = \frac{\partial W}{\partial \gamma_{12}}$$

when the terms in τ_{12} and τ_{21} have been collected together in view of (11). But it is also correctly given by (13) when the shearing strains are expressed symmetrically in terms of the ϵ_{ij} as in (14). In this case the symmetry (11) of the stress tensor is contained in (13).

In most applications material linearity is also postulated: the strain energy density is written as a quadratic form

$$W = \frac{1}{2} C_{ij}^{mn} \epsilon_{ij} \epsilon_{mn} \tag{15}$$

with a set of elastic moduli verifying the symmetry conditions

$$C_{ij}^{mn} = C_{ji}^{mn} = C_{ij}^{nm} = C_{mn}^{ij} \tag{16}$$

The matrix of this quadratic form is positive definite. Hence, the stress-strain relations

$$\tau_{ij} = C_{ij}^{mn} \epsilon_{mn} \tag{17}$$

can be inverted to yield

$$\epsilon_{mn} = F_{mn}^{pq} \tau_{pq} \tag{18}$$

Substitution of (18) into (17) shows that

$$C^{mn}_{ij} F^{pq}_{mn} = \delta^p_i \delta^q_j \tag{19}$$

and conversely, substitution of (17) into (18) that similarly

$$F^{pq}_{mn} C^{ij}_{pq} = \delta^i_m \delta^j_n \tag{20}$$

It appears then that the quadratic form (15) is numerically equal to

$$\phi = \frac{1}{2} F^{rs}_{pq} \tau_{rs} \tau_{pq} \tag{21}$$

which will be called the stress-energy density, with the property

$$\epsilon_{pq} = \frac{\partial \phi}{\partial \tau_{pq}} \tag{22}$$

This duality between W and Φ applies more generally whenever the stress-strain relations (13) can be solved in principle for the strains; that is whenever the Hessian matrix

$$\left| \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{pq}} \right|$$

is nonsingular in the range of strains that occur. The complementary energy density ϕ is then that function of the stresses defined by the Legendre transformation

$$\phi(\tau_{ij}) = \tau_{ij} \epsilon_{ij} - W \tag{23}$$

Discretization of the continuum problem can be introduced in many different ways and leads to a wide variety of mathematical models for finite element analysis.

The most widely used is the discretization of the sole displacement field; the discretization of the strain field following through rigorous application of (2). The whole weight of the approximation falls in this case on the equilibrium equations (8) that are only averaged as in a Galerkin process on each finite element region by the shaping (or weighting) functions chosen for the displacement field. This procedure minimizes the functional "total energy" (strain energy plus potential energy of applied loads) with respect to the finite displacement degrees of freedom.

A dual procedure is the discretization of the sole stress field, wherein one complies with the requirement that equilibrium should always remain satisfied. The discretized strain field following from rigorous application of the stress-strain relations (22) does not in general comply with equations (2). In other words the integrability conditions for the existence of a displacement field are generally not satisfied. The best approximation corresponds to a minimum of the functional "complementary energy" (stress-energy plus potential energy of the loads reacting against imposed displacements) and corresponds to a Galerkin procedure applied to equations (2). As a result the picture of the displacement field is "blurred;" it is only known through the values of a finite number of functionals of that field. This feature is probably responsible for the relative lack of

success of corresponding equilibrium models of finite elements, although in conjunction with displacement models they provide a powerful tool to assess energy convergence.

MATRIX FORMULATION OF THE BASIC VARIATIONAL PRINCIPLES

Displacements, body forces, surface tractions, strains and stresses can be ranged in row vectors, presented here as transposed column vectors:

$$u' = (u_1 u_2 u_3)$$

$$X' = (X_1 X_2 X_3)$$

$$t' = (t_1 t_2 t_3)$$

$$\epsilon' = (\epsilon_{11} \epsilon_{22} \epsilon_{33} \quad \gamma_{12} \gamma_{23} \gamma_{31}) \qquad (\gamma_{ij} = \gamma_{ji})$$

$$\tau' = (\tau_{11} \tau_{22} \tau_{33} \quad \tau_{12} \tau_{23} \tau_{31}) \qquad (\tau_{ij} = \tau_{ji})$$

Then, with the help of a matrix differential operator

$$D' = \begin{pmatrix} D_1 & 0 & 0 & D_2 & 0 & D_3 \\ 0 & D_2 & 0 & D_1 & D_3 & 0 \\ 0 & 0 & D_3 & 0 & D_2 & D_1 \end{pmatrix}$$

and a corresponding matrix of direction cosines for the outward normal

$$N' = \begin{pmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{pmatrix}$$

the basic equations of linear elasticity theory obtained in the previous section take the following forms

$$\epsilon = Du \qquad \text{strain-displacement relations} \tag{24}$$

$$D'\tau + X = 0 \qquad \text{volume equilibrium equations} \tag{25}$$

$$t = N'\tau \qquad \text{surface equilibrium equations} \tag{26}$$

Linear stress-strain relations of type (17) or (18)

$$\tau = H\epsilon \qquad \text{or} \qquad \epsilon = H^{-1}\tau \qquad (H = H') \tag{27}$$

involve a positive definite matrix H of elastic moduli.

Consider now a simply connected domain R, bounded by a surface ∂R, on which all surface tractions are supposedly specified. The total energy is the functional

$$f(u) = \int_R WdR - \int_R u'XdR - \int_{\partial R} u'tdS \tag{28}$$

where the strain energy density is calculated in terms of displacements as

$$W = \frac{1}{2}(Du)'H(Du) \tag{29}$$

Let ηv denote a perturbation of the displacement field, where η is a small parameter; then

$$f(u + \eta v) = f(u) + \eta a(u,v) + \eta^2 \int_R \frac{1}{2}(Dv)'H(Dv)dR$$

with

$$a(u,v) = \int_R (Dv)'H(Du)R - \int_R v'XdR - \int_{\partial R} v'tdS \tag{30}$$

Noting that $H(Du) = \tau$ is the stress field associated with the displacement field u and integrating by parts

$$a(u,v) = -\int_R v'(D'\tau + X)dR + \int_{\partial R} v'(N'\tau - t)dS \tag{31}$$

Obviously this expression vanishes for any displacement perturbation if the stresses τ satisfy the equilibrium equations (25) and (26). Conversely, if $a(u,v)$ vanishes for all perturbations belonging to a sufficiently arbitrary class, the equilibrium equations must be satisfied. Thus $f(u)$ has a relative minimum when and only when the associated stress field is in equilibrium with the prescribed loads X and t.

In such a case it must be observed that

$$\begin{aligned}\int_R WdR &= \frac{1}{2}\int_R (Du)'H(Du)dR = \frac{1}{2}\int_R (Du)'\tau dR \\ &= -\frac{1}{2}\int_R u'D'\tau dR + \frac{1}{2}\int_{\partial R} u'N'\tau dS \\ &= \frac{1}{2}\int_R u'XdR + \frac{1}{2}\int_{\partial R} u'tdS\end{aligned} \tag{32}$$

a result known as Clapeyron's theorem. Substituted into (28) it gives the following equivalent result: if the stresses associated with u are in equilibrium

$$f(u) = -\frac{1}{2}\int_R (Du)'H(Du)dR$$

Clapeyron's theorem furnishes a direct proof of the uniqueness of the displacement field whose associated stresses are in equilibrium with prescribed loads X and t. For if u and w are two such fields, the field u-w has stresses satisfying homogeneous equilibrium conditions ($X \equiv 0$, $t \equiv 0$). Then, from Clapeyron's theorem

$$\int_R \{D(u - w)\}'H\{D(u - w)\}dR = 0$$

and, by virtue of the positive definiteness of H,

$$D(u - w) = 0$$

This shows that the strain field is uniquely determined. For displacements themselves u and w can only differ by a rigid body displacement field with small rotations; thus, to that extent, the solution is to the problem is unique. Complete uniqueness occurs when, instead of specifying traction forces on the entire boundary, enough displacement specifications are introduced to prevent all rigid body movements. It should also be observed that, whenever rigid body motions are allowed, the prescribed loads must satisfy global equilibrium conditions. Even multiply-connected domains R can be subdivided in a set of simple connected sub-domains E_i, the finite elements. All the preceding considerations apply with the addition of transition conditions at the interfaces; they appear when performing the integration by parts in (30) as additional surface integrals. Denoting by I_+ and I_- both faces of a given interface I between two adjacent finite elements, a typical additional term in (31) is

$$\int_{I_+} v_+'N_+'\tau_+ dS + \int_{I_-} v_-'N_-'\tau_- dS$$

Physical integrity of the structure implies that the displacement field be continuous at the interface

$$u_+ = u_- \tag{33}$$

and, similarly, only perturbation fields respecting structural integrity are allowed, whence

$$v_+ = v_- = v$$

Transitional equilibrium conditions at the interface are then obviously

$$N_+'\tau_+ + N_-'\tau_- = 0$$

or

$$N_+'(\tau_+ - \tau_-) = 0 \tag{34}$$

if due account is taken of the opposite character of the outward normals. If advantage is taken of the presence of the interface to apply external traction forces on it, those conditions generalize to

$$N_+'(\tau_+ - \tau_-) = t \tag{35}$$

It is easily shown that, provided H is continuous at an interface, the continuity of u and the transition conditions (34) imply continuity of the stress tensor and the strain tensor.

The principle of minimum total potential

$$f(u) = \text{minimum} \tag{36}$$

and the transitional conditions (33) provide a natural basis for a piecewise discretization of the displacement field in finite elements.

The dual principle involves a functional

$$g(\tau) = \int_R \phi dR - \int_{\partial R} u'N'\tau dS \tag{37}$$

$$\phi = \frac{1}{2}\tau'H^{-1}\tau \tag{38}$$

based solely on a stress field τ, the boundary specifications consisting in prescribing the displacements u everywhere on $\partial\mathcal{R}$. Moreover, only such stresses are allowed that satisfy the volume equilibrium equations (25) with prescribed body loads. If $\eta\sigma$ denotes a stress perturbation, we have

$$g(\tau + \eta\sigma) = g(\tau) + \eta b(\tau,\sigma) + \eta^2 \int_R \frac{1}{2}\sigma'H^{-1}\sigma dR$$

with

$$b(\tau,\sigma) = \int_R \sigma'H^{-1}\tau dR - \int_{\partial R} u'N'\sigma dS \tag{39}$$

In order that $\tau + \eta\sigma$ should also satisfy the equilibrium conditions (25), we must have

$$D'\sigma = 0 \tag{40}$$

We remove this constraint on the σ field by means of a vector lagrangian multiplier v, note that $H^{-1}\tau = \varepsilon$ is the strain field associated with τ and rewrite the bilinear form as

$$b(\tau,\sigma) = \int_R (\sigma'\varepsilon + v'D'\sigma)dR - \int_{\partial R} u'N'\sigma dS \tag{41}$$

Then, after an integration by parts,

$$b(\tau,\sigma) = \int_R \sigma'(\epsilon - Dv) + \int_{\partial R} (v' - u')N'\sigma dS \tag{42}$$

Obviously this expression vanishes for all perturbations if the stress field τ is such that its associated strain field is compatible; that is, if a displacement field v exists such that $\epsilon = Dv$ and $v = u$ on ∂R.

Conversely, if the expression vanishes for all perturbations belonging to a sufficiently arbitrary set, a displacement field with such properties must exist. Thus $g(\tau)$ has a relative minimum when the associated strain field is compatible.

In such a case we note that

$$\int_{\partial R} u'N'\tau dS = \int_{\partial R} v'N'\tau dS$$

and, since v is defined in R while τ is in equilibrium with X,

$$\int_{\partial R} u'N'\tau dS = \int_R v'D'\tau dR + \int_R \tau' Dv dR$$

$$= -\int_R v'X dR + \int_R \tau'\epsilon dR$$

Finally, noting that

$$\int_R \tau'\epsilon dR = \int_R \tau' H^{-1}\tau dR = 2\int_R \phi dR$$

the result can be expressed again in the form of a Clapeyron theorem

$$\int_R \phi dR = \frac{1}{2}\int_R v'X dR + \frac{1}{2}\int_{\partial R} u'N'\tau dS \tag{43}$$

This furnishes a proof of uniqueness. For if τ and θ are two stress fields in equilibrium with X, v a displacement field associated to τ and w one associated to θ, both being equal to the specified u on ∂R, the field $\tau - \theta$ is in equilibrium without body loads $(X=0)$ and its associated displacement field $v - w$ vanishes on $\partial R (u = 0)$. Hence, by application of Clapeyron's theorem,

$$\int_R (\tau - \theta)' H^{-1} (\tau - \theta) dR = 0$$

and, by virtue of the positive definiteness of H^{-1},

$$\tau - \theta \equiv 0$$

In presence of finite elements the role of the transition conditions is obviously interchanged. Conditions (34) are now the a priori conditions completing the a priori equilibrium conditions to be satisfied by the stress field, while conditions (33) become natural transition conditions implemented by the variational principle

$$g(\tau) = \min. \tag{44}$$

MATHEMATICAL MODELS OF CONFORMING DISPLACEMENT ELEMENTS

The displacement field is discretized by assuming

$$u(x) = P(x)a \tag{45}$$

within the domain E of the finite element and on its boundary ∂E. The functions constituting the elements of the 3xn matrix P(x) are usually chosen to be polynomials. The parameters α_i

$$(\alpha_1 \alpha_2 \ldots \alpha_n) = a'$$

are the unknowns to be determined so that equilibrium be approximated in some "best" sense; in our case by minimization of the total potential. Normally the discretization is such that (45) contains all rigid body degrees of freedom with arbitrary amplitude and

$$\epsilon = Du = (DP(x))a$$

contains at least an arbitrary state of uniform strain.

Shaping Functions.

To satisfy rigorously the transition conditions (33) between elements, the behavior of u(x) is examined on each face $\partial_\alpha E$ of the boundary ∂E; e.g. by using a parametric representation of $\partial_\alpha E$. This analysis furnishes the number of degrees of freedom of u(x) on each $\partial_\alpha E$ and a generalized displacement coordinate is attached to each such degree of freedom. If the column matrix q_α denotes the generalized coordinates pertaining to $\partial_\alpha E$, the knowledge of q_α determines completely u(x) on $\partial_\alpha E$. The analysis furnishes more precisely a relationship

$$q_\alpha = M_\alpha a \tag{46}$$

between generalized coordinates and parameters and an identification

$$P(x)a \equiv Q_\alpha(x) q_\alpha \qquad \text{for} \qquad x \in \partial_\alpha E$$

valid for any a, so that

$$P(x) = Q_\alpha(x) M_\alpha \qquad \text{for} \qquad x \in \partial_\alpha E$$

The elements of $Q_\alpha(x)$ are the "boundary displacement modes" on $\partial_\alpha E$. The set of relationships (46) for all the faces results in the matrix relation

$$q = Ma \tag{47}$$

where q contains all the boundary generalized displacements. Naturally some of the elements of q are common to several faces in order to have uniqueness for displacements at vertices or along ribs.

As a result of the definition of the generalized boundary coordinates the transition conditions (33) are satisfied along a face $\partial_\alpha E$, provided the same boundary modes exist for the adjacent element, and the associated q_α take the same value across the interfaces. If the knowledge of a determines completely the displacements on the boundary ∂E of the element, it is not necessarily true that the elements of q are independent. In other words it is not always true that some value of a always corresponds to an arbitrarily given q. This raises the question of the possibility of inverting (47). The simplest, but not unfrequent case, is that of a nonsingular matrix M, implying in particular that the number of boundary displacement coordinates n (q) be equal to the number n(a) of parameters. Then, inverting (47) and substituting into (45), there comes another definition of the discretization of the displacement field:

$$u(x) = Q(x)q \tag{48}$$

where

$$Q(x) = P(x)M^{-1} \tag{49}$$

are the so-called "shaping functions". In this case the independent boundary coordinates are sufficient to determine the displacements in the interior.

If $\hat{q}_\alpha$ is defined as the result of setting into q the elements of q_α equal to zero, then by construction

$$Q(x)\hat{q}_\alpha = 0 \qquad \text{for } x \in \partial_\alpha E \qquad \text{and} \qquad \text{any } \hat{q}_\alpha$$

a property that proves useful if one wishes to avoid the literal inversion of M and construct the shaping functions directly.

Bubble Functions.

A more general, but still simple, case is that where M is a non square matrix but of rank n(q). In other words $n(q) < n(a)$ and the rows of M are linearly independent. Then, the homogeneous problem $Ma = 0$ has exactly $n(a) - n(q)$ nontrivial and linearly independent solutions and (47) has, for arbitrary q, a general solution

$$a = Qq + Bb \tag{50}$$

Thus, the boundary coordinates are still independent, and the first part of (50) is a particular solution of (47), while B is a $n(a) \times (n(a) - n(q))$ matrix with linearly independent columns, each of which is a solution of the homogeneous problem; thus

$$MB = 0 \tag{51}$$

Substitution of (50) into (45) gives now

$$\mathbf{u}(x) = Q(x)q + B(x)b \tag{52}$$

with

$$Q(x) = P(x)Q \tag{53}$$

$$B(x) = P(x)B \tag{54}$$

The $B(x)$ are "bubble functions", so called because they represent a displacement field in the interior of the element that does not displace the boundary ($q = 0$); correspondingly each element of b is a bubble coordinate that governs the scale of the bubble. We note here that the shaping functions $Q(x)$ are determined except for the addition of arbitrarily scaled bubble functions.

Such a change of definition of shaping functions corresponds to the choice of another particular solution for (47); or to a change of bubble coordinates:

$$b = Uq + c$$

where U is an arbitrary $n(b) \times n(q)$ matrix. The bubble coordinates can be so devised that they vanish for a rigid body type displacement of the boundaries; we say then that they are "relatively" defined and this brings some simplifications in some of the later computations.

The Case of Linear Dependence of Boundary Coordinates.

The linear independence of the rows of M that was postulated in the previous section is equivalent to saying that the homogeneous adjoint problem

$$M'g = 0$$

has but the trivial solution $g = 0$. Assume now instead that this problem has a general solution $g = Yy$, where Y is a $n(q) \times n(y)$ matrix of linearly independent columns and y is arbitrary. Then

$$M'Y = 0 \qquad \text{or} \qquad Y'M = 0 \tag{55}$$

The necessary and sufficient condition for (47) to be solvable for a is that the boundary displacement coordinates should satisfy

$$Y'q = 0 \tag{56}$$

By suitably ordering the elements in q

$$q' = (p'r')$$

those conditions can be expressed as the dependence of $n(y)$ of the boundary displacements on the others

$$r = Rp \tag{57}$$

and the system

$$\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} I \\ R \end{pmatrix} p = Ma$$

has always a solution of the form

$$a = Qp + Bb \tag{58}$$

Now

$$u(x) = Q(x)p + B(x)b \tag{59}$$

with the same formal definitions (53) and (54), but the internal displacement field is entirely described in terms of a reduced set of independent boundary coordinates and, when present, bubble coordinates. It will be shown later how such an expansion, representing the most general case, can be used in the global discretization problem.

Generalized Forces.

In the principle of minimum total potential appear virtual work expressions for the external loads that introduce natural definitions for the generalized loads. The case of dependent boundary displacements requiring special treatment, we shall use the expansion (52). The virtual work of prescribed body loads gives

$$\int_E u'X dE = q'f_q^* + b'f_b^* \tag{60}$$

where

$$f_q^* = \int_E Q'(x)X dE \qquad f_b^* = \int_E B'(x)X dE \tag{61}$$

are, by definition, the generalized body loads conjugate respectively to the generalized displacements q and b. They are linear functionals, wherein the actual body loads are weighted by the shaping functions and bubble functions. Similarly, through the virtual work of surface tractions on the boundary

$$\int_{\partial E} u't dS = q'g_q^* \tag{62}$$

generalized boundary loads

$$g_q^* = \int_{\partial E} Q'(x)t dS \tag{63}$$

are defined as linear functionals conjugate to the boundary displacements. Here bubble functions make no contributions since they vanish identically on the boundary. The approximate solution to the displacement field will only depend on the values of the linear functionals defined in (61) and (63) and infinitely many actual distributions of body loads and surface tractions can produce the same approximate answer. This characteristic situation is summarized by saying that the knowledge of the generalized forces is only a "weak" knowledge of the actual load distribution and recognized by the "starred" notation.

Element Stiffness Matrix.

The strain energy of the element can be calculated directly from the original discretization (45) in terms of the parameters

$$\int_E W dE = \frac{1}{2}\int_E (Du)'H(Du)dE = \frac{1}{2}\, a'K_{aa}a \tag{64}$$

with

$$K_{aa} = \int_E \{DP(x)\}'H\{DP(x)\}dE \tag{65}$$

and thereafter transformed through (50) in a quadratic form in the generalized displacements

$$\frac{1}{2}\, q'K_{qq}q + q'K_{qb}b + \frac{1}{2}\, b'K_{bb}b \tag{66}$$

with

$$K_{qq} = Q'K_{aa}Q \qquad K_{qb} = Q'K_{aa}B \qquad K_{bb} = B'K_{aa}B$$

It can also be obtained directly in the form (66) by performing integrations on the basis of the expansion (52) in shaping function and bubble functions;

$$K_{qq} = \int_E \{DQ(x)\}'H\{DQ(x)\}dE$$

$$K_{qb} = \int_E \{DQ(x)\}'H\{DB(x)\}dE$$

$$K_{bb} = \int_E \{DB(x)\}'H\{DB(x)\}dE$$

This last stiffness matrix is never singular because a bubble function displacement necessarily involves nonzero deformations; this allows the bubble coordinates to be eliminated at the element level by minimization of the total energy:

$$\frac{1}{2}\, q' K_{qq} q + q' K_{qb} b + \frac{1}{2}\, b' K_{bb} b - q' \{f_q^* + g_q^*\} - b' f_b^* \quad \text{minimum}$$

The minimum conditions are

$$K_{qq} q + K_{qb} b = f_q^* + g_q^* \tag{67}$$

$$K_{bq} q + K_{bb} b = f_b^* \qquad (K_{bq} = K_{qb}') \tag{68}$$

Solving the last with respect to the bubble coordinates and substituting into the first, produces the stiffness relation

$$Kq = f^* + g^* \tag{69}$$

with

$$K = K_{qq} - K_{qb} K_{bb}^{-1} K_{bq} \tag{70}$$

$$f^* = f_q^* - K_{qb} K_{bb}^{-1} f_b^* \tag{71}$$

$$g^* = g_q^*$$

If K_{qb} and consequently K_{bq} are zero, equations (67) and (68) are actually uncoupled and (67) is equivalent to (69). This situation corresponds to a particular choice of the shaping functions that is, however, difficult to guess. The elimination that was just indicated is fully equivalent to uncoupling by means of the change of bubble coordinates

$$b = -K_{bb}^{-1} K_{bq} q + c$$

Elastic Properties of The Assembled Structure.

After elimination of bubble coordinates, each element has discretized elastic properties with respect to boundary displacements characterized by relations of type (69)

$$K_E q_E = f_E^* + g_E^* \tag{72}$$

where it should be noted that, while both f_E^* and g_E^* are external generalized loads for the element, the g_E^* are internal loads at the structurally assembled level. This is also clearly indicated by the fact that the body loads are given data from which the values of the generalized loads f_E^* are actually known, while the traction surfaces defining the g_E^* are unknown. The principle for assembling the elements consists in stating which boundary coordinates must have common values at interfaces, implementing, as was shown earlier, the exact transition conditions (33). If w denotes the column matrix of common or "nodal" displacements, the allocation of generalized displacements for each element is expressible by means of Boolean, or incidence matrices L_E

$$q_E = L_E w \tag{73}$$

Since the virtual work of internal loads at the structural level vanishes, we can equate the sum of virtual works of all forces on individual elements to the virtual work of external forces on the assembled structure:

$$\sum_E q_E'(g_E^* + f_E^*) = \sum_E q_E' f_E^* + w' y_w^*$$

The generalized forces y_w^* take into account specified loads and unknown reaction loads along the boundary ∂R of the assembled structure as well as possible external loads applied at interfaces. After cancelling the common terms and introducing (73)

$$w' \sum_E L_E' g_E^* = w' y_w^*$$

and, since this holds for any nodal displacements,

$$y_w^* = \sum_E L_E' g_E^* \tag{74}$$

Substituting g_E^* from (72) and rearranging terms

$$y_w^* + \sum_E L_E' f_E^* = \sum_E L_E' K_E q_E = \left(\sum_E L_E' K_E L_E \right) w$$

The elastic properties of the assembled structure are thus obtained in terms of the nodal displacements w, their conjugate external loads

$$y^* = y_w^* + \sum_E L_E' f_E^* \tag{75}$$

and a master stiffness matrix

$$K = \sum_E L_E' K_E L_E \tag{76}$$

The linear system

$$r^* = Kw \tag{77}$$

can then be solved for the unknown w after due account is taken of those nodal displacements that might be prescribed.

Dependent Boundary Displacements. Superelements.

Turning back to the general case of elements with dependent boundary displacements, two types of procedures are available. As suggested by (59), generalized loads can be defined as in the case pertaining to generalized forces conjugate to bubble coordinates and to the independent boundary displacements, so that the virtual work at element level can be written as

$$p'(f_p^* + g_p^*) + b'f_b^*$$

Similarly an element stiffness matrix

$$\begin{pmatrix} K_{pp} & K_{pb} \\ K_{bp} & K_{bb} \end{pmatrix}$$

can be calculated from a transformation of (65) through (58) and the bubble coordinates eliminated to produce a stiffness relation similar to (69)

$$Kp = f^* + f^*$$

not containing only the independent generalized displacements and their conjugate loads.

The first procedure consists in assembling the elements by allocating the complete boundary coordinates to nodal displacements as in (73). We will however denote here this operation by

$$q_\varepsilon = \Lambda_\varepsilon w$$

reserving the notation L_ε for the Boolean matrix that, in particular, localizes the independent coordinates:

$$p_{(miu)\varepsilon} = L_\varepsilon w$$

Then formulas (75) and (76) remain valid to calculate the external loads and master stiffness matrix. However, the linear system (77) is completed by the set of homogeneous constraints

$$Y_\varepsilon' q_\varepsilon = (Y_\varepsilon' \Lambda_\varepsilon) w = 0 \tag{78}$$

resulting from the dependency conditions (56) expressed for each element.

The second procedure consists, if possible, in assembling a small group of elements into a superelement with independent boundary coordinates. One has here to distinguish between the nodal coordinates w_e appearing at the boundary of the superelement and the internal nodal coordinates w_i. Thus, for the allocation of displacements in the elements forming the group

$$q_\varepsilon = F_\varepsilon w_e + G_\varepsilon w_i$$

and, in particular,

$$p_{(\min)E} = H_E w_e + J_E w_1$$

The set of dependency constraints is

$$\begin{pmatrix} Y_1' F_1 \\ Y_N' F_N \end{pmatrix} w_e + \begin{pmatrix} Y_1' G_1 \\ Y_N' G_N \end{pmatrix} w_1 = 0$$

The superelement is possible if the matrix that premultiplies w_1 has linearly independent rows. For then the set of dependency constraints can be solved for w_1

$$w_1 = W w_e + A\hat{b} \qquad \hat{b} \text{ arbitrary} \tag{79}$$

The last term represents the general solution for $w_e = 0$ and clearly each (linearly independent) column of A represents a possible "assembled bubble mode" of the superelement. Turning back to (58), we can express the parameters of each constitutive element as

$$a_E = Q_E p_E + B_E b_E = Q_E H_E w_e + Q_E J_E w_1 + B_E b_E$$

$$= Q_E (H_E + J_E W) w_e + B_E b_E + Q_E J_E A\hat{b} \tag{80}$$

that is in terms of the independent boundary displacements of the superelement, a set of local bubble coordinates and a set of assembled bubble coordinates. From then on the situation is comparable to that of the section pertaining to bubble functions and the only difference in further treatment is the necessity of extending integrals to the union of the domains E or to the boundary of the superelement. External forces applied at internal interfaces can be accounted for but are generally not present.

The difficulties encountered with conforming displacement models for Kirchhoff plate bending theory originated the search for superelements. A first one is an arbitrary quadrilateral subdivided into four triangles by its diagonal and using a complete cubic deflection field [6]. The second is an assemblage of 3 triangles also using a complete cubic deflection field [7].

MATHEMATICAL MODELS OF DIFFUSIVE EQUILIBRIUM ELEMENTS

The stress field alone is discretized in the form

$$\tau(x) = R(x)c + S(x)s \tag{81}$$

where c and s are column matrices of stress parameters. Each column of R(x) is related to a nonzero body loading mode, so that the set of body loading modes is

$$X = -D'R(x)c \tag{82}$$

while the stresses generated by S(x) are in equilibrium without body loads

$$D'S(x) = 0 \tag{83}$$

general the elements of $R(x)$ and $S(x)$ are taken to be polynomials of low degree and $S(x)$ generates at least an arbitrary state of uniform stress.

Generalized Loads.

Along each face $\partial_\alpha E$ of the boundary, the assumption (31) generates a set of independent surface traction modes, to each of which a generalized load is attached. The chosen set of generalized loads constitutes a column matrix g_α. Hence, if N_α denotes the operator N associated with the outward normal along $\partial_\alpha E$ and $T_\alpha(x)$ the matrix of identified surface traction modes, the following identity must hold for arbitrary c and s:

$$N_\alpha' R(x)c + N_\alpha' S(x)s \equiv T_\alpha(x) g_\alpha \qquad \text{for } x \in \partial_\alpha E \tag{84}$$

In fact, each definition of a generalized load, as intensity factor of a surface traction mode, introduces its dependence on the values of c and s; whereby we are provided with a relation of the form

$$g_\alpha = G_\alpha c + C_\alpha s \tag{85}$$

and the identity (84) corresponds to the identities

$$N_\alpha' R(x) \equiv T_\alpha(x) G_\alpha \qquad N_\alpha' S(x) \equiv T_\alpha(x) C_\alpha \qquad \text{for} \qquad x \in \partial_\alpha E \tag{86}$$

The knowledge of g_α determines completely and uniquely the surface traction distribution along $\partial_\alpha E$.

Denote by g the column matrix of all the sub-columns g_α; then the set of relations (85) for the complete boundary ∂E is of the form

$$g = Gc + Cs \tag{87}$$

where G and C are called the "load connection matrices". This relation is similar to (47) but, in contrast with it, one element of g may only belong to one face of the boundary.

Generalized Displacements.

The generalized boundary displacements are conjugate to the generalized loads by virtual work consideration. Along a given face

$$\int_{\partial_\alpha E} u' t dS = \int_{\partial_\alpha E} u' N_\alpha' \tau dS = \int_{\partial_\alpha E} u' \{N_\alpha' R(x)c + N_\alpha' S(x)s\} dS$$

Availing ourselves of the identity (84) along that face

$$\int_{\partial_\alpha E} u' t dS = \{\int_{\partial_\alpha E} u' T_\alpha(x) dS\} g_\alpha = (q_\alpha^*)' g_\alpha$$

Consequently the generalized displacements along $\partial_\alpha E$ are linear functionals of the unknown displacements; the surface traction modes playing the role of weighting functions:

$$q^*_\alpha = \int_{\partial_\alpha E} T'_\alpha(x) u \, dS \tag{88}$$

The virtual work of all surface tractions will finally be represented by

$$\int_{\alpha E} u' t \, dS = g' q^* \tag{89}$$

where q^* is the column matrix of all sub-columns q^*_α.

Similarly, if body loading modes are considered, conjugate generalized displacements can be defined by analyzing the virtual work

$$\int_E u' X \, dE = -\{\int_E u' D' R(x) \, dE\} c = (b^*)' c \tag{90}$$

To each body loading coordinate in c a conjugate generalized displacement in b^* is thus attached and

$$b^* = -\int_E \{D' R(x)\}' u \, dE \tag{91}$$

As in general the discretized deformation field

$$\varepsilon(x) = H^{-1} \tau(x)$$

will not be integrable, it is only normal to expect that our knowledge of the displacement field will remain the "weak" one provided by the values taken by some set of linear functionals. Compared to displacement discretization the situation with respect to strong and weak knowledge of fields is reversed.

Application of the Principle of Minimum Complementary Energy.

We wish to extend this principle to the situation where the body loads, instead of being fixed, are variable. Such a situation is suggested by (82) where the body loads depend on adjustable parameters c. Those can be considered as generalized external forces. Let ψ denote the stress energy and consider its variation

$$\delta\psi = \int_E \delta\phi \, dE = \int_E \tau' H^{-1} \delta\tau \, dE = \int_E \varepsilon' \delta\tau \, dE$$

The variations $\delta\tau$ are now constrained by the equilibrium equations

$$D'\tau = -X = +D'R(x)c$$

so that

$$D'\delta\tau - D'R(x)\delta c = 0$$

The constraints are again removed by a lagrangian vector multiplier v, and we manipulate the resulting expression

$$\delta\Psi = \int_E (\epsilon'\delta\tau + v'D'\delta\tau - v'D'R(x)\delta c)dE$$

by partial integration on the second term

$$\delta\Psi = \int_E \delta\tau'(\epsilon - Dv)dE + \int_{\partial E} v'N'\delta\tau dS - \{\int_E v'D'R(x)dE\}\delta c$$

If c is kept constant ($c = 0$), we fall back on the original situation and recall that $\epsilon = Dv$ and $v = u$ prescribed on ∂E.

The result is then the vanishing of the first variation of the complementary energy

$$\delta(\Psi - \int_{\partial E} u'N'\tau dS) = 0$$

and equivalent to the treatment given in the section pertaining to matrix formulation of the basic variational principles. If c is variable, we have now the more general result

$$\delta V = \int_{\partial E} v'N'\delta\tau dS - \{\int_E u'D'R(x)dE\}\delta c \qquad (92)$$

This is now applied to the discretization (81). The stress energy is

$$\Psi = \frac{1}{2}\int_E \tau'H^{-1}\tau dE = \frac{1}{2}\, c'F_{cc}C + c'F_{cs}s + \frac{1}{2}\, s'F_{ss}s \qquad (93)$$

a quadratic positive definite form with flexibility matrices

$$F_{cc} = \int_E R'(x)H^{-1}R(x)dE$$

$$F_{cs} = \int_E R'(x)H^{-1}S(x)dE$$

$$F_{ss} = \int_E S'(x)H^{-1}S(x)dE$$

The first right hand side term becomes

$$\int_{\partial E} u'N'\delta\tau dS = \{\int_{\partial E} u'N'R(x)dS\}\delta c + \{\int_{\partial E} u'N'S(x)dS\}\delta s$$

or, considering the definitions given for the generalized boundary loads and displacemer

$$\int_{\partial E} u'N'\delta\tau dS = (q^*)'\delta g = (q^*)'(G\delta c + C\delta s)$$

Finally, the last term can be transformed by (91) to yield

$$-\{\int_E u'D'R(x)dE\}\delta c = (b^*)'\delta c \qquad (9$$

Substitution of (93), (94) and (95) into (92) and identification of the coefficients of the variations δs and δc produces

$$F_{ss}s + F_{sc}c = C'q^* \qquad (F_{sc} = F'_{cs}) \qquad (96)$$

$$F_{cs}s + F_{cc}c = G'q^* + b^* \qquad (97)$$

Equation (96) alone, solved for s and substituted into (87) furnishes a stiffness relation completely analogous to that of a displacement type of element

$$g + (CF_{ss}^{-1}F_{sc} - G)c = (CF_{ss}^{-1}C')q^* \qquad (98)$$

The body loads are here supposedly given: that is, c is known, and appears as external generalized

loads given by the second term. The stiffness matrix of the element involves only the C load connection matrix.

Equation (97), which is in the nature of a Castigliano formula allows, if desired, an estimation of the generalized displacement b^*, conjugate to the generalized body loads. Taking (96) into account it can also be written in the form

$$b^* = (F_{cc} - F_{cs}F_{ss}^{-1}F_{sc})c + (F_{cs}F_{ss}^{-1}C' - G')q^*$$

Kinematical Freedoms of Equilibrium Elements.

Stiffness matrices are always singular because the homogeneous system $Kq = 0$ has, amongst its nontrivial solutions, at least the rigid body freedoms of the element

$$Kq_{(i)} = 0 \qquad (i = 1,2,\ldots,r) \tag{99}$$

When the stiffness matrix derives from a displacement model, this property is a direct consequence of the fact that such rigid body modes were built-in on purpose into the discretization (45). Denoting by $a_{(i)}$ the r linearly independent columns, defined except for scale, that represent a rigid body displacement field

$$u_{(i)}(x) = P(x)a_{(i)}$$

we note that, by definition,

$$DP(x)a_{(i)} \equiv 0$$

and, in view of the definition (65), that consequently

$$K_{aa}a_{(i)} = 0 \tag{100}$$

Now in relation (50), the combined matrix (QB) is non singular so that

$$a_{(i)} = Qq_{(i)} + Bb_{(i)}$$

can be solved uniquely for the $q_{(i)}$ and $b_{(i)}$ yielding another representation of the rigid body modes; in particular of course $q_{(i)} = Ma_{(i)}$.

From their relationship with K_{aa} we find for the other stiffness matrices

$$K_{qq}q_{(i)} + K_{qb}b_{(i)} = 0$$

$$K_{bq}q_{(i)} + K_{bb}b_{(i)} = 0$$

and, finally (99) after elimination of the $b_{(i)}$. In this case there are no other nontrivial solutions to $Kq = 0$.

In the case of equilibrium models, the solutions of $Kq^* = 0$ are identical to those of $C'q^* = 0$ because F^{-1} is positive definite. Here the $q^*_{(i)}$ vectors, representing rigid body modes, are obtained directly by inserting fields $u = v_{(i)}(x)$ of rigid body displacements into the definitions (88), and into definition (91) for the corresponding $b^*_{(i)}$. Then, starting with the equilibrium equations satisfied by any $\tau(x)$ field in (81)

$$D'\tau(x) - D'R(x)c = 0$$

premultiplying by $v'_{(1)}(x)$ and integrating over E, we obtain after integrating by parts on the first term

$$-\int_E \tau' D v_{(1)} dE + \int_{\partial E} v'_{(1)} N' \tau dS$$
$$-\{\int_E v'_{(1)} D'R(x) dE\} c = 0$$

Now $Dv_{(1)} \equiv 0$ and the definitions (89) and (91) can be used in the other terms, giving

$$g' q^*_{(1)} + c' b^*_{(1)} = 0 \tag{101}$$

a virtual work statement of global equilibrium of generalized forces. Substituting (87) and noting that the relation holds for any b and c, there follows

$$C' q^*_{(1)} = 0 \tag{102}$$

$$b^*_{(1)} + G' q^*_{(1)} = 0 \tag{103}$$

It is now clear from (102) that the $q^*_{(1)}$ are solutions of

$$Kq^* = CF_s^{-1} C' q^* = 0$$

In "good" equilibrium elements there are no other solutions. When others exist like, $q^* = k^*$ say, and if a corresponding body displacement is taken from $b^* = -G'k^*$, we can conclude from (96) and (97) that s and c again vanish. Thus a nonrigid body type of displacement exists that produces no stresses; we can say that the model contains a kinematical deformation mode or "mechanism". The presence of mechanisms in equilibrium models presents difficulties of similar nature to the existence of dependency relations between boundary coordinates in displacement models.

It is typical of mechanisms that they can appear or are inhibited by the process of assembling elements together. It is, therefore, of interest to reconsider this process for equilibrium elements.

Structural Mechanisms.

The process of assembling can again be represented by localizing operators

$$q^*_\xi = L_\xi w^* \tag{104}$$

Equating once more the sum of virtual works of external loads on the disconnected elements and

the total virtual work of loads external to the assembled structure

$$\sum_E g_E' q_E^* = y_w' w^*$$

account was taken here that the contribution $\Sigma\, c_E' b_E^*$ is common to both sides. To maintain strict equilibrium, the generalized loads y_w conjugate to w^* and applied on the boundary of the assembled structure, or may be at some interfaces, must correspond to the exact surface tractions modes defined along the corresponding faces of elements. Substitution of (104) and identification of the coefficients of w^*, produces

$$y_w = \sum_E L_E' g_E \tag{105}$$

We note here in passing that because a given element of y_w belongs to a single interface, these equations are all individually of the type

$$\gamma_+ + \gamma_- = \eta$$

and that, since each generalized load in such a relation governs the amplitude of the same surface traction mode, exact transitional equilibrium or "diffusivity" is secured.

The g_E are now taken from the stiffness relations (98)

$$g_E = -(CF_{ss}^{-1}F_{sc} - G)_E c_E + K_E q_E$$

$$K_E = C_E (F_{ss}^{-1})_E C_E'$$

and substituted to arrive finally at the structural stiffness relations

$$y = y_w + \sum_E L_E' (CF_{ss}^{-1}F_{sc} - G)_E c_E = \Big(\sum_E L_E' K_E L_E \Big) w^* \tag{106}$$

The translation of the body loads on each element into equivalent nodal forces is clearly indicated. If one wishes to solve this linear system by the standard stiffness method, the conditions to be satisfied by the y loads should be examined; that is, the linearly indepenent solutions of the corresponding homogeneous problem must be found. Now for the assembled stiffness matrix

$$K = \sum_E L_E' K_E L_E = \sum_E (C_E' L_E)' (F_{ss}^{-1})_E (C_E' L_E) = AF^{-1}A' \tag{107}$$

where F^{-1} is a postive definite matrix constructed with the $(F_{ss}^{-1})_E$ as diagonal blocks and A is partitioned as follows:

$$A = (C_1'L_1 C_2'L_2 \ldots C_N'L_N)$$

Again the solutions of the homogeneous problem $Kw^* = 0$ are the same as those of

$$A'w^* = 0 \tag{108}$$

and can be written in the form

$$w^* = Yr \qquad r \text{ arbitrary}, \quad A'Y = 0$$

where the columns of Y are a set of linearly independent solutions. If the equilibrium elements are "good", the only solutions are the global rigid body modes of the structure. Otherwise Y also contains a set of independent mechanisms for the assembled structure. The conditions imposed on the loads for solving the set of equations $y = Kw^*$, namely

$$Y'y = 0$$

would then contain, in addition to the usual global equilibrium conditions, conditions for avoiding excitation of the mechanisms. Equilibrium superelements are combinations of a small number of elements where such conditions can be satisfied by suppressing all or part of the loads on the internal nodes.

BOUNDS TO STRAIN ENERGY

The advantage of the dual analysis, one based on conforming displacement models, the other on diffusive equilibrium elements, is the numerical assessment of energy convergence. It can first be shown that if the prescribed displacements are zero, the first approach furnishes a lower bound, the second an upper bound to strain energy. To this effect each field of stress or strain, both being always related by the elastic moduli of the material

$$\tau = H\epsilon \qquad \epsilon = H^{-1}\tau \tag{109}$$

is considered as some element m of a Hilbert space; the scalar product being defined as

$$(m_1, m_2) = \int_R \epsilon_1' H \epsilon_2 \, dR = \int_R \epsilon_2' H \epsilon_1 \, dR = (m_2, m_1)$$

or, equivalently,

$$(m_1, m_2) = \int_R \epsilon_1' \tau_2 \, dR = \int_R \epsilon_2' \tau_1 \, dR \tag{110}$$

the last equality expressing the Betti-Rayleigh reciprocity principle. The square of the norm becomes then naturally twice the strain or stress energy.

$$\pi_2 = (m,m) = \int_R \epsilon' H \epsilon dR = \int_R \tau' H^{-1} \tau dR \tag{111}$$

Furthermore one obtains through the metric properties a definition of the "distance" between two fields

$$\rho(m_1, m_2) = \|m_1 - m_2\|$$

or

$$\rho^2(m_1, m_2) = (m_1 - m_2, m_1 - m_2) \tag{112}$$

Consider now the definition of a "compatible" field, an element of the space denoted by $\tau = c$: In each finite element the strains derive from a displacement field

$$\epsilon = Du \qquad \text{in each } E \tag{113}$$

The displacement fields satisfy the transition conditions at interfaces

$$u_+ = u_- \tag{114}$$

The displacements satisfy the prescribed boundary values on $\partial_1 R$

$$u = 0 \qquad \text{on } \partial_1 R \tag{115}$$

Such compatible fields are obviously generated by conforming displacement models. Since linear combinations of compatible fields are obviously also compatible, the set of compatible fields is a subspace C of Hilbert space.

Consider next the definition of a "self-stressing field" denoted generically by $m = a$: in each finite element the stresses satisfy the homogeneous equilibrium conditions

$$D'\tau = 0 \qquad \text{in each } E \tag{116}$$

They satisfy the homogeneous transition conditions at interfaces

$$N_+'\tau_+ + N_-'\tau_- = 0 \tag{117}$$

They satisfy homogeneous boundary conditions on $\partial_2 R$

$$N'\tau = 0 \quad \text{on} \quad \partial_2 R \qquad (\partial_1 R \cup \partial_2 R = \partial R) \tag{118}$$

Such fields are obviously generated by diffusive equilibrium models. As linear combinations of self-stressing fields are themselves self-stressing fields, their set is a subspace A of Hilbert space. The most important property of subspaces C and A is that they are orthogonal. Thus, if $c \in C$ and $a \in A$

$$(c,a) = 0 \tag{119}$$

Indeed, from the definitions,

$$(c,a) = \int_R (Du)'\tau dR = \int_{U\partial E} u'N'\tau dS - \int_R u'D'\tau dR$$

Grouping the union of all boundaries ∂E in a set of interfaces plus the structural boundary ∂R, the result follows directly from (116), (114), and (117), and (115) or (118).

In order to take into account the applied loads we must introduce a particular equilibrium field a_o that satisfies the nonhomogeneous equations:

$$D'\tau + X = 0 \qquad \text{in each } E \tag{116'}$$

$$N'_+\tau_+ + N'_-\tau_- = t \qquad \text{at interfaces} \tag{117'}$$

$$N'\tau = t \qquad \text{on } \partial_2 R \tag{118'}$$

Thus, if a_o is such a field and a and arbitrary self-stressing field, the most general solution to (116'), (117') and (118') is a field $a_o + a \in (a_o + A)$.

The exact solution to our problem is a field s that lies at the intersection of subspace C and the translated subspace $A + a_o$. By an integration by parts it is easily found that

$$(c,a_o) = (c,a + a_o) = \int_{IU\partial_2 R} u'tdS + \int_R u'XdR = \langle f(a_o),u(c)\rangle \tag{120}$$

the right-hand side denoting symbolically the virtual work of the loads with which the field a_o is in equilibrium against the displacements associated with the field c. The unicity of the solution s is easily demonstrated. Should there be another solution $\hat{s}$, then $s - \hat{s}$ would at the same time be a self-stressing field and a compatible field. Hence, applying theorem (119)

$$(s - \hat{s},\ s - \hat{s}) = 0$$

from which follows

$$s - \hat{s} = 0$$

because of the positive definite character of the norm.

Similarly, since $s - a_o$ is a self-stressing field and s a compatible field,

$$(s - a_o, s) = 0$$

or

$$(s,s) = (s,a_o) = \langle f(a_o),u(s)\rangle \tag{121}$$

which is Clapeyron's theorem. The fact that the exact solution belongs to both C and $A + a_o$ leads

naturally to the concept of approaching it from both sides by minimizing the distance between a point of c of C and $a_0 + a$ of $A + a_0$:

$$\rho(a_0 + a, c) = (a_0 + a - c, a_0 + a - c) = \text{minimum}$$

$$c \in C$$

$$a \in A$$

Expanding the right hand side as follows

$$(a_0 + a, a_0 + a) + (c - a_0, c - a_0) - (a_0, a_0) - 2(c, a)$$

we can discard the third term as being constant and the last which is zero on account of (119). Thus, since the first and second terms are each positive, we obtain separately

$$\begin{aligned} (a_0 + a, a_0 + a) &= \min \\ &\quad a \in A \\ (c - a_0, c - a_0) &= \min \\ &\quad c \in C \end{aligned} \tag{122}$$

The first is the principle of minimum complementary energy, reduced here to minimum of stress energy because the boundary data on displacements are homogeneous. The second, in the equivalent form.

$$\begin{aligned} \frac{1}{2}(c,c) - (c,a_0) &= \min \\ &\quad c \in C \end{aligned} \tag{123}$$

is the principle of minimum total energy.

In any approximation by discretization of the displacement field, in particular in terms of conforming displacement type of finite elements, the best solution to (123) is seeked in terms of a finite expansion

$$c = \sum_1^n \alpha_i c_i \qquad c_i \in C \tag{124}$$

with given members c_i of C and unknown coefficients α_i. For convergence and considering all possible mesh sizes, the elements (124) should constitute a denumerable set everywhere dense in C. Substituting (124) into (123), the optimal values $\hat{\alpha}_i$ of the coefficients are given by the linear system

$$\hat{\alpha}_i (c_i, c_j) = (a_0, c_j) \qquad (j = 1,2,\ldots,n) \tag{125}$$

Thus the best approximation $\hat{c} = \Sigma \hat{\alpha}_i c_i$, satisfies equations

$$(\hat{c},c_j) = (a_o,c_j) \qquad (j = 1,2,\ldots,n)$$

Multiplying each by $\hat{\alpha}_j$ and adding, we obtain

$$(\hat{c},\hat{c}) = (\hat{c},a_o) = \langle f(a_o),u(\hat{c})\rangle \tag{126}$$

This very important property shows that the best approximation also satisfies Clapeyron's theorem. it is the analogue of (121).

Now, since the best approximation does not in general give to (123) its minimum, that on the other hand is reached by the exact solution:

$$(\hat{c},\hat{c}) - 2(\hat{c},a_o) \geq (s,s) - 2(s,a_o)$$

This result, combined with the Clapeyron theorems, furnishes the energy bound

$$(\hat{c},\hat{c}) \leq (s,s) \tag{127}$$

and the corresponding virtual work bound

$$(\hat{c},a_o) \leq (s,a_o) \tag{128}$$

Turning now to problem (122), we discretize the self-stressing field as

$$a = \sum_1^n \beta_i a_i \qquad a_i \in A$$

as occurs in a finite element analysis with diffusive equilibrium models. Substitution into (122) produces for the best coefficients the set of linear equations

$$\hat{\beta}_i (a_i,a_j) = - (a_o,a_j) \qquad (j = 1,2,\ldots,n)$$

Hence, denoting by $\hat{a} = a_o + \Sigma\hat{\beta}_i a_i$ the best approximation,

$$(\hat{a},a_j) = 0 \qquad (j = 1,2,\ldots,n)$$

Multiplying each by $\hat{\beta}_j$ and adding, we find

$$(\hat{a},\hat{a} - a_o) = 0 \qquad \text{or} \qquad (\hat{a},\hat{a}) = (\hat{a},a_o) \tag{129}$$

This result, analogous to (126) and (121), can first be interpreted as a statement of Pasternak's reduction principle. Taking the case of the exact solution first, it says that the stress energy can be evaluated as half the scalar product of the exact strains by any statically admissible stress field. This follows immediately from the definition of the scalar product (s,a_o). It is remarkable that this property also holds for any best approximation.

But the same result can also be interpreted again as a Clapeyron theorem despite the fact that neither a_o nor $\hat{a}$ will generally furnish an integrable deformation field. It must be recalled that, although the minimum complementary energy principle does not lead to a strong knowledge of displacements, it furnishes a weak knowledge in the form of functionals of the displacement field. To illustrate this in the present context consider first the result (120)

$$(c,a_o) = \langle f(a_o),u(c)\rangle$$

In this equation $f(a_o)$ denotes the fixed set of external forces in equilibrium with the stress field a_o. Changing the scale of the forces, without altering their relative distribution, we have by linearity

$$(c,\gamma a_o) = \gamma\langle f(a_o),u(c)\rangle$$

Considering γ as a generalized load, its conjugate generalized displacement is

$$q_\gamma = \langle f(a_o),\ u(c)\rangle = (c,a_o)$$

It is a linear functional of the displacement field, where the weighting functions are given by the distribution of external loads $f(a_o)$; the value of the functional is also given by the scalar product (c,a_o). In case we deal with an arbitrary field m, instead of a compatible one c, the generalized displacement conjugate to γ is numerically given by

$$q_\gamma = (m,a_o)$$

Also it still is the linear functional

$$q_\gamma = \langle f(a_o),\ u(m)\rangle$$

but, this time, the detailed (strong) knowledge of the displacement field $u(m)$ associated to m is unavailable. In this sense (129) can be interpreted as the Clapeyron theorem

$$(\hat{a},\hat{a}) = \langle f(a_o),u(\hat{a})\rangle \tag{130}$$

where $u(\hat{a})$ is, in a generalized sense, the displacement field related to $\hat{a}$. Since the best approximation $\hat{a}$ does not in general correspond to the minimum in (122), reached only by the exact solution,

$$(\hat{a},\hat{a}) \geq (s,s) \tag{131}$$

Then we have also by (121) and (129)

$$(\hat{a},a_o) \geq (s,s_o) \tag{132}$$

The strain energy bounding derived from (127) and (131)

$$(\hat{c},\hat{c}) \leq (s,s) \leq (\hat{a},\hat{a}) \tag{133}$$

is thus expressed equivalently by the bounding of generalized displacements

$$\langle f(a_o),u(\hat{c})\rangle \leq \langle f(a_o),u(s)\rangle \leq \langle f(a_o),u(a)\rangle \tag{134}$$

The consideration of several, independent, loading modes on a given structure raises the interesting problem of determining similar bounds to mutual generalized influence coefficients [1,3].

Problems where non zero displacements are prescribed on $\partial_1 R$ can be solved by the superposition principle. In a first stage, account is taken of the prescribed loads and the displacement boundary conditions are kept homogeneous as in the preceding treatment. In a second stage, the loads are suppressed and one takes only into account the boundary displacements. The solution of the second stage belongs to the subspace A of self-stressing states and the translated subspace $c_o + c$, where c_o is any particular compatible field satisfying the prescribed displacements on $\partial_1 R$. The minimum distance principle is again observed to split into

$$\frac{1}{2}(a,a) - (a,c_o) = \text{minimum} \qquad a \in A$$

and

$$(c_o + c,\ c_o + c) = \text{minimum} \qquad c \in C$$

which are particular forms of the minimum principles of complementary energy and total energy. Detailed calculations of the approximations by discretization, completely analogous to the foregoing, establish an energy bounding

$$(\hat{a},\hat{a}) \leq (s,s) \leq (\hat{c},\hat{c}) \tag{135}$$

that is the reverse of (133). Also

$$(s,s) = (s,c_o) \qquad (\hat{a},\hat{a}) = (\hat{a},c_o) \qquad (\hat{c},\hat{c}) = (\hat{c},c_o)$$

results than can either be interpreted as duals of the Pasternak reduction principle or as Clapeyron theorems involving the virtual work of the reaction loads against the prescribed displacements. For the last equation this interpretation requires the definition of the linear functional representing a generalized reaction load associated to $\hat{c}$. Then (135) is equivalent to

$$\langle f(\hat{a}),u(c_o)\rangle \leq \langle f(s),u(c_o)\rangle \leq \langle f(\hat{c}),u(c_o)\rangle \tag{136}$$

REFERENCES

1. Weber, C., "Eingrenzun von verschiebungen und zerrungen mit hilfe der minimalsatze," ZAMM, Vol. 22, p. 130, 1942.

2. Prager, W. and Synge, J. L., "Approximations in elasticity based on the concept of function space," Quart. Appl Math., Vol. 5, 1947.

3. Fraeijs de Veubeke, B., "Upper and Lower Bounds in Matrix Structural Analysis," AGARDograph No. 72, Pergamon Press, 1964.

4. Fraeijs de Veubeke, B., "Displacement and Equilibrium Models in the Finite Element Method," Chap. 9 of Stress Analysis, ed. O. C. Zienkiewicz and G. S. Holister, Wiley, 1965.

5. Sander, G. and Fraeijs de Veubeke, B., "Upper and Lower Bounds to Structural Deformations by Dual Analysis in Finite Elements," Air Force Flight Dynamics Laboratory Technical Report AFFDL-TR-66-199, 1967, Wright Patterson Air Force Base, Ohio.

6. Fraeijs de Veubeke, B., "A Conforming Finite Element for Plate Bending," Int. J. of Solids and Structures, Vol. 4, pp. 95-108, 1968.

7. Clough, R. W., and Tocher, J. L., "Finite Element Stiffness Matrices for Analysis of Plates in Bending," Proc. Conf. Matrix Methods in Struct. Mech., Air Force Institute of Technology, Wright Patterson A. F. B., Ohio 1965.

8. Sander, G., Beckers, P. and Nguyen, H. D., "Digital Computation of Stresses and Deflections in a Box Beam - A Performance Comparison Between Finite Element Models and Idealization Patterns," Air Force Flight Dynamics Laboratory Technical Report AFFDL-TR-69-4, 1969.

9. Sander, G., "Application of the Dual Analysis Principle," IUTAM 1970 Colloquim on High Speed Computing of Elastic Structures, Colloques et Congres de l'Universite de Liege, 1970.

10. Zienkiewicz, O. C., Finite Element Method in Engineering Science, Wiley, 1971.

11. Fraeijs de Veubeke, B. and Hogge, M. A., "Dual Analysis for Heat Conduction Problems by Finite Elements." To appear in Int. J. for Num. Meth. in Engin.