INFLUENCE OF INTERNAL DAMPING ON AIRCRAFT RESONANCE

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1. INTRODUCTION

Assumptions concerning the effects of damping are generally based on a purely viscous model in which the energy dissipated per cycle is proportional to frequency or on a 'structural' hypothesis which makes this energy independent of frequency. The latter approach is the most widely used in flutter calculations. However, by the application of the Fourier integral technique, it is shown that the transient behaviour associated with 'structural damping' is not physically acceptable. Nevertheless, structural damping can be retained as an approximation provided that it be restricted to a given frequency band.

A Maxwell model, involving a single 'hidden' coordinate, furnishes a more satisfactory type of viscoelastic damping. Here, the transient behaviour is fully described by a heredity function that shows how the damping force depends on the past history of the motion. The concept of the heredity function extented to an eventually infinite, number of internal freedoms can provide a

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correct formulation. Experimental evidence is, however, needed to establish the exact form of the heredity function. These points are discussed in some detail in the first part of this paper.

The second part is devoted to the analysis of resonance in one degree of freedom under these various damping laws. In the case of hereditary damping a difficulty arises. The effective spring stiffness of the system becomes a function of the forcing frequency and the concept of a natural undamped frequency is not as clear-cut as it is in the simple viscous or structural damping cases. For a simple Maxwell model two slightly different undamped natural frequencies can be defined and phase resonance is shown to occur at some intermediate value of the forcing frequency.

In the third part resonance in many degrees of freedom is discussed with the help of matrix formulation. Here again hereditary damping results in an effective stiffness matrix whose elements are functions of the frequency. The natural undamped modes are then best defined as those corresponding to phase resonance. Any other definition would be of artificial character and would prevent the experimental determination of such modes, thus defeating one of the essential purposes of resonance testing.

A convenient concept for discussing the forced response in general and resonance in particular is that of a 'characteristic phase lag'. Originally devised for the special case of purely viscous damping [1], it is extended to cover all types of damping and provides and expansion of the response in orthogonal 'response modes'. At each phase resonance one of the orthogonal 'response modes' becomes identical with a natural undamped mode as previously defined. It is shown that at phase resonance the total reactive energy is zero and stationary with respect to small arbitrary variations of the excitation forces. The most difficult problem is to obtain reasonably pure natural modes while using only a

limited number of shaking points. In this connection several principles can be applied: local phase resonance, pseudo-resonance in neighbouring modes, and the stationary property of the partial reactive energy. These are discussed in terms of the orthogonal response modes.

Under certain restrictive assumptions the inertia matrix, the effective stiffness matrix, and the damping matrix can be made diagonal simultaneously. The system is then effectively split into sybsystems with a single degree of freedom.

These assumptions, already considered by Lord Rayleigh [2] in the case of pure viscous damping, are easily extended to cover all types of damping. One is the basis of a method of shaker force adjustment previously evolved by Lewis and Wrisley [3]. The more general one was reformulated in France by Basile [4]. It is the writer's opinion that such assumptions are unduly restrictive and stand no change of being corroborated by experimental evidence.

Terminology

Square matrices are generally denoted by a capital Roman letter as follows:

M	matrix of inertia coefficients
K	matrix of stiffness coefficients
R	matrix of equivalent viscous damping coefficients
A-iB	matrix of dynamic flexibilities.

Column matrices are denoted by a lower case Roman letter, thus:

q	column matrix of coordinates
f	column matrix of excitations
y	column matrix of excitation amplitudes
x=a-ib	column matrix of response amplitudes (complex)
r	column matrix of response amplitudes
z	column matrix of natural mode amplitudes.

The transpose of a matrix is denoted by a 'prime', and the derivative with respect to time by a dot.

The elements of a matrix are indicated by subscripts, e.g.,

q_i is the i^{th} element of column q

A_{ij} is the element of the i^{th} line and j^{th} column of A.

Modes are indicated by a subscript between brackets, e.g.,

$z_{(r)}$ is the r^{th} natural mode (a column matrix).

The convolution or Duhamel integral between two time-functions is denoted by an asterisk, e.g.,

$$\phi * q_i = \int_0^t \phi(t-\tau) q_i(\tau) d\tau \ .$$

2. MECHANICAL AND MATHEMATICAL MODELS FOR THE STUDY OF ELASTIC SPRING BEHAVIOUR

2.1 *Simple Viscous Internal Damping*

The simplest mechanical model of a structural unit where spring action is accompanied by energy dissipation is shown in Figure 1. It consists of a purely elastic spring, with stiffness k_1, and a dashpot in parallel. The force developed by the dashpot is proportional to the velocity $\dot{q}_1(t)$ of the displacement. The force-displacement relation of such a unit is therefore of the type

$$f_1 = k_1 q_1 + c_1 \dot{q}_1 \ . \qquad (1)$$

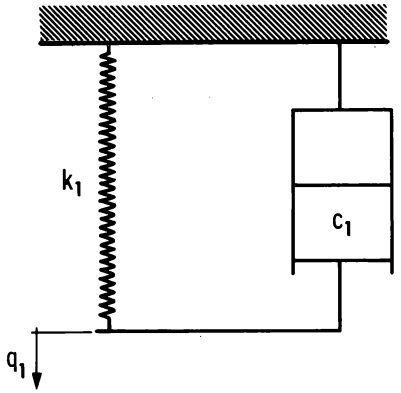

Figure 1 - Structural Unit with Simple Viscous Damping

This equation furnishes immediately the reaction force of the unit when the displacement is known as a function of time.

Conversely, when the force is known as a function of time, the solution of the differential equation (1) yields

$$q_1(t) = q_1(o) + \frac{1}{c_1} \int_o^t e^{-\tau/t_1} f_1(t-\tau)\,d\tau \qquad (2)$$

where $t_1 = c_1/k_1$ is the 'time constant' of the unit.

For instance, when a constant force f_1 is suddenly applied at $t = o$ to the unstrained unit ($q_1(o) = o$),

$$q_1 = \frac{f_1}{k_1} (1 - e^{-t/t_1})$$

and the displacement tends ultimately to the static displacement f_1/k_1. When the force suddenly applied is a harmonic function of time

$$f_1 = \mathrm{Re} \left\{ y_1 e^{i\omega t} \right\}$$

we find

$$q_1 = \mathrm{Re}\left\{\frac{y_1}{k_1 + i\omega c_1}\left(e^{i\omega t} - e^{-t/t_1}\right)\right\} .$$

Here again the motion consists of a transient associated with the time constant t_1 of the unit and a steady state part which is the forced vibration or frequency response of the unit. The amplitude and phase relationship between force and motion in the forces response is also obtained by the reverse procedure of imposing on the unit a harmonic displacement

$$q_1 = \mathrm{Re}\{x_1 e^{i\omega t}\} \tag{3}$$

and determining the force directly from equation (1)

$$f_1 = \mathrm{Re}\left\{(k_1 + i\omega c_1)x_1 e^{i\omega t}\right\} . \tag{4}$$

Equations (3) and (4) are actually simpler to discuss; they also make comparison with the behaviour of other structural units easier. From them we obtain

$$q_1 = x_1 \cos\omega t$$

$$f_1 = x_1(k_1 \cos\omega t - \omega c_1 \sin\omega t) .$$

The elimination of the time between these equations leads to the hysteresis curves in an (f_1, q_1) diagram. It is convenient to make these diagrams non dimensional by using the reduced variables

$$\frac{q_1}{x_1} = \cos\omega t$$

$$\frac{f_1}{k_1 x_1} = \cos\omega t - \omega\frac{c_1}{k_1}\sin\omega t .$$

This amounts to taking the amplitude of vibration as the unit of displacement and the static force necessary to produce this displacement as the unit of force. The family of hysteresis

curves (Figure 2) is then dependent on a single parameter $(\omega c_1/k_1)$. All curves have a common vertical tangent at the points (1,1) and (-1,-1).

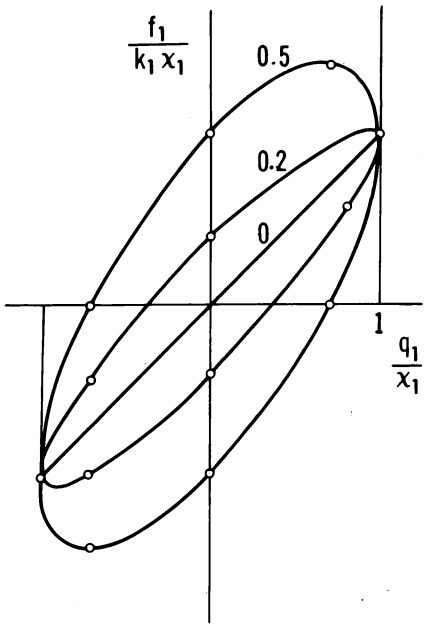

Figure 2 - Hysteresis Curves of Structural Unit with Simple Viscous Damping for Different Values of the Parameter $\omega c_1/k_1$

The energy dissipated during a cycle is

$$\int_0^{\frac{2\pi}{\omega}} f_1 \dot{q}_1 dt = \pi x_1^2 c_1 \omega \ .$$

For a given amplitude of vibration it is seen to increase proportionally to the frequency.

2.2 *Hereditary Internal Damping*

Viscous damping may be retained as a source of energy dissipation in more complicated models. The Maxwell model of Figure 3 incorporates a second spring in series with the dashpot and forms really a system having two degrees of freedom. Its behaviour is fully described by the two equations:

$$f_1 = k_1 q_1 + c_2(\dot{q}_1 - \dot{q}_2) \tag{6}$$

$$k_2 q_2 = c_2(\dot{q}_1 - \dot{q}_2) \ . \tag{7}$$

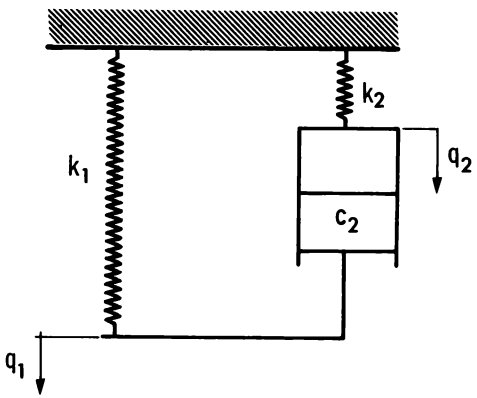

Figure 3 - Maxwell Type of Structural Unit

Since we are not interested in the internal degree of freedom q_2, we solve equation (7) for q_2 and substitute the result in equation (6) to obtain

$$f_1 = k_1 q_1 + \int_0^t \phi(t-\tau)\dot{q}_1(\tau)d\tau \tag{8}$$

where

$$\phi(t) = k_2 \exp\left(-\frac{k_2}{c_2}t\right) , \tag{9}$$

with the underlying assumption that for t = o the unit is unstrained.

In equation (1) the damping term is directly proportional to the instantaneous velocity of displacement. In the present case it is seen to depend on the past history of this velocity. For this reason it is suitably called 'hereditary damping' and the function $\phi(t)$ is known as the 'hereditary function' of the damping.

In solving problems of transient behaviour for such units the operational form of the force-displacement relationship is useful.

With the usual definition of the Laplace transform of a function x(t)

$$\bar{x}(s) = \int_o^\infty e^{-st} x(t)dt$$

and assuming $q_1(o) = o$ and $q_2(o) = o$, the basic equations (6) and (7) have the following Laplace transforms

$$\bar{f}_1(s) = (k_1 + sc_2)\bar{q}_1 - sc_2\bar{q}_2$$

$$(k_2 + sc_2)\bar{q}_2 = sc_2\bar{q}_1$$

whence, after elimination of $\bar{q}_2$

$$\bar{f}_1 = \left(k_1 + \frac{c_2 k_2 s}{k_2 + sc_2} \right) \bar{q}_1 \ . \qquad (10)$$

In this operational relation the coefficient of $\bar{q}_1$, namely

$$z_1(s) = k_1 + \frac{c_2 k_2 s}{k_2 + sc_2} \qquad (11)$$

is known as the 'operational modulus' of the unit and its reciprocal the 'operational flexibility'. (The names operational impedance and operational admittance are misleading, since the electrical analogies, from which they are derived, are actually violated.)

Solving equation (10) for $\bar{q}_1$ and re-interpreting we obtain

$$q_1(t) = \frac{f_1(t)}{k_1+k_2} + \frac{1}{c_2}\left(\frac{k_2}{k_1+k_2}\right)^2 \int_0^t e^{-\tau/t_2} f_1(t-\tau)\,d\tau \qquad (12)$$

where

$$t_2 = c_2\left(\frac{1}{k_1} + \frac{1}{k_2}\right) .$$

A comparison of this general solution with equation (2) shows a significant difference in behaviour due to the presence of an instantaneous displacement associated with a spring modulus (k_1+k_2). Such a behaviour is physically evident from a simple inspection of the structure of the unit.

From equation (12) the response of the unit to a suddenly applied constant force is readily found to be

$$q_1(t) = \frac{f_1}{k_1}\left(1 - \frac{k_2}{k_1+k_2}\, e^{-t/t_2}\right)$$

and tends ultimately to the same static displacement as in the previous model.

The forced response to harmonic excitation is of greater extent. Using the reverse procedure we put

$$q_1 = \mathrm{Re}\{x_1 e^{i\omega t}\} \text{ and } q_2 = \mathrm{Re}\{x_2 e^{i\omega t}\}$$

in equations (6) and (7), and eliminating x_2, we obtain

$$f_1 = \mathrm{Re}\left\{\left(k_1 + i\,\frac{\omega c_2 k_2}{k_2+i\omega c_2}\right) x_1 e^{i\omega t}\right\} .$$

In accordance with general theory, the complex modulus of the forced response

$$z_1(\omega) = k_1 + i\,\frac{\omega c_2 k_2}{k_2+i\omega c_2} \qquad (13)$$

is also obtained from the operational modulus (11) by substitution of $i\omega$ for s.

From equation (4) the complex modulus in the case of simple viscous damping is found to be

$$z_1(\omega) = k_1 + i\omega c_1 \ . \tag{14}$$

When equation (13) is split into its real and imaginary parts it may be given a similar form, namely

$$z_1(\omega) = k_e + i\omega c_e \tag{15}$$

where

$$k_e = k_1 + k_2 \frac{\omega^2 c_2^2}{k_2^2 + \omega^2 c_2^2} \tag{16}$$

$$c_e = c_2 \frac{k_2^2}{k_2^2 + \omega^2 c_2^2} \ . \tag{17}$$

The behaviour of the hereditary damping unit is thus reduced to that of a simple viscous damping unit with characteristics depending on the frequency. This procedure is convenient for purposes of comparison and since moreover it corresponds to a decomposition of the complex modulus into real and imaginary parts it is also physically significant.

It is seen (Figure 4) that the equivalent spring stiffness k_e increases with frequency to the asymptotic value (k_1+k_2), whilst the equivalent coefficient of viscous damping c_e decreases from c_2 to zero. The energy dissipated per cycle, $\pi x_1^2 \omega c_e$, also tends to zero for very high frequencies. Per unit time, however, it would tend ultimately to the finite value $\frac{1}{2} x_1^2 k_2^2 / c_2$.

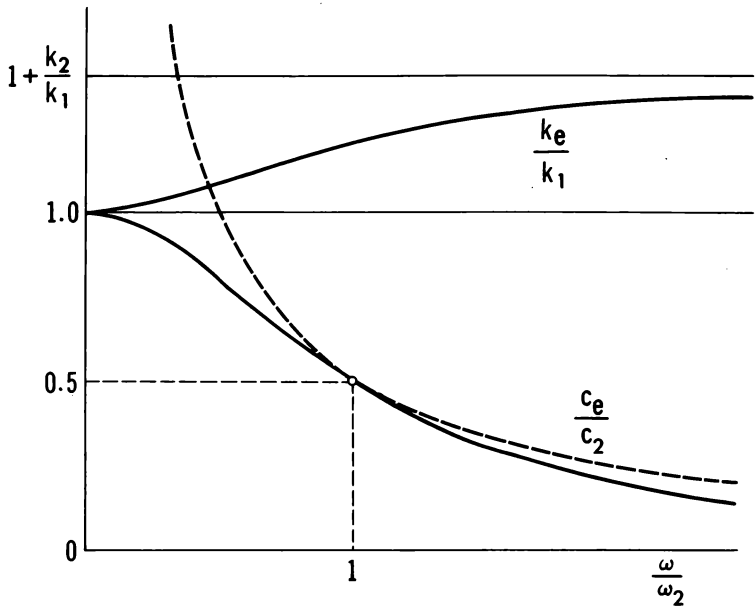

Figure 4 - Effective Spring Stiffness Re and Equivalent Viscous Damping Coefficient c_e for Maxwell Unit

There is a characteristic circular frequency of the unit for which the energy dissipated per cycle passes through a maximum. The maximum of the product ωc_e is easily found to occur for the value

$$\omega_2 = \frac{k_2}{c_2}$$

where

$$c_e = \frac{1}{2} c_2 \ .$$

This value is conveniently taken as a reference in the non-dimensional diagram of Figure 4, so that the curve

$$\frac{c_e}{c_2} = \frac{1}{1 + \left(\frac{\omega}{\omega_2}\right)^2} \qquad (18)$$

becomes valid for all Maxwell units.

The behaviour of c_e as a function of the frequency is also reflected in the hysteresis curves of Figure 5. At zero frequency we have a straight line of pure spring behaviour; for $\omega = \omega_2$ an elliptical curve of maximum area; and, when the frequency tends to infinity, we have again a pure spring behaviour associated with the modulus (k_1+k_2) and resulting in a straight line of greater slope.

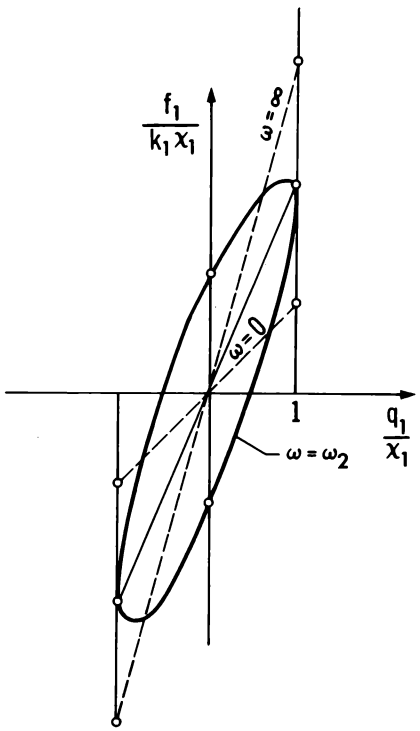

Figure 5 - Hysteresis Curves for Maxwell Unit

The Maxwell model is only a simple example of a unit with hereditary damping. More complicated models are easily imagined. For example, several Maxwell units with different time constants can be connected in parallel. A large number of hidden coordinates will facilitate the correlation between the theoretical behaviour of k_e and c_e as functions of the frequency and experimental evidence. The transient behaviour of such units will still depend on a relation of the form of equation (8) with a suitable hereditary function $\phi(t)$. On the basis of equation (9) we can write as a generalized form of the hereditary function, involving a continuous relaxation spectrum:

$$\phi(t) = \int_0^\infty F(r)\, e^{-rt} dr$$

or, in operational form,

$$\bar{\phi}(s) = \int_0^\infty \frac{F(r)}{r+s}\, dr \ .$$

Then, from the operational form of equation (8), namely

$$\bar{f}_1 = z_1(s)\bar{q}_1 \tag{19}$$

where

$$z_1(s) = k_1 + s\bar{\phi}(s) \tag{20}$$

we can deduce the corresponding operational modulus

$$z_1(s) = k_1 + s \int_0^\infty \frac{F(r)}{r+s}\, dr \ .$$

The following example is given by Biot:

$$F(r) = \frac{2}{\pi} k_1 g_1 \frac{1}{r} \quad \text{for } r > \varepsilon$$

$$= 0 \quad \text{for } r < \varepsilon \ .$$

Then,

$$z_1(s) = k_1 - \frac{2}{\pi} k_1 g_1 \ln \frac{\varepsilon}{\varepsilon+s} .$$

After replacing s by $i\omega$, the complex modulus of the frequency response appears to be

$$z_1(\omega) = k_1\left(1 + \frac{2}{\pi} g_1 \ln\sqrt{1 + \frac{\omega^2}{\varepsilon^2}}\right) + ik_1 g_1 \frac{2}{\pi} \tan^{-1}$$

For small ε the imaginary part tends very rapidly to the asymptotic value $k_1 g_1$. Since g_1 would be small, the real part, or effective spring stiffness, is also practically constant in a wide frequency band, though it ultimately grows to infinity. This behaviour is approximately in accord with the so-called 'structural damping' law considered in more detail in the next section. The corresponding hereditary function is

$$\phi(t) = \frac{2}{\pi} k_1 g_1 \int_{\varepsilon}^{\infty} \frac{e^{-rt}}{r} dr = -\frac{2}{\pi} k_1 g_1 \, Ei(-\varepsilon t) .$$

2.3 *Structural Damping*

If the complex impedance is constant in the whole frequency range

$$z_1(\omega) = k_1(1 + ig_1) \tag{21}$$

and the hysteresis curve (Figure 6) is also independent of frequency. This 'structural damping law' is widely used in flutter calculations. However, it cannot be followed in all its implications because, as will now be shown, it leads to a physically inacceptable transient behaviour. This behaviour can be deduced from the frequency response by the Fourier integral technique. In doing this it must be remembered that the imaginary part of the complex impedance is necessarily an odd function of the frequency. Consequently equation (21) holds for positive values of ω and must be extended to negative values by changing the sign of the imaginary part.

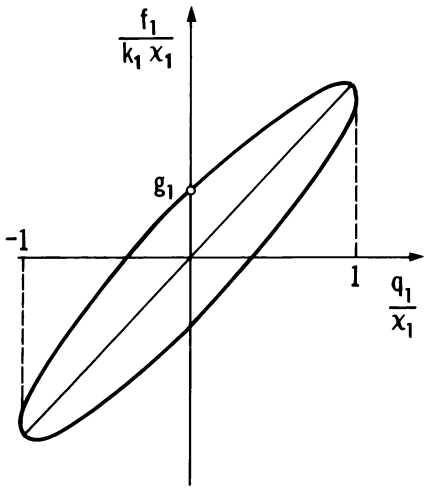

Figure 6 - Hysteresis curve of structural damping

Let us begin by assuming that $q_1(t)$ follows some periodic law expressed by a convergent Fourier series of period $T = 2\pi/\omega$ in the form

$$q_1(t) = \sum_{-\infty}^{\infty} C_n e^{-in\omega t}$$

where

$$C_n = A_n + iB_n \text{ for positive values of } n$$

$$C_n = A_n + iB_n \text{ for negative values of } n$$

so that also

$$q_1(t) = C_o + 2 \sum_{1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \quad .$$

In order to find the corresponding force exerted by the unit we must multiply each term of the complex Fourier series by $k_1(1+ig_1)$ if n is positive, by $k_1(1-ig_1)$ if n is negative and simply by k_1 for the term corresponding to n = 0. The force is then given by

$$f_1(t) = k_1 g_1(t) + k_1 g_1 \sum_1^\infty \left\{ i(A_n + iB_n) e^{-in\omega t} - i(A_n - iB_n) e^{-in\omega t} \right\}$$

$$= k_1 g_1(t) - k_1 g_1 \sum_1^\infty (2A_n \sin n\omega t - 2B_n \cos n\omega t) .$$

The last series is known as the 'allied Fourier series' to the series for $q_1(t)$ and, according to the theory for the summation of such series [5], the result can be expressed in the form

$$f_1(t) = k_1 g_1(t) + \frac{k_1 g_1}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} \omega q_1(\tau) \cot \frac{\omega(\tau - t)}{2} \, d\tau .$$

The integral must naturally be understood as a 'principle Cauchy value'. Passing to the limit for $\omega \to o$ $(T \to \infty)$, or using the corresponding analysis for the summation of 'allied Fourier integrals', we obtain

$$f_1(t) = k_1 q_1(t) + \frac{1}{\pi} k_1 g_1 \int_{-\infty}^{\infty} q_1(\tau) \frac{d\tau}{\tau - t} . \qquad (22)$$

It is now necessary to verify this general law of 'structural damping' by recomputing the frequency response.

Putting $q_1 = x_1 e^{i\omega\tau}$ and $f_1 = y_1 e^{i\omega t}$ we have

$$z_1(\omega) = k_1 + \frac{1}{\pi} k_1 g_1 \int_{-\infty}^{\infty} e^{i\omega(\tau - t)} \frac{d\tau}{\tau - t}$$

or, with the change of variable $u = \omega(\tau - t)$

$$z_1(\omega) = k_1 + \frac{1}{\pi} k_1 g_1 \left[\int_{-\infty}^{\infty} \frac{\cos u \, du}{u} + i \int_{-\infty}^{\infty} \frac{\sin u \, du}{u} \right] .$$

The first integral is zero by reason of symmetry. The second is well known to be equal to π. We are thus led back to equation (21) and we may note that in the case when ω is negative, the limits of the second integral will be reversed, confirming that the imaginary part of z_1 will have to change sign.

Besides the severe restrictions put on the behaviour of $q_1(t)$ to ensure the existence of the integral, the general form obtained for structural damping is physically absurd. It would imply that the force exerted by the unit does not only depend on the past history of the motion but on its future as well. Though structural damping does not appear to enjoy a sound physical basis, it is an acceptable approximation in some definite frequency band. This was already apparent from the example of the generalized hereditary function. It is also true when compared with the simple Maxwell model. To show this we deduce from equation (21) the equivalent coefficient of viscous damping

$$c_e = k_1 g_1 \frac{1}{\omega}$$

and compare it with equation (18). This comparison is easier if we write

$$\frac{c_e}{c_2} = g_1 \frac{k_1}{k_2} \left(\frac{\omega}{\omega_2} \right)^{-1} .$$

This curve can be made to touch the curve defined by equation (18), provided

$$g_1 = \frac{1}{2} \frac{k_2}{k_1} \tag{23}$$

and the contact point is then at $\omega = \omega_2$ (Figure 4).

If the simple Maxwell model is acceptable, equation (23) furnishes a correlation between the observed increase in spring stiffness with frequency and the value which should be taken for the coefficient g when using a structural damping law. The difference between isothermal and adiabatic moduli of metallic materials is quoted to be of the order of 1%. This would give

$$g_1 = 0.005 .$$

This figure is probably of the correct order of magnitude. It must however be remembered that other sources of damping,

like slip in riveted joints, are present, so that the value of g , which integrates the other effects, may well rise to about 0.02. The increasing use of bonded or integrally stiffened structures will tend to reduce the value of this damping coefficient to its purely internal damping contribution.

2.4 *Internal Damping in Continuous Media*

Enrico Volterra, who considered the vibrations of discrete systems with hereditary characteristics, has also extended the hereditary damping concept to continuous media [6]. The analysis shows that for an isotropic medium the stress tensor $\{\sigma_{ij}\}$ and the strain tensor $\{e_{ij}\}$ are related by two heredity functions $\phi(t)$ and $\psi(t)$, so that

$$\sigma_{ij} = \delta_{ij}\left(\lambda e + \int_0^t \phi(t-\tau)\,\frac{\partial \ell}{\partial \tau}\,d\tau\right) + 2\mu e_{ij} + 2\int_0^t \psi(t-\tau)\,\frac{\partial e_{ij}}{\partial \tau}\,d\tau . \quad (24)$$

In this relation δ_{ij} is Kronecker's delta, $e = \Sigma e_{ii}$, and λ and μ are the usual Lamé constants.

By using a single relaxation constant for each type of heredity, the operational form of the relations is found to be

$$\bar{\sigma}_{ij} = \delta_{ij}\left(\lambda + \frac{sk_\phi c_\phi}{k_\phi + sc_\phi}\right)\bar{e} + 2\left(\mu + \frac{sk_\psi c_\psi}{k_\psi + sc_\psi}\right)\bar{e}_{ij} \quad .$$

They agree, as a particular form, with more general stress-strain relations justified by M.A. Biot on the basis of thermodynamical considerations [7, 8].

Letting k_ϕ and k_ψ tend to infinity, we obtain the extension of the simple viscous damping model of Figure 1 for which

$$\bar{\sigma}_{ij} = \delta_{ij}(\lambda + sc_\phi)\bar{e} + 2(\mu + sc_\psi)\bar{e}_{ij} \quad .$$

For harmonic motion

$$e_{ij} = \mathrm{Re}\{Y_{ij}e^{i\omega t}\}$$

and this yields

$$\sigma_{ij} = \mathrm{Re}\{\delta_{ij}(\lambda+i\omega c_\phi)\Sigma Y_{ii}e^{i\omega t}+2(\mu+i\omega c_\psi)Y_{ij}e^{i\omega t}\} \ . \qquad (25)$$

The natural extension of the structural damping law would then be

$$\sigma_{ij} = \mathrm{Re}\{\delta_{ij}\lambda(1+ig_\phi)\Sigma Y_{ii}e^{i\omega t}+2\mu(1+ig_\psi)Y_{ij}e^{i\omega t}\} \ . \qquad (26)$$

When these relations are solved for the amplitudes of strain Y_{pq}, it appears that Poisson's ratio has to be replaced by a complex quantity

$$\nu(1 + ig_\nu) \qquad (27)$$

where g_ν depends on g_ϕ and g_ψ and the usual shear modulus G and traction modulus E may then be respectively replaced by

$$G(1 + ig_\psi) \qquad (28)$$

and

$$2G(1 + ig_\psi)[1 + \nu(1 + ig_\nu)] \ . \qquad (29)$$

In the special case $g_\phi = g_\psi = g$, it is found that $g_\nu = o$. Poisson's ratio then remains real, whilst the same complex factor affects both moduli, namely

$$\begin{cases} G(1 + ig) \ , & \text{and} \\ E(1 + ig) \ . \end{cases} \qquad (30)$$

As will be seen later, this leads to an important simplification in the study of the forced response of a continuous isotropic medium. It has the essential consequence that elastically uncoupled modes are also automatically uncoupled with respect to internal damping.

3. DYNAMIC AMPLIFICATION FACTORS FOR A SINGLE DEGREE OF FREEDOM

Before attempting a discussion of the influence of internal damping on aircraft resonance it will be useful to examine the resonance phenomenon for a single degree of freedom. The addition of the effects of inertia is the essential new feature we must introduce.

3.1 *Resonance in the Presence of Simple Viscous Damping*

With the addition of a mass m_1 to the model of Figure 1, equation (1) becomes

$$f_1 = k_1 q_1 + c_1 \dot{q}_1 + m_1 \ddot{q}_1 \tag{31}$$

and, under the assumption of simple harmonic displacements, when

$$q_1 = \mathrm{Re}\{x_1 e^{i\omega t}\} \ . \tag{32}$$

The external force required becomes

$$f_1 = \mathrm{Re}\{y_1 e^{i\omega t}\} \tag{33}$$

where

$$y_1 = (k_1 + i\omega c_1 - \omega^2 m_1) x_1 \ . \tag{34}$$

Whilst varying the frequency, let us keep the amplitude of excitation constant and always referred to the same phase. This is the case if y_1 is taken as a real constant. By solving equation (34) for x_1 and substituting in equation (32), the forced response is found to be

$$q_1 = \mathrm{Re}\left\{\frac{y_1}{k_1 + i\omega c_1 - \omega^2 m_1} e^{i\omega t}\right\} \ . \tag{35}$$

Noting that the low frequency or quasi-static response is

$$q_1 = \mathrm{Re}\left\{\frac{y_1}{k_1} e^{i\omega t}\right\}$$

we may rewrite equation (35) in the form

$$q_1 = \mathrm{Re}\left\{ \frac{y_1}{k_1} D\, e^{i\omega t} \right\} \tag{36}$$

where

$$D = \frac{1}{1 + i\omega \frac{c_1}{k_1} - \omega^2 \frac{m_1}{k_1}} \tag{37}$$

is the 'dynamic amplification factor'.

It is convenient at this stage to introduce the quantities

$$\omega_o = \sqrt{k_1/m_1} \quad \text{and} \quad \varepsilon = \frac{c_1}{2\omega_o m_1} = \frac{c_1}{2\sqrt{m_1 k_1}} \tag{38}$$

which are the most important characteristics of the free vibrations of the system. The free vibrations are governed by equations (31) with $f_1 = o$. With the aid of definitions (38) this is transformed to

$$\omega_o^2 q_1 + 2\varepsilon\omega_o \dot{q}_1 + \ddot{q}_1 = o$$

and the roots of its characteristic equation are

$$\omega_o [-\varepsilon \pm \sqrt{\varepsilon^2 - 1}] \ .$$

In the absence of damping (when $c_1 = o$ or $\varepsilon = o$) these roots are $\pm i\omega_o$, so that ω_o is the natural circular frequency of the undamped system. The dimensionless quantity ε along governs the complex nature of the roots and is the true measure of the dynamic effects of damping. Since the critical value, above which the roots become real, is $\varepsilon = 1$, ε may be considered as the ratio of the actual damping coefficient c_1 to its critical value.

Definitions (38) also simplify the expression (37) and we obtain

$$D = \left[1 - \left(\frac{\omega}{\omega_o} \right)^2 + 2i\varepsilon \frac{\omega}{\omega_o} \right]^{-1} \ . \tag{39}$$

The significance of D as a complex quantity is best revealed by its polar form

$$D = Ae^{-i\phi} \tag{40}$$

where

$$A = \left[\left(1 - \frac{\omega^2}{\omega_o^2}\right) + 4\varepsilon^2 \frac{\omega^2}{\omega_o^2}\right]^{-1/2} \tag{41}$$

and

$$\tan\phi = \frac{2\varepsilon \frac{\omega}{\omega_o}}{1 - \left(\frac{\omega}{\omega_o}\right)^2} \quad . \tag{42}$$

Substitution of equation (40) in equation (36) yields

$$q_1 = \frac{y_1}{k_1} A \cos(\omega t - \phi) \tag{43}$$

so that A appears as the dynamic amplification factor for the amplitude of vibration and ϕ as the phase lag angle between the motion and the excitation.

Figure 7 is a representation of D as a complex quantity in the Argand plane. Figure 8 is the classical representation of A and ϕ as functions of the frequency.

Phase resonance is defined as the condition for which the motion is in quadrature with the excitation. The condition $\phi = \pi/2$ is precisely seen to occur when $\omega = \omega_o$. Since (at least for small values of ε) the phase is very sensitive to frequency alterations in the neighbourhood of resonance, the measurement of phase is an ideal way to ascertain resonance conditions.

Amplitude resonance is defined as the condition for which A is a maximum. This occurs for a slightly lower frequency $\omega = \omega_o \sqrt{1-2\varepsilon^2}$, and the maximum of A, which is $[2\varepsilon\sqrt{1-\varepsilon^2}]^{-1}$, is not much greater than the value $[2\varepsilon]^{-1}$ it has at phase resonance.

Except for large damping, amplitude resonance, though less pure, is a good substitute for phase resonance.

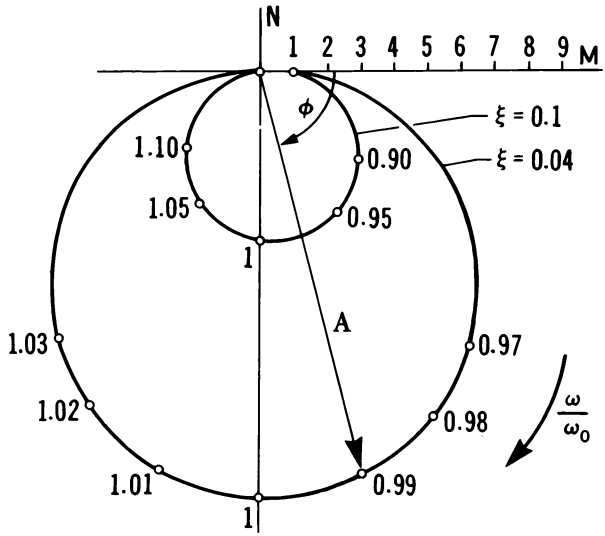

Figure 7 - Polar Representation of Dynamic Amplification Factor. Simple Viscous Damping

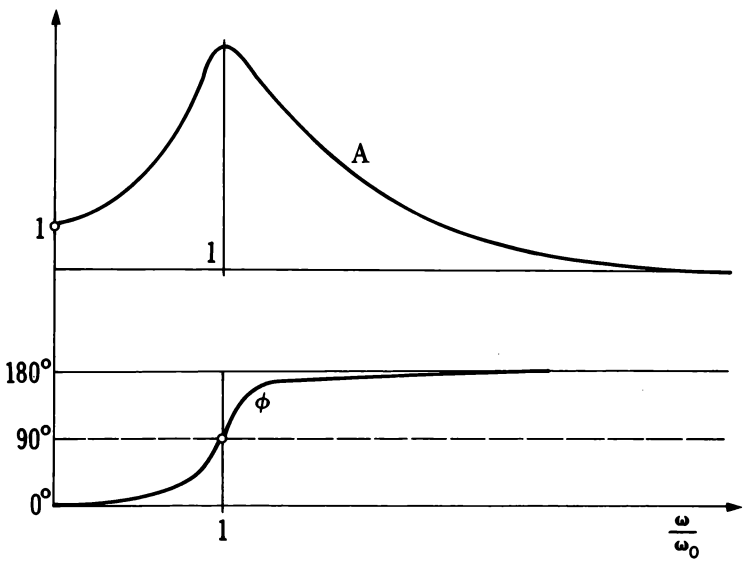

Figure 8 - Amplitude and Phase of Dynamic Amplification Factor. Simple Viscous Damping

3.2 *Resonance in the Presence of Hereditary Damping*

The addition of a mass m_1 to the simple Maxwell model of Figure 3 produces, after similar calculations, a dynamic amplification factor

$$D = \left(1 + i\,\frac{\omega c_2 k_2}{k_1(k_2 + i\omega c_2)} - \omega^2\,\frac{m_1}{k_1}\right)^{-1} . \tag{44}$$

Transformation and examination of this expression reveals two difficulties. The first is how to define the natural undamped frequency of the system. If one puts $c_2 = o$ the natural frequency is

$$\omega_o = \sqrt{k_1/m_1} . \tag{45}$$

But, when c_2 tends to infinity, the system is also undamped and its natural frequency is given by

$$\sqrt{(k_1 + k_2)/m_1} = \omega_o\sqrt{1 + \kappa} \tag{46}$$

where

$$\kappa = k_2/k_1 . \tag{47}$$

Since the ratio κ is fairly small, these two definitions of a natural undamped frequency are not very different. However, as will appear later, there will be reason to choose a third definition, giving an intermediate value.

The second difficulty is the definition of a reduced damping coefficient. The characteristic equation of the free vibrating system is of the third degree and the nature of the roots is consequently more difficult to ascertain. It will be sufficient for our purpose to introduce the dimensionless coefficient

$$\varepsilon = \frac{c_2}{\sqrt{m_1 k_1}} . \tag{48}$$

By taking (45), (46) and (48) into account, equation (44) may be written

$$D = \left(1 - \alpha^2 + i\,\frac{\varepsilon\kappa\alpha}{\kappa+i\varepsilon\alpha}\right)^{-1} \tag{49}$$

with

$$\alpha = \frac{\omega}{\omega_o} .$$

The resulting phase lag angle ϕ is given by

$$\tan\phi = \frac{\kappa^2\varepsilon\alpha}{(\kappa^2+\varepsilon^2\alpha^2)(1-\alpha^2)+\kappa\varepsilon^2\alpha^2} \quad . \tag{50}$$

The denominator of expression (50) is a quadratic in α^2. One of the roots is real and negative and has consequently no special significance. The other root is real and positive and it may be concluded that there is one value of the forced frequency corresponding to phase resonance. The position of the root with respect to the values $\alpha = 1$ and $\alpha = \sqrt{1+\kappa}$, which correspond to undamped natural frequencies, is easily found. Insertion of the first value gives to the quadratic the positive value $\kappa\varepsilon^2$. Insertion of the second gives the negative value $-\kappa^3$. Since the coefficient of α is negative this proves that phase resonance occurs between the undamped natural frequencies which were previously considered.

Since, from a physical standpoint, the definition of the undamped natural frequency is necessarily somewhat artificial, there is some virtue in adopting a precise definition based on the principle of phase resonance. In other words, the natural undamped frequencies of linear visco-elastic systems will be considered as best defined by their phase resonant frequencies. In other respects the behaviour of the phase angle with increasing frequency is qualitatively similar to that of a system with simple viscous damping. Because of the reduction in equivalent damping at high frequencies it simply approaches sooner its asymptotic value of 180°.

Consideration of amplitude resonance does not reveal any remarkable features.

3.3 *Resonance in the Presence of Structural Damping*

The dynamic amplification factor due to structural damping as defined by equation (21) is

$$D = \left(1 + ig_1 - \frac{\omega^2}{\omega_o^2}\right)^{-1} \tag{51}$$

where

$$\omega_o = \sqrt{k_1/m_1} \ .$$

The behaviour of the phase lag angle

$$\tan\phi = \frac{g_1}{1 - \left(\frac{\omega}{\omega_o}\right)^2}$$

shows phase resonance for $\omega = \omega_o$, which is thus rightly defined as the natural undamped frequency of the system. A significant difference is the existence of a small, but non-zero, phase angle at zero frequency.

The representation of D in the Argand plane is exceptionally simple. The reciprocal of D is clearly a straight line parallel to the real axis and, from inversion theory, the locus of D itself is a circle with its centre on the imaginary axis (Figure 9).

It is obvious from this circle diagram that for this type of damping amplitude resonance and phase resonance are coincident.

Kennedy and Pancu [9] have shown that in the case of structural damping $d\phi/d\omega^2$ also reaches its maximum at resonance. The method of determining resonance by measurements of the rate of change of phase with ω^2 has the advantage of avoiding the measurement of absolute phase angles, which may present difficulties for certain types of excitation. In cases other than those with structural damping, the maximum rate of phase change is not exactly coincident with phase resonance but very close to it, provided damping is small.

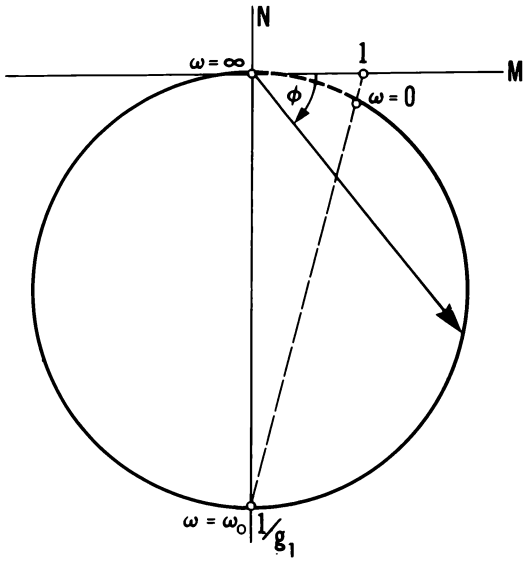

Figure 9 - Circular Locus of Dyanmic Amplification Factor for Structural Damping

4. INFLUENCE OF DAMPING ON RESONANCE IN MANY DEGREES OF FREEDOM

4.1 *Matrix Formulation of the Problem*

To discuss frequency response problems with internal damping on systems with a large but finite number of degrees of freedom it is most convenient to use matrices because of the simplicity offered by matrix formulation. The behaviour of continuous structures may be approximated closely by such methods and most of the conclusions so reached may be extended to continuous structures without further justification.

Let us return to the equation of motion of the single degree of freedom system with hereditary damping in the form

$$m_1\ddot{q}_1 + k_1 q_1 + \phi * \dot{q}_1 = f_1 \tag{53}$$

which is derived from equation (8) by the simple addition of inertia effects. The notation used for the convolution or Duhamel integral

$$\phi * \dot{q}_1 = \int_0^t \phi(t-\tau)\, \dot{q}_1(\tau)\, d\tau \tag{54}$$

is convenient for the extension required by the matrix formulation. The extension of equation (53) to many degrees of freedom is simply

$$M\ddot{q} + Kq + \Phi * \dot{q} = f \tag{55}$$

where q denotes the column matrix of the coordinates and $\dot{q}$ and $\ddot{q}$ are the column matrices of their first and second time derivatives. M is a square symmetric and positive definite matrix of inertia coefficients, K a square symmetric and positive semi-definite matrix of stiffness coefficients. Φ is a square symmetric matrix of heredity functions, whose product with $\dot{q}$ involves a convolution integral of type (54) for each product of elements. Finally, f is a column vector of external exciting forces.

If each heredity function is of type (9) we have a simple 'Maxwell damping'. Simple viscous damping may be considered a special case where each stiffness k_2 is taken to be infinite. When k_2 tends to infinity the function

$$\phi(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ k_e \exp\left(-\dfrac{k_2}{c_2}\, t\right) & \text{for} \quad t > 0 \end{cases}$$

which has the property

$$\int_0^\infty \phi(t)\, dt = c_2 \quad \text{(independent of } k_2\text{)}$$

behaves like the Dirac operator $c_2\delta(t)$ and the convolution integral reduces to

$$c_2 \int_0^t \delta(t-\tau)\, \dot{q}_1(\tau)d\tau = c_2\dot{q}_1(t)$$

in conformity with the simple viscous damping law.

The matrix Φ is then reduced to $\delta(t)C$, where C is a matrix of constant viscosity coefficients and equation (55) becomes

$$M\ddot{q} + Kq + C\dot{q} = f \ . \tag{56}$$

'Structural damping' cannot be included in equation (55) since it does not admit of a representation by a Duhamel integral.

The operational stiffness matrix corresponding to (55) is

$$s^2M + K + s\Phi(s)$$

and the frequency response of the system when

$$\left.\begin{aligned} q &= e^{i\omega t}x \\ \text{and} \qquad & \\ f &= e^{i\omega t}y \end{aligned}\right\} \tag{57}$$

derived from it by the substitution $s = i\omega$, is given by

$$[K - \omega^2M + i\omega\Phi(i\omega)]_x = y \ . \tag{58}$$

Since the elements of Φ are real functions of $i\omega$, a change of sign in ω is equivalent to changing i into -i. Consequently, when Φ is split into its real and imaginary parts, the result is of the form

$$\Phi(i\omega) = R(\omega^2) + i\omega I(\omega^2) \ . \tag{59}$$

The matrix

$$\tilde{K} \equiv K - \omega^2 I(\omega^2) \tag{60}$$

may be considered as a matrix of effective stiffnesses which are functions of the frequency, and the matrix $R(\omega^2)$ as the matrix of

equivalent viscous damping coefficients. This reduces the structure of the equations to that which would result from consideration of equation (56).

4.2 *Natural Undamped Modes*

With the use of (59), let us rewrite equation (58) in the form

$$[K - \omega^2 M + i\omega R]x = y \tag{61}$$

where K now stands for the matrix $\tilde{K}$ defined by (60), and where functional dependence of the elements of K and R on frequency is understood. In view of the discussion in Section 3.2, and since it is the purpose to obtain the natural mode shapes by resonance testing, these modes are best defined by the principle of phase resonance. Phase resonance occurs, y being a matrix of real elements (all excitations in phase), when the matrix of response amplitudes is of the type x = iz, where z has real elements.

By splitting equation (61) into real and imaginary parts, we obtain

$$-\omega R z = y \tag{62}$$

$$(K - \omega^2 M) z = o \; . \tag{63}$$

The last equation, which is homogeneous, has then by definition the natural frequencies ω_K as eigenvalues, and the corresponding natural modes $z_{(K)}$ as eigensolutions. The required excitations follow from equation (62). From (63) the square of a natural frequency is expressed by the ratio

$$\omega_K^2 = \frac{z'_{(K)} K(\omega_K^2) z_{(K)}}{z'_{(K)} M z_{(K)}} \; . \tag{64}$$

It is perhaps important to observe that, because $K(\omega_r^2)$ and $K(\omega_s^2)$ are not necessarily proprotional matrices, the modes $z_{(r)}$ and $z_{(s)}$ associated with two distinct natural frequencies ω_r and ω_s need not satisfy the usual orthogonality relation

$$\left.\begin{aligned} &z'_{(r)} M\, z_{(s)} = o \\ &\omega_r \neq \omega_s \end{aligned}\right\} \quad . \tag{65}$$

The loss of this property may lead to doubts about the adequacy of our definition of natural modes. Other assumptions, however, like the ones based on quasi-static stiffnesses K(o) or on on K(∞), whilst saving equations (65), are artificial in character and would produce definitions of modes unattainable experimentally. From a practical point of view, the variations in stiffnesses with frequency are small enough to yield negligible differences between these definitions. Hence modes obtained from resonance definition should satisfy approximately equations (65).

For pure viscous damping, where $I(\omega^2) \equiv o$ and R = C is a matrix of constants like K, and also in the case of 'structural damping', the definitions coincide and orthogonality should be satisfied exactly. These cases represent, however, an idealization of the physical situation.

4.3 *The Characteristic Phase Lags*

An analysis of the frequency response based on equation (61) is conveniently based on a type of orthogonal expansion using the concept of 'characteristic phase lag'. In its original form [1] it was developed for the pure viscous damping case. Since it does not use equations (65) it is easily extended to the general case where K and R are functions of the frequency.

At any given frequency the existence of a set of normal excitation modes and corresponding normal frequence response modes will be assumed as follows

$$\left.\begin{aligned} &y = y_{(K)} \quad \text{real} \\ &x = e^{-i\phi} K_{r_{(K)}} \quad \text{with real } r_{(K)} \end{aligned}\right\} \quad . \tag{66}$$

It is therefore postulated that in a natural mode all coordinates vibrate with the same phase lag angle ϕ_K with respect to a mode of excitation forces which are in phase.

Substitution of these expressions in equation (61) yields after separation of terms in phase and terms in quadrature:

$$[\omega\cos\phi_K R - \sin\phi_K(K - \omega^2 M)]r_{(K)} = o \tag{67}$$

and

$$[\cos\phi_K(K - \omega^2 M) + \omega\sin\phi_K R]r_{(K)} = y_{(K)} \ . \tag{68}$$

Equation (67) is homogeneous and poses an eigenvalue problem for the 'characteristic phase lags'. The eigenvalue $\tan\phi_K$ are the roots of the algebraic equation in $\tan\phi$, namely

$$t\,|\,[\omega R - \tan\phi(K - \omega^2 M)]\,| = 0 \ .$$

They are all real since

$$\tan\phi_K = \frac{\omega r'_{(K)} R\, r_{(K)}}{r'_{(K)}(K - \omega^2 M) r_{(K)}} \tag{69}$$

and, if $r'_{(K)}$ denotes the transpose conjugate of $r_{(K)}$, these are ratios of two Hermitian forms. When $\tan\phi_K$ is known the values ϕ_K are defined without ambiguity by the condition

$$0 \leq \phi_K \leq \pi \ . \tag{70}$$

It will be shown that this determination is consistent with a continuous variation of ϕ_K with the frequency.

Once a ϕ_K is determined, the corresponding mode r_{iK} follows from equation (67), except for an undetermined multiplier. The corresponding excitation mode follows from equation (69) or one of the following combinations of equations (67) and (68), namely

$$\cos\phi_K y_{(K)} = (K - \omega^2 M) r_{(K)} \tag{71}$$

and

$$\sin\phi_K y_{(K)} = \omega R r_{(K)} \ . \tag{72}$$

Obviously both $r_{(K)}$ and $y_{(K)}$ may be taken as real. They then satisfy the initial assumptions of (66); and moreover, in equation (69) we may take $r'_{(K)}$ to represent simply the transpose of $r_{(K)}$.

The energy dissipated per unit cycle in such a type of excitation is

$$E_K = \pi \sin\phi_K \; r'_{(K)} y_{(K)} = \pi\omega r'_{(K)} R \; r_{(K)} \; . \tag{73}$$

It should be small compared to the maximum deformation energy, so that the quantities

$$\gamma_K = \frac{\omega r'_{(K)} R \; r_{(K)}}{r'_{(K)} \; R \; r_{(K)}} \tag{74}$$

should be small compared to unity.

Now the fact that energy is really dissipated per unit cycle shows that the equation

$$\omega R \; r_{(K)} = 0$$

cannot be true, except perhaps for $\omega = o$ or $\omega = \infty$. It follows from equation (67), that the value $\tan\phi_K = o$ cannot occur in the frequency range, except at the limits $\omega = o$ and $\omega = \infty$. As a consequence the continuous variation of a root $\tan\phi_K$ will yield a continuous variation of ϕ_K consistent with equation (70).

Let us now introduce the frequencies λ_K defined by

$$\lambda_K^2 = \frac{r'_{(K)} R \; r_{(K)}}{r'_{(K)} M \; r_{(K)}} \; . \tag{75}$$

In view of this definition and (74), equation (69) can be written as

$$\tan\phi_K = \frac{\gamma_K}{1 - \left(\frac{\omega}{\lambda_K}\right)^2} \; .$$

Note that, since K, R and $r_{(K)}$ are dependent on frequency, λ_K and γ_K are themselves functions of the forcing frequency.

Whilst λ_K^2 always remains finite and positive (the limits $K(o)$ and $K(\infty)$ exist for the stiffnesses) as a ratio of two positive definite quadratic forms, γ_K behaves in a different manner for $\omega = o$ and $\omega = \infty$ according to the assumptions made for damping.

For hereditary damping. Finite, positive definite limits exist for R when $\omega = o$ and for $\omega^2 R$, when $\omega = \infty$. Then γ_K and $\tan\phi_K$ behave like ω near $\omega = o$ and like ω^{-1} and ω^{-3} respectively near $\omega = \infty$.

For pure viscous damping. K and R are matrices of constants and nothing is changed in their behaviours near $\omega = o$. However, near $\omega = \infty$ γ_K now tends to infinity as ω (these values are thus not small any more at high frequency) and $\tan\phi_K$ still tend to zero but only as ω^{-1}.

For structural damping. K and ωR are matrices of constant elements. Both γ_K and $\tan\phi_K$ tend to small positive quantities near $\omega = o$. Near $\omega = \infty$ γ_K remains finite, whilst $\tan\phi_K$ tends to zero as ω^{-2}.

From equation (67) again, it appears that $\tan\phi_K = \infty$ can be an eigenvalue if $r_{(K)}$ satisfies

$$(K - \omega^2 M) r_{(K)} = o \ . \tag{76}$$

This is recognized as the problem of equation (63) and justifies the assumption that

$$r_{(K)} = z_{(K)} \ , \tag{77}$$

$$\phi_K = \pi/2 \ , \tag{78}$$

$$\lambda_K^2 = \omega_K^2 \quad \text{for} \quad \omega = \omega_K \ , \tag{79}$$

$$y_{(K)} = \omega_K R z_{(K)} \ . \tag{80}$$

To sum up:

There are as many excitation modes and normal response modes as degrees of freedom in the system.

Each characteristic phase lag is zero at zero frequency (or small and positive for the assumption of structural damping), passes through $\pi/2$ for one of the natural frequencies of the system (the response mode being then identical with the natural mode) and tends to π at high frequency. It thus behaves qualitatively like the phase lag in a system of one degree of freedom.

For pure viscous damping [1] and structural damping [10] it may even be shown that $\omega^{-1}\tan\phi_K$ and $\tan\phi_K$ respectively increase monotonically with ω^2.

4.4 *Orthogonality of the Normal Response Modes*

From equation (67) it follows that a normal response mode satisfies the equation

$$\omega R r_{(K)} = \tan\phi_K (K-\omega^2 M) r_{(K)} \ .$$

Let $r_{(j)}$ be another response mode at the same frequency. Then, by pre-multiplication by $r'_{(j)}$ we obtain

$$\omega r'_{(j)} R r_{(K)} = \tan\phi_K \ r'_{(j)} (K - \omega^2 M) r_{(K)} \ .$$

Similarly, using the equation satisfied by $r_{(j)}$, transposing and post-multiplying by $r_{(K)}$, we derive

$$\omega r'_{(j)} R r_{(K)} = \tan\phi_j \ r'_{(j)} (K - \omega^2 M) r_{(K)} \ .$$

Hence, if $\tan\phi_K \neq \tan\phi_j$, the following orthogonal properties are satisfied:

$$r'_{(j)} R r_{(K)} = o \ , \tag{81}$$

$$r'_{(j)}(K - \omega^2 M)r_{(K)} = o \ . \tag{82}$$

These can still be shown to hold true if a multiple root occurs for a cheracteristic phase-lag by orthogonalizing the associated independent modes. (From matrix theory the number of independent modes is known to correspond to the degree of multiplicity of the root).

Pre-multiplying equation (68) by $r'_{(j)}$ and using the orthogonality relations we obtain

$$r'_{(j)}y_{(K)} = o \qquad j \neq K \ . \tag{83}$$

This proves a most essential property, namely that the energy input from a given excitation goes only into its associated response mode.

4.5 *Expansion of the Frequency Response to Arbitrary Excitation*

At any given frequency the $y_{(K)}$'s form a complete set, or base, of the vectorial space. Indeed, a dependency relation

$$\Sigma \alpha_K y_{(K)} = o$$

would imply, after multiplication by $r'_{(j)}$ and use of equations (83), that

$$\alpha_j [r'_{(j)}y_{(j)}] = o \quad \text{for all j's} \ .$$

Now, from equations (71) and (72)

$$[r'_{(j)}y_{(j)}]^2 = [r'_{(j)}(K-\omega^2 M)r_{(j)}]^2 + \omega^2 [r'_{(j)}Rr_{(j)}]^2 > 0$$

and consequently all α_j's are necessarily zero.

Any excitation mode may then be represented in a unique manner as an expansion

$$y = \Sigma\, \beta_K y_{(K)} \ . \tag{84}$$

It will be assumed that all exciters are kept in phase so that only real coefficients β_K are to be considered; they may be regarded as generalized coordinates for the arbitrary excitation.

The corresponding expansion of the frequency response will be

$$\kappa = \Sigma\, \beta_K e^{-i\phi_K} r_{(K)} = a - ib$$

where

$$a = \Sigma\, \beta_K \cos\phi_K r_{(K)} \tag{85}$$

is the part in phase with the excitation and

$$b = \Sigma\, \beta_K \sin\phi_K r_{(K)} \tag{86}$$

is the part lagging 90° behind.

From equations (84) and (83), and then from equation (72),

$$\beta = \frac{r'_{(K)} y}{r'_{(K)} y_{(K)}} = \sin\phi_K \frac{r'_{(K)} y}{\omega r'_{(K)} Rr_{(K)}} \tag{87}$$

or, using definitions (74) and (75),

$$\beta_K = \frac{\sin\phi_K}{\gamma_K \lambda_K^2} \frac{r'_{(K)} y}{r'_{(K)} Mr_{(K)}} \ . \tag{88}$$

It will be convenient to normalize the response modes by assigning to each the same generalized mass

$$\mu = r'_{(K)} Mr_{(K)} \ . \tag{89}$$

The phase and quadrature responses to arbitrary excitation as given by (85) and (86) will then appear in the form

$$a = \frac{1}{\mu} \Sigma \frac{\sin\phi_K \cos\phi_K}{\gamma_K \lambda_K^2} [r'_{(K)} y] r_{(K)} \tag{90}$$

$$b = \frac{1}{\mu} \Sigma \frac{\sin^2\phi_K}{\gamma_K \lambda_K^2} [r'_{(K)} y] r_{(K)} \quad . \tag{91}$$

4.6 *Dynamic Flexibility Matrices*

If the energy input terms shown in brackets $[r'_{(K)} y]$, are written as the last factor in each term of the expansions (they are scalar quantities) and if use is made of the associative properties of matrix multiplication, the excitation y may be factored out to the right. We then find

$$a = Ay \quad \text{and} \quad b = By \tag{92}$$

with

$$A = \frac{1}{\mu} \Sigma \frac{\sin\phi_K \cos\phi_K}{\gamma_K \lambda_K^2} r_{(K)} r'_{(K)} \tag{93}$$

$$B = \frac{1}{\mu} \Sigma \frac{\sin^2\phi_K}{\gamma_K \lambda_K^2} r_{(K)} r'_{(K)} \quad . \tag{94}$$

They are dynamic flexibility matrices and they appear as expansions in dyadic products of the response modes. An element $A_{ij}(\omega)$ of A is the amplitude of response in phase for the coordinate q_i, due to unit amplitude of excitation in the coordinate q_j; $B_{ij}(\omega)$ is the corresponding response in quadrature. Since

$$x = (A - iB)y$$

we have actually inverted relation (61) and found the reciprocal

$$[K - \omega^2 M + i\omega R]^{-1} = A - iB \quad . \tag{95}$$

4.7 *Stationary Character of the Reactive Energy in Pure Mode Excitation* [*10*]

Consider the energy input per cycle under arbitrary excitation, namely the 'active energy'

$$E_a = \pi y'b = \pi b'y \tag{96}$$

and define the 'reactive energy' by

$$E_r = \pi y'a = \pi a'y \ . \tag{97}$$

By substituting equation (84) for y, equations (85) and (86) for a and b, using equation (83) and then substituting equation (71) for $y_{(K)}$, we obtain

$$E_a = \pi\omega\Sigma\beta_K^2[r'_{(K)}Rr_{(K)}]$$

$$E_b = \pi\omega\Sigma\beta_K^2\cot\phi_K[r'_{(K)}Rr_{(K)}] \ .$$

Clearly E_a is a positive definite quadratic form, which corresponds to its character of really dissipated energy. On the other hand E_r is generally not positive definite. Each time a phase resonance is passed, one of the $\cot\phi_K$ coefficients passes from a positive to a negative value, thereby changing the sign of the expression.

The conditions for the reactive energy to be stationary with respect to arbitrary small variations of the generalized coordinates of the excitation are

$$\frac{\partial E_r}{\partial \beta_K} = 2\pi\omega\beta_K\cot\phi_K[r'_{(K)}Rr_{(K)}] = o \quad \text{for all values of K.}$$

Except for the trivial solution of zero excitation (all β_K's zero) the conditions can only be satisfied if

(1) All β_K's are zero except one, say β_j,

(2) $\cot\phi_j = o \quad$ or $\quad \phi_j = \pi/2$.

It follows then from equations (77) to (80) that

$\omega = \omega_j$ (a natural frequency),

$$x = -i\beta_j z_{(i)} \quad \text{(a natural mode in quadrature)}$$

$$y = \beta_j \omega_j Rz_{(j)} \quad \text{(a pure mode excitation)}.$$

Under these conditions the stationary value of E_r is zero. Hence the stationary property of the reactive energy provides a control for the ideal excitation cases and allows the experimental determination of a pure natural mode.

An alternative interesting proof is as follows. From equations (97) and (92)

$$E_r = \pi y' A y$$

and if we now consider the elements of y as the excitation coordinates, the conditions for stationary values are

$$\delta E_r = \pi(\delta y' A y + y' A \delta y) = 2\pi \delta y'(Ay) = o$$

for arbitrary variations δy. Hence

$$a = Ay = o \ . \tag{100}$$

Indeed, the vanishing of the response in phase implies a natural response mode in quadrature which can only occur at a resonant frequency. The natural frequencies of the system are therefore also solutions of the determinant

$$|A(\omega)| = 0 \tag{101}$$

and the pure excitation modes $Rz_{(j)}$ are the corresponding eigensolutions of equation (100).

4.8 *Pure Modes from Partial Excitation*

A pure excitation mode for a continuous structure would involve distributed volume forces and is clearly impossible to achieve. For a finite number of shaking points there arises the problem

of optimizing the distribution of force amplitudes between the shakers to obtain a response as close as possible to a natural mode.

The principles involved may be illustrated on a system with a finite number of degrees of freedom, provided only some of the coordinates be excited.

4.8.1 *The resonance principle*

Let $z_{(r)}$ and ω_r be the natural mode and the natural frequency aimed at in a resonance test. Then, provided (1) ω_r is well separated from the other natural frequencies, so that for $\omega = \omega_r$ we have $\sin\phi_r = 1$ and the other $\sin\phi_K$'s are small quantities of $0(\gamma)$ like the γ_K's; (2) the energy input bracket $[z'_{(r)}y]$ be of the same order (at least) as the other input brackets, we get the following situation for $\omega = \omega_r$:

(a) The dominant term in the response in quadrature is

$$b_{(r)} = \frac{[z'_{(r)}y]}{\mu\omega_r^2\gamma_r(\omega_r)}\, z_{(r)} \tag{102}$$

all other terms being small of $0(\gamma^2)$ compared to it.

(b) The term in $z_{(r)}$ disappears in the response in phase ($\cos\phi_r = o$) and the other terms are small of $0(\gamma)$ compared to the dominant term in quadrature. Consequently we may determine experimentally the natural mode $z_{(r)}$, with errors of $0(\gamma^2)$ only, by measuring the response in quadrature at resonance. It is necessary for this purpose to use pick-ups capable of discriminating between the phase and quadrature components. If simple pick-ups are used, giving only peak response amplitude, caution must be exercised in the neighbourhood of nodes of $b_{(r)}$. For if a local amplitude of $b_{(r)}$ is large, the addition of terms of $0(\gamma^2)$ in phase with it and of $0(\gamma)$ in quadrature still results in a peak amplitude correct to $0(\gamma^2)$. But in the vicinity of nodes of $b_{(r)}$ the error may be of $0(\gamma)$.

Numerical results for torsional and flexural vibrations of a continuous beam with uniform damping, excited at one end, show that the relative importance of the response in phase grows significantly with the order of resonance.

The situation which has just been described is an ideal one and, provided we do not apply it to a high order of resonance, excellent results may be obtained from tests with a single shaker.

To achieve the energy bracket requirement the shaker should be located at a station where large vibration amplitudes occur for each particular resonance condition.

However, the following question arises; how are we going to decide that we have a resonance condition? An answer to this is provided by the use of the local phase resonance principle.

4.8.2 *The local phase resonance principle*

Let us compute the variation in the response x for a small alteration of the forcing frequency, y being kept constant. More precisely let us calculate the column matrix:

$$v = \frac{dx}{d\omega^2} .$$

This will be done to a first approximation by assuming ωR and K not to vary appreciably with frequency. Then, by differentiating equation (61) with respect to ω^2, we obtain

$$-Mx + [K - \omega^2 M + i\omega R]v = o$$

and from equation (95)

$$v = (A - iB)Mx = (AMa - BMb) - i(AMb + BMa) . \qquad (103)$$

In the ideal case described earlier, the dominant term in the real part of v is easily determined from equations (90),

(91), (93) and (94) and found to be

$$\frac{da}{d\omega^2} = - \frac{[z'_{(r)}{}^{y}]}{\mu\omega_r^4\gamma_r^2(\omega_r)} z_{(r)} \quad . \tag{104}$$

The other terms are of $0(\gamma^2)$ compared to it and the terms of the imaginary part are of $0(\gamma)$.

Suppose now that the resonance condition is decided by picking out the response in phase in one of the coordinates, say the i^{th} coordinate, and by adjusting the forced frequency so as to make it vanish. This procedure may be called the local phase resonance principle and is justified if, besides condition (2) of the resonance principle, the amplitude $z_{(r)i}$ in the i^{th} coordinate is of the same order as the other $r_{(K)i}$. For then a comparison of the correction term

$$\Delta\omega^2 \frac{d}{d\omega^2} a_i = - \frac{\Delta\omega^2}{\omega_r^2} \frac{[z'_{(r)}{}^{y}]}{\mu\omega_r^2\gamma_r^2(\omega_r)} z_{(r)i}$$

with the term $a_i(\omega_r^2)$ which it must cancel (as may be found from equation (90)) shows that

$$\frac{\Delta\omega^2}{\omega_r^2} = 0(\gamma^2) \quad \text{only} \; .$$

The natural frequency will thus be measured within a relative error of $0(\gamma^2)$ and the corrections due to $\Delta\omega^2$ in the response in quadrature consists of terms of $0(\gamma^2)$ with respect to the dominant term given by equation (102).

In conclusion it may be expected that, when a natural frequency is well separated and single excitation is applied at a point of large response amplitude, local phase resonance at any other (or the same) point of large response amplitude provides a measurement of the natural frequency and the mode shape both with relative errors of $0(\gamma^2)$.

It may be observed that this principle amounts to replacing the frequency equation (101) by the approximate one

$$A_{ij}(\omega) = o$$

where j is the index of the coordinate being excited and i the index of the coordinate where the phase response is made to vanish.

4.8.3 *The pseudo-resonance principle*

In complex structures the natural frequencies are not always well separated. In theory two or several distinct mode shapes may even belong to the same natural frequency [11, 12]. In such cases and more generally in the higher frequency spectrum, where there is a tendency for natural frequencies to be closer to each other, it is impossible to obtain pure response modes without having to resort to a multiplicity of shaking points. By using several shakers it becomes theoretically possible to achieve pseudo-resonance in the response modes which are too close in frequency with the one it is wished to measure. In other words the energy brackets of these response modes may be made to vanish, thereby cancelling in the expansion (91) terms which would otherwise be relatively large because their $\sin\phi_K$ values are no longer of $0(\gamma)$.

There remains, therefore, the problem of deciding from experimental evidence whether these pseudo-resonance conditions are present.

For the sake of simplicity we will begin with the problem of two close frequencies which will be denoted by ω_1 and ω_2. Taking only the dominant terms, when the forces frequency is close to ω_1 and ω_2, we have

$$b = \frac{\sin^2\phi_1}{\mu\gamma_1\lambda_1^2}\,[r'_{(1)}y]r_{(1)} + \frac{\sin^2\phi_2}{\mu\gamma_2\lambda_2^2}\,[r'_{(2)}y]r_{(2)} \quad ,$$

$$a = \frac{\sin\phi_1\cos\phi_1}{\mu\gamma_1\lambda_1^2}\,[r'_{(1)}y]r_{(1)} + \frac{\sin\phi_2\cos\phi_2}{\mu\gamma_2\lambda_2^2}\,[r'_{(2)}y]r_{(2)} \ .$$

Supposing $\omega = \omega_1$, exactly, then $\cos\phi_1 = o$, $\sin\phi_1 = 1$, $r_{(1)} = z_{(1)}$ and the second term in b is the parasitic term to be eliminated by an excitation vector y such that $[r'_{(2)}y] = o$ and $[r'_{(1)}y]$ is of the same order as the energy brackets. It is seen that the remaining dominant term in a disappears at pseudo-resonance and that we then have almost pure phase resonance in every coordinate.

In fact, if we have two shakers we may adjust their relative force amplitude to have exact phase resonance at a point of large amplitude. Again, by a small adjustment $\Delta\omega^2/\omega_1^2$ of $0(\gamma^2)$ we may, when taking into account the terms neglected in a, provide for exact phase resonance at some other point of large amplitude. It may therefore be concluded that all requirements for accurate measurements, and in particular for pseudo-resonance in the parasitic mode, are met by using two shakers, preferably located at points of large response amplitude, and adjusting frequency and relative shaker forces to provide for phase resonance in two other (or eventually the same) points of large response amplitudes.

If we distinguish by the subscripts j_1 and j_2 the coordinates excited and by i_1 and i_2 the coordinates measured, this method amounts to replacing the pure excitation condition of equation (100) by the approximate one

$$a_{i_1} = A_{i_1j_1}(\omega)y_{j_1} + A_{i_1j_2}(\omega)y_{j_2} \ ,$$

$$a_{i_2} = A_{i_2j_1}(\omega)y_{j_1} + A_{i_2j_2}(\omega)y_{j_2} \ .$$

The extension of this principle to any number of shakers is straightforward and leads to the conditions

$$A_{(s)}y_{(s)} = o \qquad (105)$$

where $A_{(s)}$ is the submatrix of A obtained by selecting the elements common to the lines $(i_1, i_2, \ldots, i_m)$ and the columns $(j_1, j_2, \ldots, j_m)$, and $y_{(s)}$ is the column matrix of the excitation amplitudes used in the test.

We may note that in the special case where

$$i_t = j_t \qquad (t = 1, 2, \ldots, m)$$

i.e., when phase resonance is imposed at the locations of the shakers, the principle is an expression of the stationary property of the reactive energy of the partial excitation.

The practical imposition of conditions (105) becomes increasingly difficult with the number of shakers in action. More research is needed to examine the convergence of possible methods of successive approximations. One may for instance think about the possibility of extending the Gauss-Seidel process to equation (105). This would consist in practice of adjusting in succession each shaker force amplitude y_{j_t} to provide for phase resonance in coordinate q_{i_t} with a cyclic frequency adjustment as well.

Another method would be to measure experimentally (by bringing only one of the exciters to life) each column of $A_{(s)}$ and the phase responses. The frequency would then be found by numerical evaluations of $|A_{(s)}|$ until it vanishes, and the required excitations by solving numerically the compatible system fo equations (105). This procedure has actually been proposed by Traill-Nash [13].

A useful control of the pseudo-resonance condition in the neighbouring modes is provided by the purity of the response loops (quadrature versus phase response) in each coordinate when the frequency is varied in the vicinity of the natural frequency under examination. A good response loop would have to resemble that a single degree of freedom system, which is almost a pure circle, a point specially emphasized by Kennedy and Pancu [9].

4.9 *Forced Response under Special Assumptions*

If we make the assumption that

$$K(\omega^2) = [1 + \theta(\omega)]K(o) \quad \text{with} \quad \theta(o) = o \tag{106}$$

the natural modes are orthogonal with respect to both the matrices M and K(o) and

$$\left.\begin{aligned} z'_{(r)}Mz &= o \\ z'_{(r)}K(o)z_{(s)} &= o \quad r \neq s \end{aligned}\right\} \quad . \tag{107}$$

If we now make the further assumption that $R(\omega^2)$ is defined by

$$R = \gamma(\omega)K(o) \tag{108}$$

we have the extension of a case considered by Rayleigh [2], which leads to very simple results. Indeed the substitution of the assumptions in equation (67) and comparison with the equation satisfied by a natural mode shows that at any frequency we may write

$$r_{(K)} = z_{(K)}$$

and that

$$\tan\phi_K = \frac{\omega\gamma(\omega)}{1 + \theta(\omega) - [1 + \theta(\omega)_K]\,\dfrac{\omega^2}{\omega_K^2}}$$

whilst from equation (72) and the previous results

$$y_{(K)} = \frac{\omega\gamma(\omega)}{1 + \theta(\omega_K)}\,Mz_{(K)}$$

Hence the shapes of the forced response modes become independent of the frequency, the behaviour of a characteristic phase lag is obtained explicitly and the shape of the normal excitation mode depends only on the inertial matrix M.

This last feature is important because from the knowledge of the mass distribution and the measurements of the response amplitudes the optimum distribution of the shaker forces may be progressively adjusted. This is the basis of a method proposed by Lewis and Wrisley [3].

Turning back to the more general equations of motion (55), let us introduce the assumption

$$\Phi = \phi(t)K(o) \tag{109}$$

which affects both equations (106) and (108) when applied specifically to the case of forced response. (Note that in equation (55) K stands really for what is now called K(o)). Then, introducing normal coordinates $\eta_k(t)$, we obtain

$$q = \sum_{1}^{n} \eta_k(t) z_{(K)}$$

and pre-multiplying by $z'_{(r)}$ and making use of the orthogonality relations (107), we obtain the normal equations

$$[z'_{(r)}Mz_{(r)}]\ddot{\eta}_r + [z'_{(r)}K(o)z_{(r)}](\eta_r + \phi^*\dot{\eta}_r) = z'_{(r)}f \quad (r = 1,2,\ldots,n) .$$

Or, normalizing the natural modes as in equation (89) and considering equation (64) we derive

$$\ddot{\eta}_r + \frac{\omega_r^2}{1 + \theta(\omega_r)} (\eta_r + \phi^*\dot{\eta}_r) = \frac{1}{\mu} (z'_{(r)}f) . \tag{110}$$

Because of assumption (109), the normal coordinates are entirely uncoupled. In such a case a further test of pure excitation becomes possible, for when the natural mode $z_{(r)}$ is the only one excited and the excitation is suddenly cut off, $z_{(r)}$ is also a mode of decay. Moreover the analysis of the decay in any local amplitude provides information as to the exact nature of the heredity function $\phi(t)$.

In the general case where equation (109) is not valid, $z_{(r)}$ is not a mode of dacay; the coupling due to damping gradually introduces the other modes in the decay response.

The assumption made in (109) is not the most general that will lead to entirely uncoupled modes. As pointed out by Rayleigh [2] (in the particular case of simple viscous damping) the same type of reduction is possible when is a linear function of K and M, namely

$$\Phi = \phi(t)K(o) + \psi(t)M \ . \tag{111}$$

Furthermore, from matrix theory, it is possible to show that this is the most general assumption under which

$$z'_{(r)}\phi z_{(s)} = o \quad r \neq s$$

holds together with equations (107). Then K(o), M and Φ are reducible together to diagonal forms, a property known in France under the name of Basile's assumption [4]. The normal equations corresponding to assumption (111) are

$$\ddot{\eta}_r + \left(\psi + \frac{\omega_r^2}{1 + \theta(\omega_r)}\phi\right) *\dot{\eta}_r + \frac{\omega_r^2}{1 + \theta(\omega_r)}\eta_r = \frac{1}{\mu}\ (z'_{(r)}f)$$

and all the properties of the forces response are easily deduced from them.

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