CREEP BUCKLING

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List of Symbols

A	total area of column cross-section
$h/2$	half-width of web or radius of gyration
L	total length of column
x	co-ordinate along axis of column
$\xi=\pi x/L$	dimensionless co-ordinate
$w_0(x)$	initial deviation of column axis from straightness
$w(x)$	additional lateral deflexion
$z(\xi)=2(w_0+w)/h$	
a_0	initial amplitude of sinusoidal deviation
$a(t)$	amplitude of sinusoidal deviation at given time
P	axial compressive load
$\bar{\sigma}=P/A$	average compressive stress
$\sigma_E=E(\pi h)^2/(4L^2)$	Euler critical compressive stress
$\alpha=\bar{\sigma}/\sigma_E$	load ratio
σ	axial stress in flange, positive when tensile
ε	axial strain in flange
t	time
$t_1, t_2, \theta_1, \theta_2$	time constants
t_{cr}	critical lifetime of column
E, E_1	Young's modulus of elastic deformations
λ, μ	material constants
m, n	exponents of plastic deformation and of creep laws

Note.—To ensure uniformity in notation and sign convention the notation used by the original authors has been changed in several instances.

Introduction

Aircraft structures necessarily comprise elements which are under compression. At room temperature slender columns and thin plates may be considered to be stable provided the compressive load does not exceed a critical value. At the high temperatures resulting from kinetic heating at supersonic speeds material creep becomes important and introduces a new element, namely time, in the buckling problem. Unavoidable initial crookedness or eccentricity of loading will be subject to a gradual amplification from the unequal rates of creep prevailing across the thickness of the compressed element.

The growth of lateral deflexion occurs under arbitrarily small compressive loads. This indicates that, although the purely elastic properties of the material at the temperature considered still permit the evaluation of a short-time buckling load, this concept loses much of its significance as a design parameter and should be replaced by a concept of critical lifetime.

In a broad sense the critical lifetime is the time after which a compressed element ceases to be an efficient structural part, either because it collapses or because its crookedness interferes with the proper functioning of other elements. This general definition can be interpreted in various ways. In this chapter the meaning of the term "critical time" will be restricted to the time necessary for collapse under constant load and temperature conditions.

According to the creep properties postulated for the material, a column may collapse in significantly different manners. For instance, a hypothetical linearly viscoelastic material requires an unbounded time interval to reach unbounded lateral deflexions. In the restricted sense one may say that for such a material the critical time is infinite.

On the other hand, if the steady creep rate increases more than proportionally to the stress, which is a recognized behaviour of metallic materials, unbounded lateral deflexions will occur after some finite critical time.

When, furthermore, instantaneous plastic deformations are added, true instability of the column appears for some finite deflexion. At the corresponding critical time the lateral velocity becomes infinite. Moreover plastic deformations increase the lateral bending in the loading phase which also contributes to a reduction in the critical time. This effect is most pronounced for near-perfect highly loaded columns. In a perfect column there is bifurcation of the equilibrium at the tangent modulus load. Infinitesimal imperfections are sufficient to induce a finite lateral bending as the load is further increased. Thus the limiting value of the critical time, as the out-of-straightness tends to vanish, is infinite below the tangent modulus load, finite above it and tends to zero as the short-time buckling load is reached. On the other hand, without plastic deformations, the limiting value remains infinite for all loads below the Euler load, and at the Euler load it suddenly changes to zero.

The main purpose of creep buckling theories is to provide an estimate of the critical lifetime as a function of load, slenderness ratio, initial out-of-straightness parameters and the mechanical properties of the material. According to the type and destination of the aircraft or missile the required lifetime may vary between a few minutes and several hundred hours.

Creep Buckling Theories

The main source of difference between the various theories developed so far lies in the interpretation given to creep tests. This is not surprising since the information given by simple constant load tests must be extrapolated to describe the behaviour of the material under considerably different conditions. For instance, at some point on the convex side of the bent column the fibre stress history may consist in a gradual increase in compression followed by unloading up to stress reversal and increasing tension.

Extrapolations of this kind necessarily deviate from the purely phenomenological observations and involve idealizations or physical hypotheses concerning the mechanism of creep. They are moreover influenced by considerations of analytical convenience without which the analysis could not be carried through or the results easily discussed.

Aside from this the following assumptions are generally common to all theories and are listed for reference and future discussion:

(*a*) The column is of an idealized H-section consisting of two concentrated flanges of equal area $A/2$ joined by a thin web of infinite shearing rigidity whose width is denoted by h (*Fig. 1*).

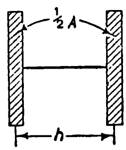

Fig. 1. Cross-section of idealized column

(*b*) Small curvature and the Bernoulli assumption result in flange strains given by the formulae

$$\varepsilon_1 = \bar{\varepsilon} + \tfrac{1}{2}h\, w_{xx} \qquad \varepsilon_2 = \bar{\varepsilon} - \tfrac{1}{2}h\, w_{xx}$$

where w is the lateral deflexion measured from the curved axis of the column before load application (*Fig. 2*). In terms of non-dimensional variables there results the kinematical equation

$$\varepsilon_2 - \varepsilon_1 = -2\,(\sigma_E/E)\,(z - z_0)_{\xi\xi}. \qquad \ldots .(1)$$

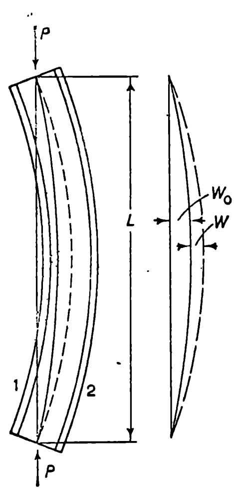

Fig. 2. Initial and additional deviations from straightness of pin-jointed column

(*c*) The compressive load acts along the line joining the pin-jointed ends. The resulting flange stresses are given by

$$\sigma_1 = -\bar{\sigma}(1+z) \qquad \sigma_2 = -\bar{\sigma}(1-z). \qquad \ldots .(2)$$

(*d*) Before the application of the load the column axis is sinusoidal in shape

$$z_0 = a_0 \sin \xi. \qquad \ldots .(3)$$

It is assumed to remain sinusoidal during the buckling process

$$z = a(t) \sin \xi. \qquad \ldots.(4)$$

The differential equation of the problem is then either satisfied by collocation at mid-span or preferably averaged by a Galerkin process.

(*e*) The load P is applied so rapidly that no creep can develop during the loading process but slowly enough so that dynamic effects are avoided.

(*f*) The temperature is constant both in time and along the column.

1. Creep buckling of linearly viscoelastic columns

The consideration of mechanical models is useful as it suggests the type of stress–strain relation one may use to describe certain anelastic material properties. KEMPNER[1] investigated the creep buckling of a column whose material follows the stress–strain law implied by the mechanical model of *Fig. 3.* It consists of a Maxwell unit of time constant $t_1 = c_1/E_1$, in series with a Kelvin unit of time constant $t_2 = c_2/E_2$, producing a relation of the form

$$\ddot{\sigma} + \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{E_1}{E_2} \cdot \frac{1}{t_2}\right) \dot{\sigma} + \frac{1}{t_1 t_2} \sigma = E_1 \left(\ddot{\varepsilon} + \frac{1}{t_2} \dot{\varepsilon}\right). \qquad \ldots.(5)$$

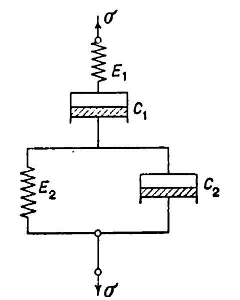

Fig. 3. Mechanical analogue of linearly viscoelastic stress–strain relation

The heredity characteristics of this model are better evidenced in the equivalent form

$$\varepsilon = (\sigma/E_1) + \int_0^t H(t-\tau)\, \sigma(\tau)\, d\tau$$

where the model was supposed to be unstrained for $t<0$. The strain consists of a purely elastic part and the cumulative effects of a time-dependent part involving a heredity function

$$H(t) = \frac{1}{t_1 E_1} + \frac{1}{t_2 E_2} \exp(-t/t_2).$$

The corresponding creep curve under constant stress

$$\varepsilon = \frac{\sigma}{E_1} + \frac{\sigma}{E_2}\{1 - \exp(-t/t_2)\} + \frac{\sigma}{E_1} \cdot \frac{t}{t_1} \qquad \ldots.(6)$$

presents the characteristic features of experimental observation (*Fig. 4*). The first term is the instantaneous elastic response OB. The second represents a primary or transient creep whose rate vanishes exponentially with the time constant t_2 and results in a final strain of magnitude σ/E_2. The third term represents a constant-rate secondary creep.

It is most important to observe that primary creep, which is responsible for the decrease in the general rate of strain, is due to straining in the Kelvin unit and is fully recoverable. When the load is removed this part of the strain gradually disappears. Such elastic after-effects are actually observed in high polymers.

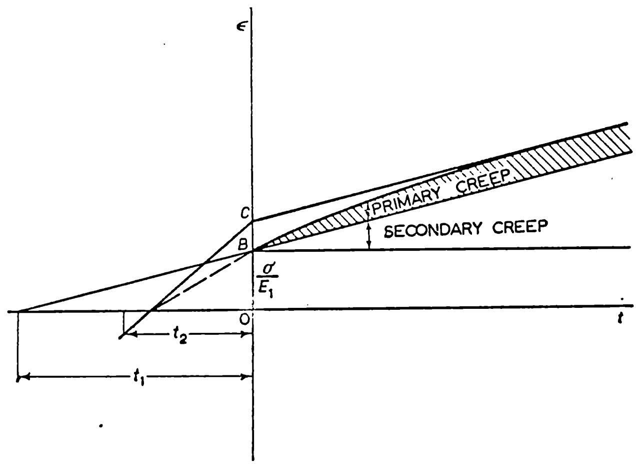

Fig. 4. Constant-stress creep curve corresponding to mechanical model of Fig. 3

The linear character of the relation between stress and strain has important consequences. First of all the analysis is easily extended to arbitrary cross-sectional shapes. This entails only the replacement of $Ah^2/4$ by the actual moment of inertia I. In other words $h/2$ may be considered as denoting the radius of gyration of the general cross-section whose total area remains designated by A. These two are the only parameters which have any bearing on the results of the analysis. Furthermore the assumption that the deformation retains a sinusoidal shape is entirely justified. The variable ξ separates and a single differential equation remains for the amplitude a (t). According to assumption (e) the amplitude immediately after load application is

$$a(0) = \frac{1}{1-\alpha}a_0. \qquad \ldots.(7)$$

Evidently the load ratio $\alpha = P/P_E = \bar{\sigma}/\sigma_E$ should be smaller than 1 to avoid initial elastic buckling. Then as time proceeds, the lateral deflexion grows continuously. Since, however, the differential equation is linear with constant coefficients this growth is of an exponential character and the deflexion remains finite for any finite time.

The same conclusion holds naturally for the more general but still linear viscoelastic relations considered by HILTON[2] and for other types of initial column imperfections. For any shape of the type $a_m \sin m\xi$ (m integer) remains congruent and the greater m the slower is the rate of increase of its amplitude. If an arbitrary initial imperfection is expanded in a convergent

Fourier series, the corresponding series for the deflexion prevailing at any time will then converge *a fortiori*. The conclusion about the non-existence of a finite critical time for linear viscoelastic columns is at variance with early statements by FREUDENTHAL[3] and by ROSENTHAL and BAER[4]. Freudenthal's treatment of the case of constant initial eccentricity was later corrected by KEMPNER and POHLE[5].

The rate of growth of the lateral deflexion is nevertheless significant with regard to the critical time in its broader sense. It is thus of interest to note the relationship between the time constants of the mechanical model and the time constants of the final result

$$\frac{a(t)}{a(o)} = (1-\alpha)\left(\frac{a(t)}{a_0}+1\right) = 1+B\{1-\exp(-t/\theta_1)\}+C\{1-\exp(-t/\theta_2)\}$$

The relations are

$$\theta_1\theta_2 = -\frac{1-\alpha}{\alpha}t_1t_2 \qquad \theta_1+\theta_2 = t_2-t_1\left(\frac{1-\alpha}{\alpha}-\frac{E_1}{E_2}\right)$$

Only one of the time constants is negative and responsible for divergence. For small load ratios the positive time constant θ_2 is almost equal to the relaxation time t_2 of the Kelvin unit, whilst the negative constant is very large and produces slow divergence. For load ratios near unity, that is, close to the Euler buckling load, the positive constant is asymptotic to $t_2+t_1(E_1/E_2)$ and the negative one tends to zero, thereby producing a very fast divergence.

2. Buckling induced by secondary creep in metallic materials

High-temperature creep tests show that for metallic materials the secondary creep rate is not proportional to the applied stress. If the transient or primary creep phenomenon is neglected the constant-stress creep law may be expressed in the form

$$\varepsilon = (\sigma/E) + S(\sigma)t. \qquad \ldots.(8)$$

From this a general law has to be derived for creep under variable stress. Following early suggestions by NADAI[6] and NADAI and DAVIS[7], this is generally done by postulating the existence of an equation of state connecting the rate of deformation with the stress and the strain

$$(\partial\varepsilon/\partial t)_{\sigma=\mathrm{const}} = f(\sigma, \varepsilon). \qquad \ldots.(9)$$

Furthermore experimental evidence supports the idea that to small increments of stress the material reacts instantaneously in a purely elastic manner

$$(\partial\varepsilon/\partial\sigma)_{t=\mathrm{const}} = (1/E). \qquad \ldots.(10)$$

These two assumptions result in the general law

$$\dot{\varepsilon} = (\dot{\sigma}/E) + f(\sigma, \varepsilon). \qquad \ldots.(11)$$

The assumptions are sometimes referred to as SHANLEY's engineering hypotheses.[8]

It follows from Eq. (8) that

$$f(\sigma, \varepsilon) = S(\sigma)$$

and the general stress–strain relation assumes the form

$$\dot{\varepsilon} = (\dot{\sigma}/E) + S(\sigma). \qquad \ldots.(12)$$

With $S(\sigma)=\sigma/(E_1 t_1)$ this law would be a particular case of the linear model of Kempner with the Kelvin unit frozen and thus the transient creep disregarded. For metallic materials and moderate stress levels a power law is acceptable:[9,10]

$$S(\sigma) = \lambda(\sigma/E)^n \qquad n>1. \qquad \ldots.(13)$$

It can represent both tensile and compressive creep, if they are symmetric, provided n is an odd integer. Otherwise the absolute value of the stress must be taken and λ must be given the sign of the stress.

In his Wilbur Wright Memorial Lecture, HOFF[11] neglected the elastic part of the deformation. It was shown by him elsewhere[12] that in creep problems in general this can give a satisfactory approximation to reality after a sufficiently long time of loading. Thus the simplification is acceptable in creep buckling analysis for the large critical times corresponding to small load ratios. However, the retention of the elastic term does not complicate the analysis and it is important in designs for short critical times. The following treatment of this case is due to KEMPNER[13].

The stress–strain law expressed by Eqs. (12) and (13), together with the kinematical and equilibrium Eqs. (1) and (2), leads to the partial differential equation

$$-2\frac{\partial}{\partial t}\left(\frac{\sigma_E}{E}\frac{\partial^2 z}{\partial \xi^2}+\frac{\bar\sigma}{E}z\right) = \lambda\left(\frac{\bar\sigma}{E}\right)^n\{(z+1)^n+(z-1)^n\} \text{ for } z>1$$

$$= \lambda\left(\frac{\bar\sigma}{E}\right)^n\{(z+1)^n-(1-z)^n\} \text{ for } z<1.$$

A solution of the type of Eq. (4) has now the character of an approximation. When it is used, the differential equation cannot be satisfied everywhere. The simplest approach, but also the least exact one, is to satisfy it at mid-span where $\sin\xi=1$. Under constant load the time variable separates and the following expression is found for the critical time

$$\lambda\left(\frac{\bar\sigma}{E}\right)^n\frac{E}{2(\sigma_E-\bar\sigma)}t_{cr}=\int_{a(0)}^{1}\frac{da}{(1+a)^n-(1-a)^n}+ \qquad \ldots.(14)$$

$$+\int_{1}^{\infty}\frac{da}{(1+a)^n+(a-1)^n}=F_n[a(0)]$$

where $a(0)$ is the amplitude immediately upon load application as given by Eq. (7). If the load is so high that the initial stress at mid-span on the convex ide is tensile ($a(0)>1$), the first integral must be discarded and the lower limit of the second raised to the actual $a(0)$ value. For the special case that n is an odd integer the two integrands are identical and the integrals may be lumped together. In particular for $n=1$ the integration is elementary and yields an infinite critical time in accordance with the linear case of section (1).

For n greater than one the generalized integral is convergent and the critical time is finite. It was computed by KEMPNER and PATEL[14] for various amplitudes of initial column imperfections and exponent values of 1·1, 1·5, 2, 3 . . . 14. In the notation of this paper their definition of a reduced critical time and out-of-straightness parameter are:

$$\tau_{cr} = 2^{n-1}F_n\,(2f\tau_0) \qquad 2f\tau_0 = a(0).$$

Since the critical time is reached at infinitely large deflexions there is some approximation involved in the small-curvature assumption during the final stages of the buckling process. In this respect, however, a refinement of the analysis should proceed simultaneously with an improvement of Eq. (4). For, by reason of symmetry, the slope of the bending curve remains vertical at mid-span and causes the approximate formula used for the curvature and the exact one

$$w_{xx}/(1+w_x^2)^{3/2}$$

to be identical at the collocation point.

3. Buckling induced by primary and secondary creep in metallic materials

Whilst the effect of primary creep was disregarded in the preceding analysis, the purpose of a theory developed by Libove[15] is to represent accurately the curved part of the creep diagram. The theory is based on experimental evidence produced for 75S-T6 aluminium alloy at 600°F by Jackson, Schwope and Shober[16].

It seems useful to make some remarks concerning the interpretation of the test results. They seem to indicate that constant load tensile tests exhibit a constant rate of creep after about 4 hr, while constant stress tests result in creep curves with a decreasing rate of deformation up to 16 hr. To reduce the load in accordance with the change in cross-sectional area, the constant stress experiments were devised on the basis of an assumed Poisson's ratio of 0·5 corresponding to deformations at constant volume. Constant stress experiments are valuable from the standpoint of basic creep theory, but obviously the constant load tests are the correct ones to use in uniaxial stress problems if one wishes to avoid the calculation of the changes in flange area.

The representation used by Libove

$$\varepsilon = (\sigma/E) - A\, e^{-B\sigma}\, t^K \qquad \ldots.(15)$$

is one which fits the constant-load compressive tests in the range

$$0 < t < 20 \text{ hr} \qquad 4500 \leqslant -\sigma \leqslant 5500 \text{ p.s.i.}$$

In accordance with the preceding considerations σ is the fictitious stress referred to the original area.

The time exponent is given as $K=0{\cdot}66$; it indicates that the rate of strain decreases during the entire time of testing. It is not known whether an asymptotic rate of strain is eventually reached in compression; nor is it intended to discuss here the physical grounds on which to distinguish between primary and secondary creep. The important point to stress with regard to the phenomenological interpretation is that this distinction should not follow the viscoelastic model in which primary creep is fully recoverable. Although there is not sufficient evidence to assess quantitatively the amount of recoverable creep, it is, according to Odqvist[17], negligible in the case of stable materials.

This in part justifies the use of a strain hardening law, such as that of Eq. (9), to extend the test result represented by Eq. (15) to variable stress conditions. The use of Eqs. (9) and (10) produces the law

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} - K\frac{(Ae^{-B\sigma})^{1/K}}{\{(\sigma/E)-\varepsilon\}^{1/K-1}} \qquad \ldots.(16)$$

used by Libove. Aside from linearity, the absence of elastic after-effects is an essential difference between this theory and that considered in Section 1.

When interpreted outside of its rather narrow range of validity Eq. (16) presents peculiarities due to the fact that the plastic rate of strain does not change sign with the stress. Thus when the flange stress on the convex side becomes tensile the law still predicts a negative rate of strain. This unfortunate feature can easily be corrected by the use of a stress function of Nadai's type

$$\varepsilon = (\sigma/E) + A(\sinh B\sigma)t^K.$$

Nevertheless Libove's theory should yield a good approximation when, as seems to be generally the case, tensile stresses develop in the flange only during a negligible fraction of the total computed critical time.

The calculation of the critical time by Libove's theory is very lengthy; it is possible only by stepwise integration. Hence no simple formula can be given for the critical lifetime.

4. Inclusion of instantaneous plastic deformations

Instead of idealizing the creep curve by an elastic segment OB followed by a constant rate creep line (*Fig. 5*) as in Section 2, HOFF[18] proposed to consider the instantaneous segment OC, where C is the intercept of the asymptote to the real creep curve. The segment BC should then be considered as an instantaneous irreversible plastic deformation. Not only does

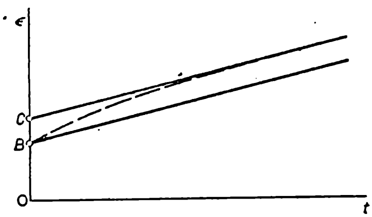

Fig. 5. Different idealizations of experimental creep curves

this theory include the instantaneous plastic deformations but it also approximates the effects of primary creep in a time-concentrated manner. ODQVIST[19] was the first to show that the problem stated in this manner, but without elastic deformations, could be integrated.

The complete stress–strain relation considered by HOFF may be given in the form

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \mu\left|\frac{\sigma}{E}\right|^m \frac{\dot{\sigma}}{E} + \lambda\left|\frac{\sigma}{E}\right|^n \qquad \ldots.(17)$$

As in Section 2, λ is a material constant which must be given the sign of the stress and which measures the intensity of the secondary creep. The instantaneous plastic deformations are given by the second term where

μ is a positive constant if $\sigma\dot{\sigma} > 0$
is zero if $\sigma\dot{\sigma} < 0$

This accounts for the fact that instantaneous unloading occurs in a purely elastic manner.

When the last term is omitted, Eq. (17) yields the tangent modulus of the instantaneous stress–strain curve (*Fig. 6*). During a tensile or compressive loading phase this modulus decreases steadily towards zero. Since during the differential creep process at least one of the flanges is undergoing such a type of loading, the combined incremental modulus of the column is also steadily decreasing. One must then expect the column to fail at some finite deflexion because the critical load of true instability is reached. This

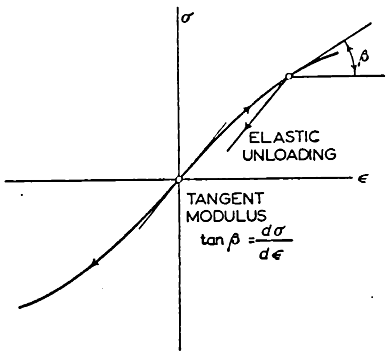

Fig. 6. Instantaneous uniaxial stress–strain curve with irreversible plastic deformations

is an essential new feature of the buckling process brought forward by the inclusion of plastic deformations. If one seeks the critical condition in the usual way by investigating the existence of a state of indifferent equilibrium, one has to satisfy the homogeneous equation in dz:

$$-2\,\frac{\sigma_E}{E}\,dz_{\xi\xi} = 2\,\frac{\bar{\sigma}}{E}\,dz + \left(\frac{\bar{\sigma}}{E}\right)^{m+1}\{\mu_2\,|\,z-1\,|^m + \mu_1\,|\,z+1\,|^m\}\,dz \qquad \ldots.(18)$$

where z (ξ) is a known shape of deflexion at a given time and dz (ξ) a virtual incremental deformation. Moreover

$$\mu_2 = \mu \text{ if } dz\ (z-1) > 0 \text{ and otherwise zero}$$
$$\mu_1 = \mu \text{ if } dz\ (z+1) > 0 \text{ and otherwise zero.}$$

As stating an eigenvalue problem for $\bar{\sigma}$, the equation may be solved approximately on the assumptions that both the deflexion and its increment are sinusoidal. Instead of proceeding by collocation at mid-span, the equation may be averaged as in the Galerkin process. In such a case it is multiplied by sin ξ and integrated from 0 to $\pi/2$, which by reason of symmetry is equivalent to integration along the whole span.

If $dz>0$ or, what is the same, $da>0$, the load in the flange on the concave side is increasing everywhere and consequently $\mu_1=\mu$. According to the amplitude of the deflexion two cases must be distinguished; they correspond to different loading conditions in the other flange.

Case 1

$a<1$, whence $z<1$; there is elastic unloading along the flange and consequently $\mu_2=0$. The averaging process yields then

$$\left\{\frac{\pi}{2}\,\frac{\sigma_E-\bar{\sigma}}{E} - \mu\left(\frac{\bar{\sigma}}{E}\right)^{m+1}\int_0^{\pi/2}(a\sin\xi+1)^m\sin^2\xi d\xi\right\}da = 0. \qquad \ldots.(19)$$

The critical condition between average compressive stress and amplitude is obtained if the bracket vanishes. For the case $m=1$ considered by Hoff

$$a = \frac{3\pi}{8}\left[\frac{2\{(\sigma_E-\bar{\sigma})/E\}-\mu(\bar{\sigma}/E)^2}{\mu(\bar{\sigma}/E)^2}\right] = \frac{3\pi}{8}\Sigma(\bar{\sigma}). \qquad \ldots.(20)$$

Instead of $3\pi/8$, the factor unity is obtained when the collocation method is used. In that case a critical load corresponding to the reduced modulus of the central section is reached. At that time the real column will not yet buckle as its other sections are stiffer than the central section. The averaging process takes this into account and produces the correction factor $3\pi/8$.

Case 2

$a>1$; there is unloading in the sections defined by $0\leqslant a\sin\xi\leqslant 1$ and there is loading in tension in the complementary segment. Hence $\mu_2=\mu$ from $\xi=\eta=\sin^{-1}(1/a)$ to $\pi/2$ and

$$\left[\frac{\pi}{2}\frac{\sigma_E-\bar{\sigma}}{E} - \mu\left(\frac{\bar{\sigma}}{E}\right)^{m+1}\left\{\int_0^{\pi/2}(a\sin\xi+1)^m\sin^2\xi d\xi + \right.\right.$$
$$\left.\left. + \int_\eta^{\pi/2}(a\sin\xi-1)^m\sin^2\xi d\xi\right)\right\} da = 0. \qquad \ldots.(21)$$

For $m=1$ the critical condition is then

$$C(a) = \frac{3\pi}{8}\Sigma(\bar{\sigma}) \qquad \ldots.(22a)$$

where

$$C(a) = a - \frac{3\pi}{8} + \frac{3}{4}\sin^{-1}\frac{1}{a} + \frac{4a^2-1}{4a^2}\sqrt{(a^2-1)}. \qquad \ldots.(22b)$$

If we take a to be large or $\eta=0$ we obtain Hoff's approximation which considers the stress reversal to extend along the whole flange. Then

$$C(a) \simeq 2a - \frac{3\pi}{8} \qquad \ldots.(23a)$$

and

$$a \simeq \frac{3\pi}{8}\frac{1}{2}\left(1+\Sigma(\bar{\sigma})\right) \qquad \ldots.(23b)$$

Again the collocation process would give this result without the factor $3\pi/8$ and it would then correspond to a critical combination of two tangent moduli reached in the centre section. The difference between the exact and the approximate values of $C(a)$ is only significant in the range $1\leqslant a\leqslant 1{\cdot}5$. If the value of a obtained from Eq. (23b) is larger than $1{\cdot}6$ it will be in excess by an error of less than $0{\cdot}02$.

The computation of the time required to reach one of these critical conditions under a constant load follows a similar procedure. The assumption of Eq. (4) is substituted in the partial differential equation of the problem

$$-2\frac{\sigma_E}{E}\dot{z}_{\xi\xi} = 2\frac{\bar{\sigma}}{E}\dot{z} + \left(\frac{\bar{\sigma}}{E}\right)^{m+1}\{\mu_2|z-1|^m + \mu_1|z+1|^m\}\dot{z} +$$
$$+ \left(\frac{\sigma}{E}\right)^n\{\lambda_2|z-1|^n + \lambda_1|z+1|^n\}$$

where $\lambda_2 = \pm\lambda$ according to whether z is greater or smaller than one. $\lambda_1 = \lambda$ since $(z+1)$ is always positive.

The equation is multiplied by $\sin \xi$ and integrated between zero and $\pi/2$. This either represents a Galerkin averaging process or the type of Fourier analysis described by Hoff. In this way an ordinary differential equation is obtained for $a(t)$ in which the variables are separable. The critical conditions appear now as those for which the lateral velocity $\dot{a}(t)$ becomes infinite. The two cases specified earlier must again be distinguished. We give only the results corresponding to the values $m=1$ and $n=3$ considered by Hoff.

Case 1

Critical condition reached for an amplitude smaller than one.

$$\lambda \left(\frac{\bar{\sigma}}{E}\right)^3 t_{cr} = M \ln \frac{a^2(4+a^2(0))}{a^2(0)(4+a^2)} - N \tan^{-1} \frac{2(a-a(0))}{4+aa(0)} \quad \ldots\ldots(24)$$

where

$$M = \frac{1}{6}\left\{\frac{\sigma_E - \bar{\sigma}}{E} - \frac{\mu}{2}\left(\frac{\bar{\sigma}}{E}\right)^2\right\} \qquad N = \frac{8}{9\pi}\mu\left(\frac{\bar{\sigma}}{E}\right)^2 \quad \ldots\ldots(25)$$

In this formula a is the critical amplitude as given by Eq. (20) or, what is the same, by $a = 4M/N$, provided the result is smaller than unity.

Case 2

Critical condition reached for an amplitude greater than one. The time lapse must be split in two parts:

$$t_{cr} = t_{lim} + t_{ad}. \quad \ldots\ldots(26)$$

The first part corresponds to the time necessary for the amplitude to increase from $a(0)$ to unity. It is naturally given by Eq. (24) where a is set equal to unity:

$$\lambda \left(\frac{\bar{\sigma}}{E}\right)^3 t_{lim} = M \ln \frac{4+a^2(0)}{5a^2(0)} - N \tan^{-1} \frac{2(1-a(0))}{4+a(0)} \quad \ldots\ldots(27)$$

The second part is the time lapse between unit amplitude and critical amplitude, during which stress reversal spreads gradually from the centre of the flange on the convex side towards the ends. The averaging process yields

$$\lambda \left(\frac{\bar{\sigma}}{E}\right)^3 t_{ad} = M \ln \frac{5a^2}{4+a^2} - N\,B(a) \quad \ldots\ldots(28a)$$

$$B(a) = \int_1^a \frac{2C(a)}{a(4+a^2)}\,da \quad \ldots\ldots(28b)$$

where a is the critical amplitude as given by Eq. (22). Equations (28) are not derived in Hoff's paper but it is shown there how the critical time may be bracketed between a lower and an upper limit. The upper limit is found by neglecting the plastic deformations which occur in the stress-reversed part. Equation (24) remains valid under this assumption though the amplitude given by Eq. (20) exceeds unity.

The lower limit involves an approximate additional time based on complete stress reversal in the outside flange from $a=1$ onward. This amounts to the substitution of Eq. (23a) into (28b) and yields

$$\lambda \left(\frac{\bar{\sigma}}{E}\right)^3 t_{\text{ad}} = M' \ln \frac{5a^2}{4+a^2} - aN \tan^{-1} \frac{2\,(a-1)}{4+a} \qquad \ldots .(29\text{a})$$

where

$$M' = M + \frac{3\pi}{32} N = \frac{1}{6} \frac{\sigma_E - \bar{\sigma}}{E} \qquad \ldots .(29\text{b})$$

and a is the approximate critical amplitude of Eq. (23b).

If under exceptional conditions $a(0)$ is already larger than unity, the critical time is obtained as a difference between two additional times

$$\lambda \left(\frac{\bar{\sigma}}{E}\right)^3 t_{\text{cr}} = M \ln \frac{a^2\{4+a^2(0)\}}{a^2(0)\,(4+a^2)} - N\,[\mathrm{B}(a) - \mathrm{B}\{a(0)\}]$$

and a similar bracketing is again possible.

The analyses of the critical times must be completed by calculation of the zero-time amplitude $a(0)$ from the initial amplitude a_0 prevailing before load application. This is a problem in inelastic column theory which, in general, has no simple solution. Three phases may occur during the loading process:

Phase I

Compressive loading takes place along both flanges. This necessarily implies that the initial amplitude is smaller than unity. Since the loading is so rapid that the creep deformations may be neglected the integrated stress–strain relations

$$\varepsilon = \frac{\sigma}{E} - \frac{\mu}{m+1} \left|\frac{\sigma}{E}\right|^{m+1}$$

are valid everywhere. The equation of equilibrium is then

$$-2\,\frac{\sigma_E}{E}\,(z-z_0)_{\xi\xi} = 2\,\frac{\bar{\sigma}}{E}\,z + \frac{\mu}{m+1} \left(\frac{\bar{\sigma}}{E}\right)^{m+1} \{(1+z)^{m+1} - (1-z)^{m+1}\}$$

It has to be solved for $\bar{\sigma}$ increasing monotonically from zero.

The averaging process is always applicable; however, for $m=1$ the situation simplifies because the equation becomes linear and the assumption of Eq. (3) implies Eq. (4) and leads to the simple relation

$$a = \frac{\sigma_E/E}{(1/E)\,(\sigma_E - \bar{\sigma}) - \mu(\bar{\sigma}/E)^2}\, a_0. \qquad \ldots .(30)$$

Phase I ends when unloading begins at the centre of the convex side where $\sigma_2 = \bar{\sigma}\,(a-1)$.

The unloading condition expresses that σ_2 has reached a minimum there

$$\frac{d\sigma_2}{d\bar{\sigma}} = a - 1 + \bar{\sigma}\,\frac{da}{d\bar{\sigma}} = 0 \qquad \ldots .(31)$$

from which one obtains with the aid of Eq. (30)

$$a_0\,\frac{\sigma_E}{E}\left\{\frac{\sigma_E}{E} + \mu\left(\frac{\bar{\sigma}}{E}\right)^2\right\} = \left\{\frac{\sigma_E - \bar{\sigma}}{E} - \mu\left(\frac{\bar{\sigma}}{E}\right)^2\right\}^2 \qquad \ldots .(32)$$

This equation permits the computation of the limiting value of $\bar{\sigma}$ marking the end of the first phase.

Since a_0 is smaller than unity, before load application the left side is smaller than the right side. When $\bar{\sigma}$ increases the left side is positive increasing, and the right side positive decreasing. Equality occurs before the denominator of Eq. (30) vanishes. Thus the column cannot buckle during the first phase. The amplitude cannot even exceed unity since the tensile stresses which would then be present could not be produced without previous unloading.

Phase II

If the load is increased beyond the limiting value of Eq. (32), account must be taken of the elastic unloading in part of the right flange. The propagation of the strain reversal along the flange leads to complicated calculations unless reversal is assumed to take place instantaneously, at the beginning of the phase all along the flange. This approximation, introduced by Hoff, seems acceptable in view of the rapid increase in the size of the reversed region with the amplitude of deflexion. It leads to a quadratic equation for the amplitude as a function of load and initial deviation.

Buckling is possible during this phase, the critical condition being naturally given by Eq. (20).

Phase III

There is occasionally a third phase which starts as soon as the amplitude exceeds unity. The real situation in the right flange may then become quite complicated. Compressive stresses may still increase near the ends of the column and decrease in adjacent regions whilst a central part is now undergoing tensile loading. The approximation introduced by Hoff is here somewhat less satisfactory, giving strain discontinuities at the phase change. Since tensile plastic deformations are introduced immediately along the whole flange, in this approximation the critical condition is given by Eq. (23b).

Adding to Hoff's considerations we wish to observe that a column with vanishingly small out-of-straightness may be subject to creep buckling in a finite critical time. The analysis of phase I shows that smaller imperfections a_0 result in higher limiting $\bar{\sigma}$ values. When the initial deviation tends to zero this limiting value tends to the tangent modulus critical value $\bar{\sigma}_t$ which makes the denominator of Eq. (30) vanish. Thus if one considers the limiting case obtained by letting a_0 vanish the following situation arises: If the load is less than the tangent modulus load, for which bifurcation of the equilibrium occurs in a perfect column, there will be no bending and the column will simply shorten as a result of the creep strains. In the other creep buckling theories this conclusion holds for loads up to the Euler buckling load. If, however, the tangent modulus load is reached the asymptotically straight column will start to bend.[20] The analysis must then be carried out starting with phase II and zero initial deviation, giving under Hoff's assumption

$$\frac{2}{3\pi}\mu\left(\frac{\bar{\sigma}}{E}\right)^2 a^2 - \left\{\frac{\sigma_E-\bar{\sigma}}{E} - \frac{\mu}{2}\left(\frac{\bar{\sigma}}{E}\right)^2\right\} a + \frac{1}{\pi}\mu\left\{\left(\frac{\bar{\sigma}}{E}\right)^2 - \left(\frac{\bar{\sigma}_t}{E}\right)^2\right\} = 0. \quad \ldots (33)$$

Unless buckling occurs upon load application in this phase, or unless the third phase is entered, this equation permits the calculation of $a(0)$ for an initially perfect column under a service load $\bar{\sigma}$ higher than the tangent modulus load.

For columns with small initial deviations the situation which is most likely to occur in a practical design will be that of a service load smaller than the limiting value in phase I. Substitution of the service load in Eq. (30) will then give the $a(0)$ value required in the computation of the critical lifetime.

The extension of Hoff's theory to other exponent values is desirable. Although straightforward in principle, it may lead to lengthy developments. PATEL[21] has given theoretical expressions for the critical time for general integral values of the exponents. He uses collocation at the centre section instead of the more exact averaging method. The correlation between $a(0)$ and a_0 has not been investigated.

NUMERICAL EXAMPLES AND GRAPHS

(1) For 2024-T4 (formerly 24S-T3) aluminium-alloy at 600°F typical values of the material constants are cited by Hoff: $m=1$; $n=3$; $E=7{\cdot}4\times10^6$ p.s.i.; $\mu=85{\cdot}5625$; $\lambda=10080$ min^{-1}. Take a column of slenderness ratio $2L/h=50$ with the Euler buckling characteristics

$$\sigma_E/E = (\pi/50)^2 = 39{\cdot}4786\times10^{-4} \quad \sigma_E = 29{,}214 \text{ p.s.i.}$$

The critical tangent modulus stress is obtained by equating to zero the denominator of Eq. (30), which gives

$$\bar{\sigma}_t/E = 31{\cdot}167\times10^{-4} \quad \bar{\sigma}_t = 23{,}064 \text{ p.s.i.}$$

With an initial deviation $a=0{\cdot}06$ the limiting average compressive stress of phase I results from Eq. (32). A trial and error process yields

$$\bar{\sigma}/E = 24{\cdot}2\times10^{-4} \quad \bar{\sigma} = 17{,}910 \text{ p.s.i.}$$

A service load of 15,000 p.s.i. will therefore result in a loading confined to phase I. The zero-time amplitude will then be given by Eq. (30) with $\bar{\sigma}/E = 20{\cdot}27\times10^{-4}$; one obtains

$$a(0) = 0{\cdot}1509.$$

Equation (20) gives a critical amplitude

$$a_{cr} = 11{\cdot}696$$

which is larger than one. Its substitution in Eq. (24) yields an upper limit to the critical time

$$t_{cr} = 16{\cdot}3 \text{ min.}$$

To find the lower limit the t_{lim} value is first computed from Eq. (27)

$$t_{lim} = 11{\cdot}95 \text{ min.}$$

The critical amplitude under conditions of stress reversal in the right flange is then obtained from Eq. (23b)

$$a_{cr} = 5{\cdot}259$$

and is substituted in Eq. (29a) to obtain

$$t_{ad} = 3{\cdot}90 \text{ min.}$$

Hence

$$t_{lim} + t_{ad} = 15{\cdot}85 \text{ min.}$$

Application of the formulae (22b) and (28a, b) gives practically the same critical amplitude but a value

$$t_{\mathrm{ad}} = 4{\cdot}11 \text{ min}$$

and a critical time value

$$t_{\mathrm{lim}} + t_{\mathrm{ad}} = 16{\cdot}06 \text{ min}$$

effectively bracketed between the previous estimates. In this example taken from one of HOFF's papers[18], the short-time buckling load is found to occur in the third phase. Its value of 22,450 p.s.i. is seen to be lower than the tangent modulus load, whilst for a near-perfect column it would lie somewhere between the tangent modulus load and the (higher) reduced modulus load.

(2) It is of interest to compare these values with the results given by an analysis which neglects primary creep and plastic deformations. We thus apply KEMPNER's analysis[13] with the same values of E, n and λ and the same service load.

If we first keep the same initial crookedness $a_0{=}0{\cdot}06$, Eq. (7) gives for zero time amplitude

$$a(0) = 0{\cdot}1233.$$

Then from Eq. (14)

$$t_{\mathrm{cr}} = 20{\cdot}27 \text{ min.}$$

If on the other hand we keep the zero-time amplitude of Hoff's analysis, we find

$$t = 18{\cdot}64 \text{ min.}$$

These values give an idea of the separate effects of neglecting primary creep and plastic deformations during the creep process only or during the load application as well.

(3) More generally the effects of initial crookedness and load ratio in Kempner's analysis are best shown if Eq. (14) is rewritten in the form

$$\lambda \left(\frac{\sigma_E}{E}\right)^{n-1} t_{\mathrm{cr}} = 2\,\frac{1-\alpha}{\alpha^n}\,F_n\left(\frac{a_0}{1-\alpha}\right).$$

This is illustrated in the case $n{=}3$ by the constant load ratio curves of *Fig. 7*, where this type of reduced critical time is plotted on a logarithmic scale v. a_0.

(4) We also wish to give an example of creep buckling for the limiting case of an initially straight column. The material constants are those of the first example.

A column of slenderness ratio 25 is chosen, for which

$$(\sigma_E/E) = 157{\cdot}914\times10^{-4} \quad \sigma_t/E = 89{\cdot}449\times10^{-4}.$$

The short-time buckling load of this column is reached when Eqs. (20) and (33) yield the same amplitude, provided it is smaller than one. By trial and error this is the case for

$$\bar{\sigma}/E = 97{\cdot}7\times10^{-4}.$$

A service load corresponding to

$$\bar{\sigma}/E = 92\times10^{-4}$$

will then give according to Eq. (20)

$$a_{\mathrm{cr}} = 0{\cdot}9664$$

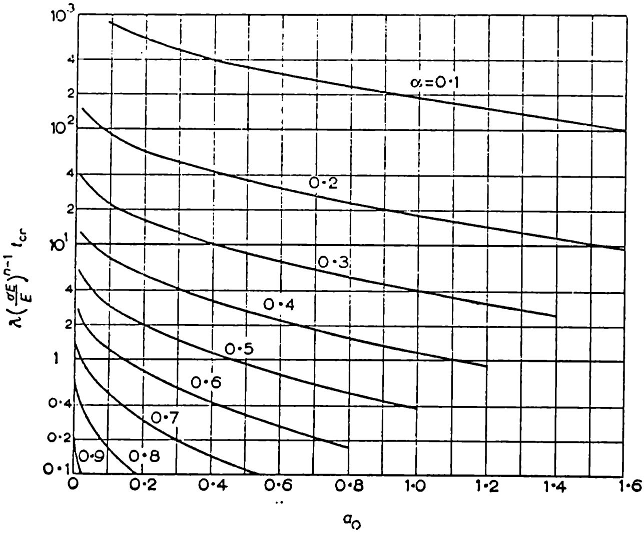

Fig. 7. Reduced critical time vs. initial amplitude of deviation and load ratio for pin-jointed H-section columns. Primary creep neglected, no plastic deformations. Stress exponent of secondary creep n=3. Analysis according to KEMPNER[11]

and Eq. (33) a zero time amplitude

$$\acute{a}(0)=0{\cdot}0434.$$

The critical time as computed from Eq. (24) is

$$t_{cr} = 16 \text{ sec.}$$

One sees how sensitive the lifetime of such a column will be. It ranges from infinity to zero in a loading range

$$\bar{\sigma}_t = 66{,}192 \text{ p.s.i. to } \bar{\sigma}_{buc} = 72{,}298 \text{ p.s.i.}$$

and is already reduced to 16 sec for

$$\bar{\sigma} = 68{,}080 \text{ p.s.i.}$$

(5) It is difficult to make a numerical comparison with Libove's theory both on account of the lengthy computations and because of the lack of data for the material constants of his representation as applicable to 24S-T3. For idealized columns of 75S-T6 at 600°F Libove quotes the values

$$E=5{\cdot}2\times10^6 \text{ p.s.i. } K=2/3 \;\; A=0{\cdot}264\times10^{-6} \text{ hr}^{-1} \;\; B=1{\cdot}92\times10^{-3} \text{ p.s.i.}^{-1}.$$

The relation between his reduced critical time and his initial straightness parameter which are in our notation

$$\gamma_L t_{cr} = \left(\frac{AEB}{2K}\,\frac{\alpha}{1-\alpha}\right)^{1/K} \exp\left\{\frac{B}{K}\,\bar{\sigma}\left(1+\frac{a_0}{1-\alpha}\right)\right\} t_{cr}$$

$$\gamma_R/\gamma_L = \exp\left(-\frac{B}{K}\cdot\frac{a_0}{1-\alpha}\right)$$

is given in *Fig. 5* of the paper quoted.

Libove extended his theory to solid rectangular sections.[22] For the material constants given above a plot, similar to that of *Fig. 7*, was given by MATHAUSER and LIBOVE[23] and is reproduced here as *Fig. 8*.

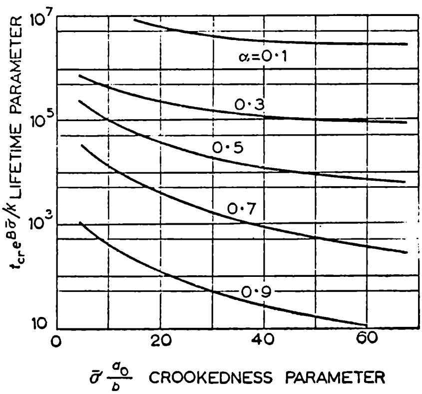

Fig. 8. Reduced critical time vs. crookedness parameter and load ratio for pin-jointed, solid rectangular section columns, of width b, according to LIBOVE[22]

COMPARISON WITH EXPERIMENTS

Extensive tests were conducted at the National Advisory Committee for Aeronautics on creep buckling of rectangular-section columns. The material constants of Libove's creep law for 75S-T6 aluminium alloy at temperatures between 300° and 600°F may be found in a technical note by MATHAUSER and BROOKS[24] together with results on fifty-four specimens under varying conditions. An interesting observation is that collapse occurs suddenly at a finite critical deflexion, confirming the presence of instantaneous plastic deformations. Despite the fact that Libove's theory does not account for this it predicts critical times which are generally shorter than the experimentally observed ones. The largest discrepancies are associated with the largest average compressive stresses. This would seem to indicate that the exponential law has been extrapolated with too much confidence.

Other tests conducted at the Polytechnic Institute of Brooklyn on fifteen 2024-T4 rectangular section columns of slenderness ratio 111 at 600°F have indicated the disturbing influence of higher harmonics in the initial deviation. An estimate of the fundamental and third harmonic was obtained at room temperature from strain-gauge measurements along the column. Unfortunately the measurements could not be taken at the testing temperature just prior to load application. For this reason some smaller discrepancies are attributed to variations in the creep deformations with preload intensity during the heating period. Correlation between these tests and theory awaits extended information on the properties of the material used.

The growth of the third harmonic during the buckling process was observed with the aid of dial gauges in tests by HULT[26], and compared to the theoretical growth when the initial shape is a pure fundamental. The columns were of extruded pure aluminium bars machined into H-sections and tested at a temperature of 77·5°C (171°F). No quantitative comparison is given with theory.

All tests show qualitative agreement with the various creep buckling theories concerning the influence of the various parameters. The chief obstacle to quantitative comparisons is generally the lack of sufficient data on creep properties. Theories based on strain hardening hypotheses like Libove's are capable of handling correctly the observed primary and secondary creep effects. They are, however, difficult to amplify and do not lend themselves easily to numerical discussion. Theories based on the Odqvist–Hoff approach have interesting features of simplicity and take plastic deformations into account. However, the idealization of the creep curve they imply poses a problem when critical lifetimes are to be predicted which do not extend far beyond or are even shorter than the primary creep period.

Creep Buckling Under More General Assumptions

The theories developed so far have concentrated on a highly idealized case. To enhance their applicability to practical design problems extensions are necessary, some of which are outlined below.

A good feature of the idealized section is its coincidence with an efficient design. Extension to the solid rectangular section as done by Libove simplifies the production of experimental columns for correlation tests. The unequally flanged H-section has practical importance for the extensions of the theory to reinforced cover sheets and has been considered by Hult[26].

More important is the presence of end restraints in columns which are members of a framework. Consideration of the end restraints is almost an elementary problem in the linear viscoelastic case. But with non-linear creep laws serious new difficulties appear, especially in view of the fact that congruence of the deflected shapes is likely to be seriously violated.

Consideration should also be given to the axial loading history. Variations in the axial load may be caused by the external forces but also by unequal temperature distribution in the structure, a characteristic feature of thermo-elastic buckling. Thus a decrease in axial load may be encountered when the temperature distribution is equalizing. It should be observed that the creep rates will generally be different in sections along the column and will also vary with time according to the temperature distribution and variation.

In this connexion Ness[27] has presented a theoretical investigation of the time dependent buckling of a uniformly heated column of linear viscoelastic material.

Finally creep buckling occurs in other types of structural members. Test data are already available for plates[28] and multi-webbed box-beams[29] and have been announced for circular cylinders.[30] No theories have been developed yet to predict the observed lifetimes.

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