UPPER AND LOWER BOUNDS IN MATRIX STRUCTURAL ANALYSIS

B. M. Fraeijs de Veubeke
Universities of Liège and Louvain, Belgium

CONTENTS

1. INTRODUCTION

The numerical calculation of static influence coefficients for a complex structure is necessarily based on simplifying assumptions. The results of such calculations are of value if the accuracy of the approximation is known. Within some measure of probability a quantitative estimate of the accuracy can be gained through previous experience with a similar structure. A theoretically more satisfactory situation arises if the calculation procedures are such that both upper and lower bounds are obtained for the coefficients. The maximum possible errors are then known quantitatively and no further investigation is needed should they lie inside the margin acceptable to the designer. Should this margin be exceeded, further refinements are required in the structural idealization.

The analytical procedures leading to upper and lower bounds are well established for the small deflexions case. It is not mandatory, though preferable for obvious reasons of economy, that the two procedures be applicable to the same idealization of the given structure. For the sake of completeness a short account of these procedures is given in the Appendix A of the present paper. Their chief characteristic is that they should be "pure". By "pure" is meant that no assumptions are allowed that would violate *both* the compatibility conditions and the equilibrium equations.

The approach yielding lower bounds must be purely compatible and, to this purpose, built on a continuous single-valued field of small displacements. The approach yielding upper bounds must satisfy everywhere the equilibrium conditions.

One must here distinguish carefully between

(1) the structural decomposition into elements: plates, beams . . . assumed to behave in accordance with simplified displacement or stress fields,

(2) the assumptions involved in the procedures for connecting the elements together, especially with regard to continuity of the displacements or stresses,

(3) the procedures adopted for solving the redundancies.

The terminology "displacement method" and "force method" used in the literature too often refers only to the last aspect of the overall problem and is not entirely relevant to the distinction made here between a "purely compatible approach" and "a purely equilibrium approach".

A displacement method of solution, or stiffness method, can be used to solve the connexion problem between elements and still violate some continuity of displacement between panels. A force method of solution can be applied and still violate local equilibrium conditions.

The important consideration to apply, if one wishes to be certain that the coefficients are upper or lower bounds, is to ensure that the geometry of the elements, their assumed displacement or stress field and the geometry of their connexions produce either a purely compatible overall field or a pure overall equilibrium field. Once this is ascertained it will be seen that, in either case, both the force method of resolution or the stiffness method are generally applicable, though perhaps with different degrees of efficiency.

2. DUAL TREATMENT OF THE TRIANGULAR PANEL

The subdivision of panels in triangular elements is an attractive proposition for complicated geometries (taper, sweepback, skew ribs . . .). The triangular panel is also the natural shape for the purely compatible or purely equilibrium connexions between *uniform* stress–strain fields.

2.1. *Uniform Field*

A linear two-dimensional displacement field

$$\begin{aligned} u &= a + x\varepsilon_x + y(\tfrac{1}{2}\gamma_{xy} - \omega) \\ v &= b + x(\tfrac{1}{2}\gamma_{xy} + \omega) + y\varepsilon_y \end{aligned} \tag{2.1}$$

depends on six parameters: two translations a,b, a rotation ω accounting for the rigid body displacements, and three uniform strain components $(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$.

The corresponding uniform stress field is given by

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = M \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{2.2}$$

with a stress–strain matrix M, which for isotropic panels is

$$M = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix} \tag{2.3}$$

The reciprocal relations are

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = M^{-1} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix} \frac{1}{E} \qquad \begin{matrix} (2.4) \\ (2.5) \end{matrix}$$

Retaining the assumption that there are no body forces, the uniform stress field satisfies the internal equilibrium equations [see Eqs. (A.6)]. So, with regard to its own interior domain, the uniform stress–strain field is both purely compatible and a purely equilibrium field.

The duality between compatibility and equilibrium arises solely when such uniform fields are edged and pieced together.

2.2. *Compatible Net of Triangular Fields*

Along any straight boundary line both the u and v components vary linearly and are entirely determined by the values taken in two different points. Consequently, if two linear displacement fields of type (2.1) are made to coincide in two points, they are coincident along the whole straight line joining these points. This property leads to the construction of a compatible net of triangular fields by enforcing the coincidence of the displacements at each common vertex. A natural requirement for such a construction is then that the parameters of each field be expressed in terms of the six displacement components at the vertices. Let the vertices be numbered (p,q,r) in a counter-clockwise sense. Then

$$A = \begin{vmatrix} 1 & x_p & y_p \\ 1 & x_q & y_q \\ 1 & x_r & y_r \end{vmatrix} = y_{rp}x_{qr} - x_{rp}y_{qr} \qquad (2.6)$$

is equal to twice the area of the triangle, where

$$x_{pq} = x_p - x_q \quad \text{etc.}$$

An elementary calculation yields for the strain components

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \frac{1}{A} Nu \qquad (2.7)$$

where N is the rectangular (3×6) matrix

$$N = \begin{pmatrix} y_{qr} & y_{rp} & y_{pq} & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_{qr} & -x_{rp} & -x_{pq} \\ -x_{qr} & -x_{rp} & -x_{pq} & y_{qr} & y_{rp} & y_{pq} \end{pmatrix} \qquad (2.8)$$

and u the column matrix of displacements, whose transpose or row matrix will be written

$$u' = (u_p \quad u_q \quad u_r \quad v_p \quad v_q \quad v_r) \qquad (2.9)$$

The strain energy of the panel of uniform thickness t is conveniently obtained from Clapeyron's theorem (see Eq. (A.5) for the energy density)

$$U = \frac{1}{2}\, t\, \frac{A}{2} \begin{pmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

Substitution of Eqs. (2.2) and (2.7) gives

$$U = \frac{t}{4A}\, u'N'MNu = \tfrac{1}{2} u'Ku \tag{2.10}$$

with $\qquad K = (t/2A)\, N'MN$ a symmetrical (6×6) matrix $\qquad$ (2.11)

The elements of the u matrix are to be considered as generalized coordinates for the field. The corresponding generalized forces (or the so-called corner forces) are obtained from Castigliano's formulae. If f denotes their column matrix, the variational procedure equivalent to Castigliano's theorem gives

$$\delta U = \tfrac{1}{2}\delta u'Ku + \tfrac{1}{2}u'K\delta u = \delta u'Ku = \delta u'f$$

Hence

$$f = Ku \tag{2.12}$$

and K is the so-called "stiffness matrix". When formula (2.11) is expanded the identity of K with the forms given by Turner or Argyris can easily be checked. The elements of the f matrix will be denoted as follows

$$f' = \begin{pmatrix} H_p & H_q & H_r & V_p & V_q & V_r \end{pmatrix}$$

The relations between stresses and displacements follows from Eqs. (2.2) and (2.7)

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{1}{A}\, MNu \tag{2.13}$$

and from this can be derived the equation

$$f = Ku = \frac{t}{2A}\, N'MNu = \frac{t}{2}\, N' \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \tag{2.14}$$

yielding an interpretation of the generalized forces. For example, it is found that

$$H_p = \frac{t}{2}\,(\sigma_x y_{qr} + \tau_{xy} x_{rq})$$

It can then be stated[5] that each generalized force at a vertex is equal to half the resultant of stresses acting on the opposite edge or to the resultant of stresses acting on half of the adjacent edges (Fig. 1).

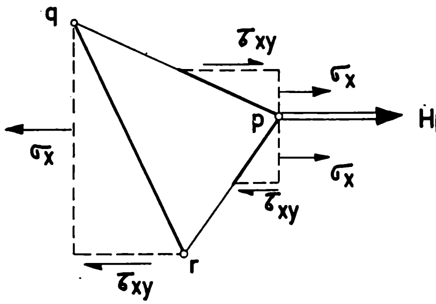


Fig. 1

Obviously a compatible net of triangular fields is not an equilibrium field. The equilibrium equations between adjacent fields are of a global character; they state that the vectorial sum of all generalized loads meeting in a common vertex is equal to the external load applied there. They do not prevent the stresses to vary discontinuously across a common edge.

2.3. *Equilibrium Net of Triangular Fields*

In this case it is the continuous transmission of stresses that is required and this can only be achieved by relaxing compatibility requirements. The equilibrium requirements are met by taking as generalized displacement coordinates the displacements at the middle of each edge of the triangle. The mid-points define the vertices of a "skeleton triangle" (hatched in Fig. 2), whose area is four times smaller than that of the real panel:

$$\tfrac{1}{4}A = y_{rp}x_{qr} - x_{rp}y_{qr} \tag{2.15}$$

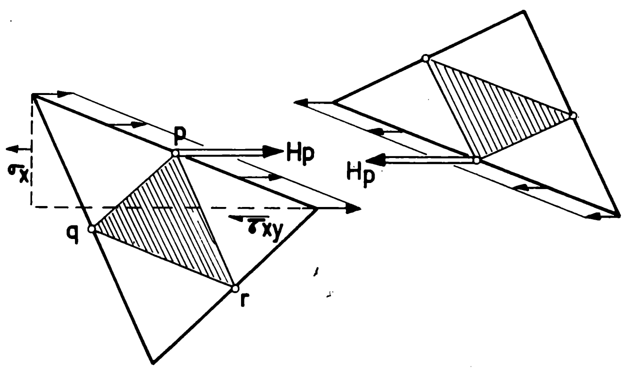


Fig. 2

Otherwise the theory of par. 2.2 applies without modifications and, keeping the definition (2.8) of the N matrix, one obtains the stiffness matrix

$$K^* = \frac{8t}{A} N'MN \tag{2.16}$$

and the relationships

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \frac{4}{A} Nu \quad \text{and} \quad f = 2tN' \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \tag{2.17}$$

From this last result follows that each generalized load applied to a mid-point is the resultant of the stresses acting along the corresponding side (Fig. 2).

Whereas in the compatible net a vertex or node is common to several fields, a node of an equilibrium net is only common to two panels. The global equilibrium condition is, barring external loading, equivalent to stating that the generalized forces at the node are equal and opposite (Fig. 2). In view of the fact that the stresses are constant along each edge this implies that the stresses are transmitted continuously as was required. Any external load, represented by a generalized resultant applied at a node, will really have to consist of a uniformly distributed load along the edge.

If local equilibrium is satisfied in an equilibrium net, compatibility is now generally violated for two reasons:

(1) the strain along a common edge can have different values in each panel;

(2) the orientation of the edge can be different.

This possibility of rotation of one panel with respect to the adjacent one leads to peculiar difficulties to be discussed later.

3. THE STIFFNESS METHOD OF RESOLUTION

The essential steps of the stiffness method of resolution are briefly recalled:

(1) Addition of the stiffnesses of the individual elements in a stiffness matrix for the complete structure. This step expresses the single-value of the displacement vector at a node where the elements are assembled and the additivity of the corresponding internal generalized loads, balancing the external load;

(2) Elimination of externally unloaded nodes, where deflexions are not to be determined. Suppose Eq. (2.12) has been extended by step 1 to represent the relations between the external loads matrix f and the displacement matrix u for the complete structure. If the ith element of f is zero, we have

$$\sum_j K_{ij} u_j = 0$$

This equation is used to express u_i as a linear combination of the other displacements. The jth column of the original stiffness matrix is thereby modified by the addition of the ith column multiplied by the factor $-K_{ij}/K_i$. The ith column itself becomes a column of zeros and is deleted together with the ith row from which Eq. (2.18) was derived.

Steps 1 and 2 can be taken progressively as an intertwined process of growth and reduction until the complete stiffness matrix $K_{(f)}$ (free-structure matrix) is obtained for the loaded nodes.

(3) Elimination of all rigid body modes to produce a final non-singular stiffness matrix $K_{(s)}$ (supported structure matrix). This step consists in prescribing certain displacements to be zero in order that the structure be at least isostatically, or even hyperstatically, supported. Rows corresponding to the suppressed displacements are deleted in the equation $f = K_{(f)}u$. They are used afterwards to know the reaction loads due to the remaining independent loads.

(4) Inversion of the non-singular matrix $K_{(s)}$ to produce the matrix of influence coefficients.

No difficulties are encountered in implementing those steps in the case of compatible triangular fields associated to beam elements or, more generally, in the case of fields where the displacements are defined at the corners. The treatment of equilibrium fields is by no means restricted to the use of the dual "force" method and there are some obvious advantages to be gained from attempting to solve them also by the stiffness procedure. The same basic programme can be used in the computer and the sometimes delicate choice of convenient self-strainings is avoided. To achieve this, however, some care must be exercised with respect to the addition of beam elements and the elimination of rigid-body modes:

(a) *Addition of a beam element* (Fig. 3). The total load g_p applied by the fields (and possibly by an external source) to the isolated beam

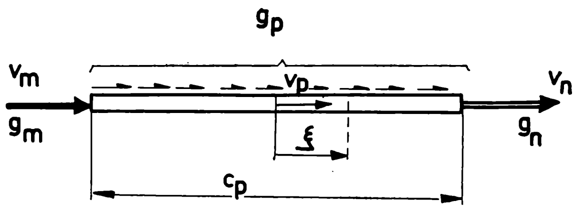


Fig. 3

element is in the form of a uniform shear flow g_p/c_p. There are also end loads g_m and g_n with conventional positive sense identical to the sense chosen for the end displacements v_m and v_n. The internal tension

T in a cross-section positioned by the non-dimensional coordinate $-\frac{1}{2} \leq \xi \leq \frac{1}{2}$ is to be calculated by equilibrium considerations

$$T = -g_m - \tfrac{1}{2}g_p - \xi g_p \tag{3.1}$$

The (complementary) strain-energy of the beam, assumed here for simplicity to have a uniform cross-sectional area, is

$$U = \frac{c_p}{2ES}\int_{-\frac{1}{2}}^{\frac{1}{2}} T^2\, d\xi = \frac{c_p}{2ES}\left[(g_m + \tfrac{1}{2}g_p)^2 + \frac{1}{12}g_p^2\right] \tag{3.2}$$

A simple application of Castigliano's formulae yields

$$v_m - v_n = \frac{\partial U}{\partial g_m} = \frac{c_p}{ES}(g_m + \tfrac{1}{2}g_p) \tag{3.3}$$

as relative displacement of the end-sections, and

$$v_p - v_n = \frac{\partial U}{\partial g_p} = \frac{c_p}{2ES}(g_m + \tfrac{2}{3}g_p) \tag{3.4}$$

It should be noted that v_p is not the displacement of the mid-span cross-section; it is a generalized displacement corresponding to the uniform shear flow. It is equal to the ordinary average of all cross-section displacements as calculated by integration of the strain. A best fit in compatibility is reached when this v_p is identified with that belonging to the field edges which are loading the beam. In that respect it would have been more appropriate, when discussing the equilibrium nets of triangular fields, to define the displacements as averages along each edge. However, since the field is uniform the average is identical with the displacement of the middle of the edge.

While the addition of a beam segment to a compatible net introduces no additional nodes, in the present case the end-sections of the beam are new points where displacements must be introduced and where concentrated external loads can be applied. To obtain the component-stiffness matrix due to the beam, Eqs. (3.3) and (3.4) are solved for g_m and g_p; the result substituted in Eq. (3.2) yields the strain-energy

$$U = \frac{2ES}{c_p}(v_m^2 + v_n^2 + 3v_p^2 + v_m v_n - 3v_p v_m - 3v_p v_n) \tag{3.5}$$

The other Castigliano's formulae give the load-displacement relations expressed by means of the required stiffness matrix in the form

$$\begin{pmatrix} g_m \\ g_p \\ g_n \end{pmatrix} = \frac{2ES}{c_p}\begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_m \\ v_p \\ v_n \end{pmatrix} \tag{3.6}$$

These equations are also directly obtainable from Eqs. (3.3) and (3.4) together with (3.1) for $\xi = \frac{1}{2}$ and $T = g_n$. It should be noted that at the point where beam end sections meet together equilibrium will automatically be satisfied by the stiffness addition process.

(b) *Elimination of rigid body modes.* The possibility of relative panel rotation already mentioned is generally checked by the presence of connected beam segments. The case can, however, arise of structural deformation modes due to panel rotations without strain energy. They will be called rigid deformation modes and are best determined by inspection of the kinematics of the skeleton triangles.

Two cases must be distinguished. In the first, passing from the free to the supported structure, the nature of the support prevents the rigid deformation modes as well as the rigid body displacements and no special treatment is required.

In the second case some rigid deformation modes remain even after the structure is supported. They are characterized by non-zero displacement vectors satisfying the matrix equations

$$K_{(s)} w_{(r)} = 0 \qquad r = 1, 2 \ldots t$$

The theory of linear equations systems gives as necessary and sufficient conditions for solving $f = K_{(s)} u$ that the loading f verifies the virtual work equations

$$w'_{(r)} f = 0 \tag{3.7}$$

Suppose now that the rigid deformation modes are just suppressed by preventing the first t displacements (this will generally require a renumbering of the displacements and is only advocated for simplicity of exposition). The first t components of f and u are now isolated in submatrices $f_{(1)}$ and $u_{(1)}$, their complements denoted by $f_{(2)}$ and $u_{(2)}$ and the matrix $K_{(s)}$ subdivided in four component blocks so that

$$\begin{aligned} f_{(1)} &= K_{11} u_{(1)} + K_{12} u_{(2)} \\ f_{(2)} &= K_{21} u_{(1)} + K_{22} u_{(2)} \end{aligned}$$

By making $u_{(1)} = 0$ we have, since K_{22} is by assumption non-singular

$$u_{(1)} = 0 \qquad u_{(2)} = K_{22}^{-1} f_{(2)} \tag{3.8}$$

$$f_{(1)} = K_{12} K_{22}^{-1} f_{(2)} \tag{3.9}$$

The last equation furnishes the reactions in the rigid links suppressing the $u_{(1)}$ displacements as linear functions of the independent loads of the $f_{(2)}$ matrix.

A unit load on the structure is now considered as a group formed by a unit vector of the $f_{(2)}$ matrix associated with the corresponding set of $f_{(1)}$ loads so that the reactions in the links disappear and the $u_{(1)}$ displacements can be freed. Under this circumstance the general expression for the displacements contains arbitrary contributions from the rigid deformation modes

$$u_{(1)} = \sum_{1}^{t} a_r w_{(r)(1)}$$

$$u_{(2)} = K_{22}^{-1} f_{(2)} + \sum_{1}^{t} a_r w_{(r)(2)}$$

The analysis shows that the situation is unsatisfactory except for those influence coefficients that would happen to be independent of the arbitrary coefficients a_r.

4. THE QUADRILATERAL PANEL SUB-DIVIDED BY DIAGONALS

A remarkable case where the relative rotations between panels does not affect the loading possibilities and constitutes a convenient new building block is the quadrilateral panel sub-divided into four triangular fields by the internal diagonals (Fig. 4). The four skeleton triangles

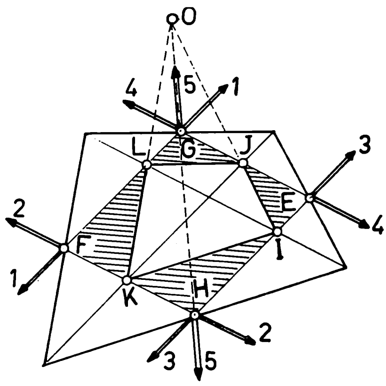


Fig. 4

articulated in points I, J, K and L, form a kinematically deformable chain. Yet the 2×4 load components applied on the outside vertices E, F, G and H are only restricted by three conditions expressing their statical equivalence to zero (rigid-body equilibrium).

One proof consists in obtaining $2 \times 4 - 3 = 5$ independent states of loading which can be transmitted through the kinematical chain. Four of them are obvious and consist in the equal and opposite force pairs numbered from 1 to 4. The transmission is here based on the fact that the sides of the skeleton triangles are aligned two by two to form a skeleton parallelogram.

A fifth independent state of loading has been analysed for clarity on Fig. 5. The pair of forces numbered 5 can be transmitted through the sides IJ and KL, keeping each skeleton triangle in equilibrium, provided the extensions of the lines IJ, KL and HG intersect in a

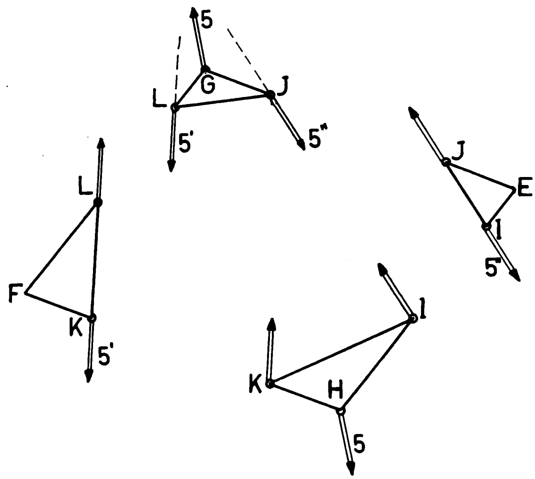


Fig. 5

common point O. Suppose this to be true; then by Ceva's theorem applied to the triangle FGH cut by the transversal KLO:

$$\frac{GO}{GH} \cdot \frac{LF}{LG} \cdot \frac{KH}{KF} = 1$$

Again from Ceva's theorem applied to the triangle GHE cut by the transversal IJO:

$$\frac{GO}{GH} \cdot \frac{JE}{JG} \cdot \frac{IH}{IE} = 1$$

Whence a necessary and sufficient condition that O be a common intersection is that we should have

$$\frac{LF}{LG} \cdot \frac{KH}{KF} = \frac{JE}{JG} \cdot \frac{IH}{IE}$$

This, however, is true since $LF = IH$, $LG = IE$, $KH = JE$ and $KF = JG$.

Another way to look at this property of the quadrilateral is to prove that the kinematical deformability of the chain can take place with each skeleton rotating about the outside vertex. The geometrical proof is similar. There follows that no virtual work of the loads applied to the outside vertices is ever involved and no other restrictions placed on them than static equilibrium with respect to rigid body motion. For the purpose of building up the structure the skeleton parallelogram of the quadrilateral can be considered as rigid.

Obviously a sixth state of loading is provided by a pair of opposite forces along EF. This is, however, easily seen to be a linear combination of the preceding states.

5. THE FORCE METHOD OF RESOLUTION

A drawback of the stiffness method, as applied to equilibrium fields, is a relatively large increase in the number of nodes requiring a correspondingly large number of eliminations. Experience should tell whether the accuracy of the computations is thereby seriously affected.

The opposite situation occurs when one wishes to apply the force method of resolution simultaneously to compatible fields and equilibrium fields. We shall here assimilate the force method to the complementary energy method and, since the prescribed displacements are usually zero, it will further reduce to the minimum principle of the strain energy expressed in terms of stresses.

The stress field is built up by adding a convenient particular equilibrium field with a unit load, ignoring the compatibility restrictions, and a complete independent set of self-strainings. The most convenient self-strainings are those confined to a minimum number of elements.

In that respect the compatible triangular nets are very simply dealt with. A complete set of independent self-strainings is obtained by considering the interactions between pairs of triangles (Fig. 6) or between a triangle and a beam segment (Fig. 7). The effect of such

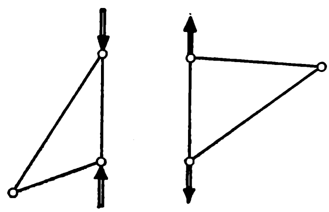


Fig. 6

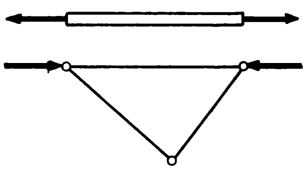


Fig. 7

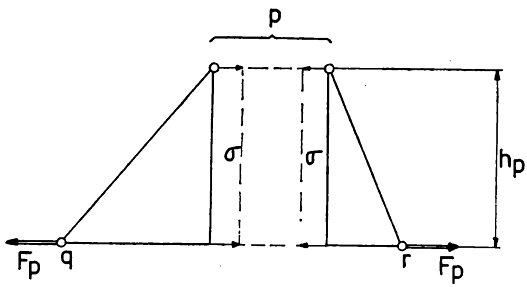


Fig. 8

a self-straining on a triangular field is to produce a state of simple traction in a direction parallel to the interacted edge. This is easily verified by Eq. (2.14) with the relation (Fig. 8)

$$\tfrac{1}{2}th_p\sigma = F_p$$

Turning to the cartesian axes of reference, we find

$$\sigma_x = \frac{2F_p}{th_p}\left(\frac{x_{qr}}{c_p}\right)^2$$

$$\sigma_y = \frac{2F_p}{th_p}\left(\frac{y_{qr}}{c_p}\right)^2$$

$$\tau_{xy} = \frac{2F_p}{th_p}\left(\frac{x_{qr}}{c_p}\frac{y_{qr}}{c_p}\right)$$

Adding the effect of the self-strainings F_q and F_r along the other edges and substituting into

$$U = \tfrac{1}{2}\frac{tA}{2E}\left(\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2 - 2\nu(\sigma_x\sigma_y - \tau_{xy}^2)\right)$$

The following standard form of the strain energy is obtained

$$U = \frac{1}{Et}\left\{\frac{(F_pc_p + F_qc_q + F_rc_r)^2}{A} - 2A(1+\nu)\left(\frac{F_pF_q}{c_pc_q} + \frac{F_qF_r}{c_qc_r} + \frac{F_rF_p}{c_rc_p}\right)\right\}$$

The use of this standard form in adding together all the energy contributions implies that the particular equilibrium field be locally resolved in a (F_p, F_q, F_r) system. In that case each force of this system is the sum of a component from the particular equilibrium field and one hyperstatic unknown.

The beam energies are of the type $(c_pN^2)/(2ES)$ where the traction load N is the sum of a component fixed by the particular equilibrium state and unknowns from the interaction with adjacent fields.

The disadvantage of a large number of self-strainings is partially offset by their simplicity and the possibility of eliminating them gradually before the structure is completed. Indeed a self-straining intensity X can be eliminated by a Menabrea type of equation $\partial U/\partial X = 0$ as soon as the elements concerned by this self-straining have been added to the structure.

The equilibrium net of triangular fields contains fewer self-strainings but of a more complex character. One example is shown on Fig. 9. As already observed[6] the setting up of the simplest independent self-strainings by the computer itself represents a major step in the automation of the computations.

The state of stress in a triangle can this time be regarded as a superposition of three F states (Fig. 10) similar to those of Fig. 10 used in the compatible case. By virtue of Eq. (2.17) we have here $th_p\sigma = F_p$. Hence, with the exception of a numerical factor the strain energy of the superposed fields is the same as before.

$$U = \frac{1}{4Et}\left\{\frac{(F_pc_p + F_qc_q + F_rc_r)^2}{A} - 2A(1+\nu)\left(\frac{F_pF_q}{c_pc_q} + \frac{F_qF_r}{c_qc_r} + \frac{F_rF_p}{c_rc_p}\right)\right\}$$

To make use of this standard form, both the particular equilibrium and the effects of the various self-strainings to which a triangle would happen to be subjected to must be resolved in an F system. Note incidentally that the partial derivatives of this expression would yield the elongations of the skeleton triangle sides under the F forces. The

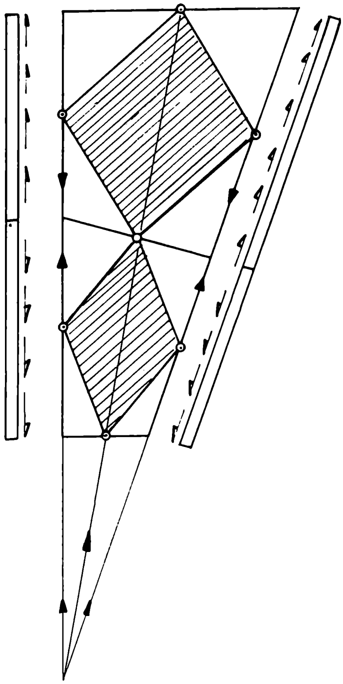


Fig. 9

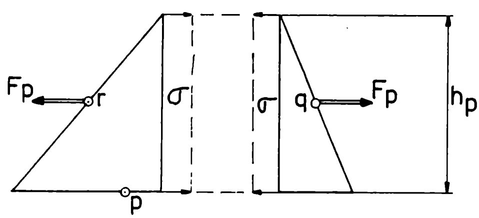


Fig. 10

strain energy of a beam segment as a function of loads was already dealt with. The loads appear again as linear combinations of a term due to the particular equilibrium condition considered and the self-straining terms.

REFERENCES

1. Trefftz, E. Ein gegenstück zum Ritzschen verfahren, *Proceedings of the 2nd Int. Cong. for Appl. Mech.*, Zürich (1926).
2. Weber, C. Eingrenzung von verschiebungen und zerrungen mit hilfe der minimalsätze, *Z.A.M.M.*, **22,** 130 (1942).
3. Prager, W. and Synge, J. L. Approximations in elasticity based on the concept of function space, *Quart. Appl. Math.*, **5** (1947).
4. Fraeijs de Veubeke, B. Sur certaines inégalités fondamentales et leur généralisation dans la théorie des bornes supérieures et inférieures en élasticité, *Rev. Universelle des Mines*, XVII, No. 5 (1961).
5. Turner, M. J. The direct stiffness method of structural analysis. Structures and Materials Panel, AGARD, Aachen, 1959.
6. Denke, P. H. A general digital computer analysis of statically indeterminate structures. Structures and Materials Panel, AGARD, Aachen, 1959.
7. Gallagher, R. H. A correlation study of methods of matrix structural analysis. Oral progress report to the Structures and Materials Panel of AGARD, Paris, 1961.
8. Argyris, J. H. and Kelsey, S. Note on the theory of aircraft structural analysis, *Zeitschrift für Flugwissenschaften*, 7 heft 3 (1959).

APPENDIX A

*Principles for the Determination of Upper and Lower Bounds to Influence Coefficients**

1. LINEARIZED THEORY

The principles apply only to the fully linearized theory of elasticity. This means that

(a) Rotations and strains are small enough to justify the linear relationships

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{A.1}$$

between strain components $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ and displacement vector (u,v). No distinction need be made between Lagrangean and Eulerian coordinates: the cartesian coordinates (x,y) are either those of the initial reference state or those of the deformed state.

(b) The strain energy density $W\ (\varepsilon_x, \gamma_{xy}, \varepsilon_y)$ relating stresses and strains

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x} \qquad \tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}} \qquad \sigma_y = \frac{\partial W}{\partial \varepsilon_y} \tag{A.2}$$

* For simplicity the two-dimensional case is considered. The extension to three dimensions is trivial.

is taken to be a quadratic, positive definite, homogeneous form. The stress–strain relations are thus of the homogeneous linear type (generalized Hooke's law).

(c) The complementary energy density Φ $(\sigma_x,\tau_{xy},\sigma_y)$, whose general definition is through the Legendre contact transformation

$$\Phi = \sigma_x\varepsilon_x + \tau_{xy}\gamma_{xy} + \sigma_y\varepsilon_y - W \tag{A.3}$$

is also a quadratic homogeneous form (in the stress components). This is an obvious consequence of the assumption of a generalized Hooke's law.

Differentiating Eq. (A.3)

$$d\Phi = \left(\sigma_x - \frac{\partial W}{\partial \varepsilon_x}\right) d\varepsilon_x + \left(\tau_{xy} - \frac{\partial W}{\partial \gamma_{xy}}\right) d\gamma_{xy} + \left(\sigma_y - \frac{\partial W}{\partial \varepsilon_y}\right) d\varepsilon_y + \varepsilon_x\, d\sigma_x + \gamma_{xy}\, d\tau_{xy} + \varepsilon_y\, d\sigma_y$$

and using Eqs. (A.2) there follows

$$\varepsilon_x = \frac{\partial \Phi}{\partial \sigma_x} \qquad \gamma_{xy} = \frac{\partial \Phi}{\partial \tau_{xy}} \qquad \varepsilon_y = \frac{\partial \Phi}{\partial \sigma_y} \tag{A.4}$$

Equations (A.4) are dual to Eqs. (A.2); they state again the generalized Hooke's law, resolved this time with respect to strains. Euler's theorem on homogeneous quadratic forms is applicable both to W and to Φ. In each case, by virtue of Eqs. (A.2) or (A.4), the same result is obtained:

$$W = \Phi = \tfrac{1}{2}(\sigma_x\varepsilon_x + \tau_{xy}\gamma_{xy} + \sigma_y\varepsilon_y) \tag{A.5}$$

This is conveniently referred to as the local form of Clapeyron's theorem: the energy densities are both numerically equal to half the sum of the products of stresses by corresponding strains.

(d) The internal equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{A.6}$$

and the surface equilibrium equations

$$l\sigma_x + m\tau_{xy} = p \qquad l\tau_{xy} + m\sigma_y = q \tag{A.7}$$

are linear. Here (X,Y) and (p,q) denote cartesian components of body forces and surface forces, while (l,m) are the direction cosines of the outward normal at the surface.

(e) As a consequence of the linear character of all equations the principle of superposition applies.

2. SCALAR PRODUCT OF TWO FIELDS

By "field" is implied an arbitrary field of stresses and corresponding strains, related by the generalized Hooke's law. This field need not satisfy either the compatibility conditions Eqs. (A.1) or the equilibrium Eqs. (A.6) and (A.7). It should, however, be integrable in the sense that the total strain energy:

$$\tfrac{1}{2}\iint(\sigma_x\varepsilon_x + \tau_{xy}\gamma_{xy} + \sigma_y\varepsilon_y)\, dS$$

should exist.

If two fields are distinguished by the subscripts A and B, consider the following mixed expression

$$(A,B) = \iint(\sigma_{x,A}\varepsilon_{x,B} + \tau_{xy,A}\gamma_{xy,B} + \sigma_{y,A}\varepsilon_{y,B})\, dS \qquad \text{(A.8)}$$

The integrand can be transformed as follows:

$$I = \sigma_{x,A}\frac{\partial\Phi}{\partial\sigma_{x,B}} + \tau_{xy,A}\frac{\partial\Phi}{\partial\tau_{xy,B}} + \sigma_{y,A}\frac{\partial\Phi}{\partial\sigma_{y,B}}$$

The last member is clearly the bilinear form attached to Φ and is equivalent to

$$I = \sigma_{x,B}\frac{\partial\Phi}{\partial\sigma_{x,A}} + \tau_{xy,B}\frac{\partial\Phi}{\partial\tau_{xy,A}} + \sigma_{y,B}\frac{\partial\Phi}{\partial\sigma_{y,A}}$$

Hence we obtain the commutativity property

$$(A,B) = (B,A) \qquad \text{(A.9)}$$

which is recognized as a statement of the Betti–Rayleigh reciprocity principle. On the other hand if α and β denote scalar multipliers we obviously have:

$$(\alpha A,\beta B) = \alpha\beta(A,B)$$

and also

$$(A,B + C) = (A,C) + (B,C)$$

Expression (A.8) has the properties of a scalar product, it is conveniently referred to as the scalar product of the two fields.

By virtue of Clapeyron's theorem the norm of a field

$$(A,A) \geq 0$$

is equal to twice the strain energy; it vanishes if and only if the field itself vanishes.

3. POTENTIAL ENERGY AND COMPLEMENTARY POTENTIAL ENERGY OF THE EXTERNAL LOADS

The following notations will prove to be useful by their concision:

C will denote a particular compatible field. This implies that the strains can be derived from continuous single-valued displacement functions

(u,v) according to Eqs. (A.1) and that those displacements take values

$$u = \bar{u} \qquad v = \bar{v} \qquad \text{on } c_2 \tag{A.10}$$

prescribed on parts of the boundary.

H will denote a general homogeneous compatible field. Here the values prescribed to the displacements on c_2 are zero. That this field is "general" means either that it is the most general one, or, if approximate solutions are sought, that it contains unknown functions or unknown parameters to be determined by the application of the energy theorems.

According to these definitions and to the superposition principle the general compatible field of a problem can be denoted by

$$C + H.$$

E will denote a particular equilibrium field. This implies that the stresses satisfy the equilibrium Eqs. (A.6) and (A.7) on the complementary part c_1 of the boundary.

A will denote a general self-straining field. It is an equilibrium field with the prescribed loads set equal to zero. It also contains unknown functions or parameters.

According to these definitions the general equilibrium field of a problem can be denoted by

$$E + A$$

S will denote the exact field of the problem, or the solution. It is both a compatible and an equilibrium field.

In a scalar product, if one field is compatible and the other an equilibrium field, the value of the product is expressible in terms of the displacements of the compatible field and the external loads with which the other field is in equilibrium. Take the case of the product (C,E). Since C is a compatible field, the product can be written as

$$(C,E) = \iint \left\{ \sigma_{x,E} \frac{\partial u_c}{\partial x} + \tau_{xy,E} \left(\frac{\partial u_c}{\partial y} + \frac{\partial v_c}{\partial x} \right) + \sigma_{y,E} \frac{\partial v_c}{\partial y} \right\} dx\, dy$$

If we now integrate by parts

$$(C,E) = \int_{c_1+c_2} \{ (l\sigma_{x,E} + m\tau_{xy,E})u_c + (l\tau_{xy,E} + m\sigma_{y,E})v_c \}\, ds$$

$$- \iint \left\{ u_c \left(\frac{\partial \sigma_{x,E}}{\partial x} + \frac{\partial \tau_{xy,E}}{\partial y} \right) + v_c \left(\frac{\partial \tau_{xy,E}}{\partial x} + \frac{\partial \sigma_{y,E}}{\partial y} \right) \right\} dx\, dy$$

However, the stresses of the equilibrium field verify Eqs. (A.6) and also Eqs. (A.7) on the part c_1 of the boundary, while the displacements of C verify Eq. (A.10) on c_2. Hence

$$(C,E) = -P_c - Q_e \tag{A.11}$$

where

$$P_c = -\iint (u_c X + v_c Y)\, dx\, dy - \int_{c_1} (p u_c + q v_c)\, ds$$

is the potential energy of the prescribed loads associated with the displacements of the field C, and

$$Q_e = -\int_{c_2} \{\bar{u}(l\sigma_{x,E} + m\tau_{xy,E}) + \bar{v}(l\tau_{xy,E} + m\sigma_{y,E})\}\, ds$$

is the complementary potential energy of the reactions of field E associated with the prescribed displacements. From this result, which has a general character, one finds immediately the following similar properties:

$$(H,E) = -P_h \tag{A.12}$$

$$(C,A) = -Q_a \tag{A.13}$$

$$(H,A) = 0 \tag{A.14}$$

This last result is specially useful and can be stated as the following theorem: "The scalar product of a homogeneous compatible field and a self-straining field vanishes."

In most problems the prescribed boundary displacements are zero and the particular displacement field associated with C does not appear. This simplification will be used henceforward. Should it arise, the problem involving non-zero prescribed displacements can be treated by the same techniques as presented below.

4. PRINCIPLE OF MINIMUM TOTAL ENERGY

Under the assumption that $C = 0$, the minimum total energy principle can be stated as follows:

$$\mu = \tfrac{1}{2}(H,H) - (H,E)$$

takes its minimum value for $H = S$.

Proof: Since the solution S is also a compatible field, the difference between S and H is a compatible and homogeneous field. This is written

$$H = S + \delta H$$

Substitution of this into the expression of μ, expansion of the product and rearrangement of the terms produces

$$\mu = \tfrac{1}{2}(S,S) - (S,E) + \tfrac{1}{2}(\delta H,\delta H) + (S - E,\delta H)$$

Now $S - E$ is the difference between two particular equilibrium fields and is consequently a self-straining field. Its product with a compatible homogeneous field vanishes, so the last term vanishes. The two first terms are the value taken by μ when $H = S$. The third term is positive unless δH vanishes. This completes the proof.

5. PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

Since the prescribed boundary displacements are assumed to be zero there is no complementary potential energy. The principle states that the true solution makes the complementary strain energy (the strain energy expressed in terms of stresses) a minimum. In terms of a scalar product

$$\lambda = \tfrac{1}{2}(E + A, E + A) \quad \text{is minimum for} \quad E + A = S$$

If we subtract from $E + A$ the particular equilibrium field S we obtain a self-straining field; this can be written

$$E + A = S + \delta A$$

Substitution of this into the scalar product and rearrangement produces

$$\lambda = \tfrac{1}{2}(S,S) + \tfrac{1}{2}(\delta A,\delta A) + (S,\delta A)$$

Again the last term vanishes as the product of a self-straining field and a homogeneous compatible field. The first term is the value taken by λ for $E + A = S$. The second term is positive unless δA vanishes. This completes the proof.

6. CLAPEYRON'S EXTERIOR THEOREM

Since S is a homogeneous compatible field and $S - E$ a self-straining field we have

$$(S, S - E) = 0$$

This can also be written in the form

$$\tfrac{1}{2}(S,S) = \tfrac{1}{2}(S,E) = -\tfrac{1}{2}P \tag{A.15}$$

the last equality following from property (A.12) so that P denotes the potential energy of the prescribed loads under the displacements of the true solution. This is also the usual theorem of Clapeyron according to which the strain energy is equal to half the virtual work done by the loads under the true displacements.

7. UPPER AND LOWER BOUNDS TO DIRECT INFLUENCE COEFFICIENTS

From the principle of minimum total energy and Clapeyron's theorem (A.15) follows

$$\tfrac{1}{2}(H,H) - (H,E) \geq \tfrac{1}{2}(S,S) - (S,E) = -\tfrac{1}{2}(S,S) = \tfrac{1}{2}P$$

From the principle of minimum complementary energy follows

$$\tfrac{1}{2}(E + A, E + A) \geq \tfrac{1}{2}(S,S) = -\tfrac{1}{2}P$$

Hence by changing signs and the sense of the last inequality

$$\tfrac{1}{2}(H,H) - (H,E) \geq \tfrac{1}{2}P \geq -\tfrac{1}{2}(E + A,\ E + A) \qquad \text{(A.16)}$$

Under restrictive assumptions used for displacements in some approximate but compatible approach, the left-hand side of the inequality will be the value estimated for half the potential energy; denote it by $\tfrac{1}{2}\bar{P}$. Denote similarly by $\tfrac{1}{2}\underline{P}$ the right-hand side which will be the value estimated under restrictive assumptions for the stresses in some equilibrium approximation to the same problem. The free parameters or functions contained in the approximations are naturally determined by the minimum principles themselves in order that the bracket for P

$$\bar{P} \geq P \geq \underline{P}$$

be as tight as possible.

If the load consists of a single force F with corresponding displacement w, the influence coefficient c is defined by

$$w = cF$$

and the potential energy can be written

$$P = -Fw = -cF^2 \quad \text{(exact solution)}$$

In a compatible approximation we would find

$$\bar{P} = -\underline{c}F^2 \quad \text{(compatible approximation)}$$

In an equilibrium approximation

$$\underline{P} = -\bar{c}F^2 \quad \text{(equilibrium approximation)}$$

Insertion of these expressions in the bracket for the potential energy produces after cancellation of the common factor F^2 and a change of sign altering the sense of the inequalities

$$\bar{c} \geq c \geq \underline{c} \qquad \text{(A.17)}$$

Thus an approximate compatible approach produces a lower bound and an approximate equilibrium approach an upper bound to a direct influence coefficient.

8. UPPER AND LOWER BOUNDS TO CROSS INFLUENCE COEFFICIENTS

Let

$$w_1 = c_{11}F_1 + c_{12}F_2 \qquad\qquad c_{21} = c_{12}$$
$$w_2 = c_{21}F_1 + c_{22}F_2$$

be the (exact) displacements associated to the two loads F_1 and F_2. If $(\underline{c}_{11}, \underline{c}_{12}, \underline{c}_{22})$ are the approximate influence coefficients resulting from a compatible approach and $(\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{22})$ those resulting from an equilibrium approach, previous results enable us to write

$$\bar{c}_{11} \geq c_{11} \geq \underline{c}_{11} \qquad \bar{c}_{22} \geq c_{22} \geq \underline{c}_{22} \tag{A.18}$$

The problem is to establish a similar connexion for the cross influence coefficient c_{12}. To this purpose we set

$$F_2 = \lambda F_1$$

and consider the potential energy P given by

$$-P = w_1 F_1 + w_2 F_2 = F_1^2 (c_{11} + 2\lambda c_{12} + \lambda^2 c_{22})$$

Again from the general bracket applicable to the potential energy there is found that

$$\bar{c}_{11} + 2\lambda\bar{c}_{12} + \lambda^2\bar{c}_{22} \geq c_{11} + 2\lambda c_{12} + \lambda^2 c_{22} \geq \underline{c}_{11} + 2\lambda\underline{c}_{12} + \lambda^2\underline{c}_{22}$$

Take the first inequality and solve it for c_{12} *assuming λ to be a positive quantity*:

$$2c_{12} \leq \frac{1}{\lambda}(\bar{c}_{11} - c_{11}) + 2\bar{c}_{12} + \lambda(\bar{c}_{22} - c_{22})$$

It follows *a fortiori* by virtue of Eqs. (A.18) that

$$2c_{12} \leq \frac{1}{\lambda}(\bar{c}_{11} - \underline{c}_{11}) + 2\bar{c}_{12} + \lambda(\bar{c}_{22} - \underline{c}_{22})$$

The positive λ giving the smallest upper bound is readily found to be

$$\lambda = \sqrt{(\bar{c}_{11} - \underline{c}_{11})/(\bar{c}_{22} - \underline{c}_{22})}$$

from which follows

$$c_{12} \leq \bar{c}_{12} + \sqrt{(\bar{c}_{11} - \underline{c}_{11})(\bar{c}_{22} - \underline{c}_{22})} \tag{A.19}$$

Had we solved the same inequality assuming a negative λ value we would have

$$2c_{12} \geq \frac{1}{\lambda}(\bar{c}_{11} - c_{11}) + 2\bar{c}_{12} + \lambda(\bar{c}_{22} - c_{22})$$

It follows again *a fortiori* by virtue of Eqs. (A.18) that

$$2c_{12} \geq \frac{1}{\lambda}(\bar{c}_{11} - \underline{c}_{11}) + 2\bar{c}_{12} + \lambda(\bar{c}_{22} - \underline{c}_{22})$$

A highest lower bound is obtained for the negative λ value

$$\lambda = -\sqrt{(\bar{c}_{11} - \underline{c}_{11})/(\bar{c}_{22} - \underline{c}_{22})}$$

from which follows

$$c_{12} \geq \bar{c}_{12} - \sqrt{(\bar{c}_{11} - \underline{c}_{11})(\bar{c}_{22} - \underline{c}_{22})} \qquad (A.20)$$

The second inequality can be treated by the same technique and produces the new bounds

$$c_{12} \leq \underline{c}_{12} + \sqrt{(\bar{c}_{11} - \underline{c}_{11})(\bar{c}_{22} - \underline{c}_{22})} \qquad (A.21)$$

$$c_{12} \geq \underline{c}_{12} - \sqrt{(\bar{c}_{11} - \underline{c}_{11})(\bar{c}_{22} - \underline{c}_{22})} \qquad (A.22)$$

If, as usually the case, it is found numerically that $\bar{c}_{12} > \underline{c}_{12}$ the best bracket is made of inequalities (A.20) and (A.21).

APPENDIX B

Examples and Numerical Results

B. M. Fraeijs de Veubeke and Guy Sander

1. STRUCTURAL MODEL

The ultimate step is the analysis of a complete and complicated structure. Very instructive results are, however, obtained concerning the validity of the dual approach by the consideration of a rather simple structural element.

A rectangular element, or panel, of length $2a$ and height $2b$, edged by beams of identical cross-section S, was analysed by various methods and the influence coefficients compared. The beams are assumed to be devoid of flexural rigidity. The two structural parameters of the structure are then

$$r = a/b$$
$$R = S/(bt)$$

where t is the constant thickness of the panel. Exploratory values of 1 and 5 for r, and 0·4–1·0–2·0 and 4·0 for R were adopted.

The reference to Model 1 or Model 2 relates to the type of support of the structure and is explained in Fig. 11.

The loading cases investigated are shown in Fig. 12 and referred to as Cases I, II, III or IV.

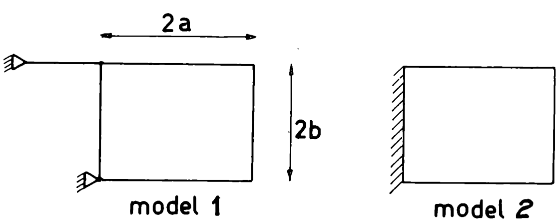


Fig. 11

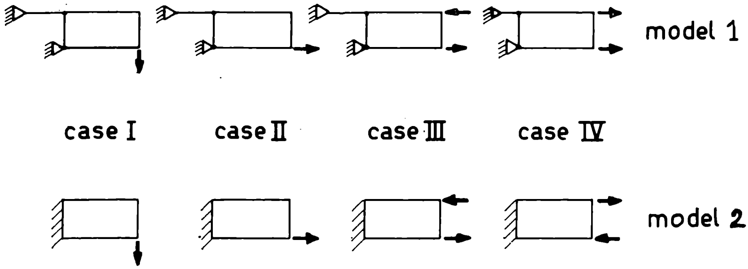


Fig. 12

2. METHODS OF ANALYSIS

The methods of analysis that were compared are the following:

(1) *Tension* 4. This denomination refers to a subdivision of the panel in four triangles by the diagonals and construction of a pure equilibrium field as explained in Sections 2.3 and 3a of the paper. It is found that the state of stress in the panel is either zero or a state of pure shear.

(2) *Tension* 16. The panel is first subdivided into four equal rectangular fields of dimensions a and b. Each rectangle is then subdivided in four triangles by the diagonals. Again a pure equilibrium field is constructed.

In contrast to the "Tension 4" case there are redundancies in the form of self-strainings. Figure 13 shows the self-strainings involved in

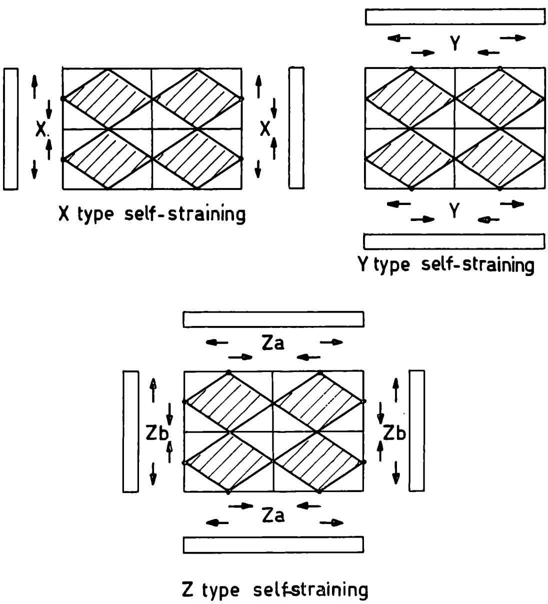
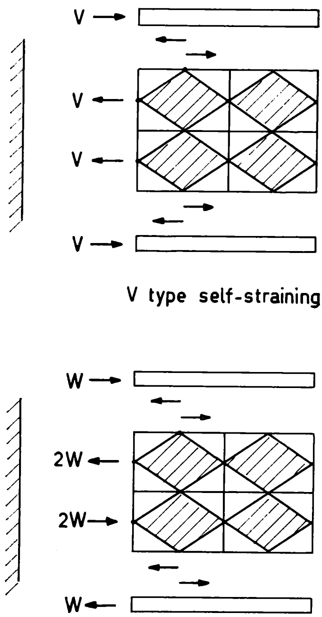


Fig. 13

the Model 1 type of support. In the case of Model 2 there are additional self-strainings depicted in Fig. 14.

"Tension 4" and "Tension 16" are the only methods used to provide upper bounds to the influence coefficients.

(3) *Turner* 16. The subdivision of the panel in triangles is the same as for "Tension 16". However, the field of displacements is purely



V type self-straining

W type self-straining

Fig. 14

compatible. It is identical to the Argyris–Turner triangularization method and the calculations were essentially made according to the Turner stiffness method.[5]

(4) *Turner* 4. It is the triangularization and stiffness method applied to the subdivision used in Tension 4.

(5) *Argyris* 4. The panel is first subdivided into the four equal rectangular fields. In each field the displacement assumption is used

$$u = u_0 + px + qy + r\,xy$$
$$v = v_0 + lx + my + n\,xy$$

The eight parameters can be expressed in terms of the corner displacements. As in the case of triangular fields continuity of the displacements for the whole panel is obtained by simply requiring identical displacements at the common corners. Corner loads and stiffness

Table 1

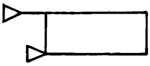
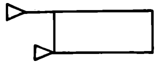
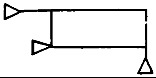
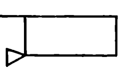
$r = \frac{a}{b}$	$r = \frac{a}{b} = 1$ 				$r = \frac{a}{b} = 5$ 			
$R = \frac{S}{bt}$	0·4	1·0	2·0	4·0	0·4	1·0	2·0	4·0
Methods	*Case I*  P							
Tension 4	0·2106	0·1196	0·0893	0·0742	9·840	4·113	2·204	1·249
Tension 16	0·2038	0·1185	0·0890	0·0741	9·182	3·989	2·171	1·241
Turner 16	0·1287	0·0991	0·0825	0·0719	2·139	1·699	1·303	0·939
Turner 4	0·1059	0·0894	0·0767	0·0690	0·810	0·765	0·561	0·606
Argyris 1	0·1099	0·0895	0·0773	0·0692				
Argyris 4	0·1238	0·0948	0·0807	0·06945				
Beam. theory	0·0986	0·0813	0·0719	0·06603	5·239	3·068	1·896	1·1630
Fict. spars	0·1374	0·1045	0·0850	0·0730	1·985	1·556	1·159	0·788
	Case II  → P							
Tension 4	0·1136	0·0454	0·0227	0·01136	0·5681	0·227	0·113	0·0568
Tension 16	0·0930	0·0415	0·0217	0·0111	0·365	0·178	0·0999	0·0529
Turner 16	0·0545	0·0315	0·0185	0·0102	0·143	0·103	0·0705	0·0434
Turner 4	0·0393	0·0256	0·0162	0·0094	0·101	0·0749	0·0534	0·0349
Argyris 1	0·0413	0·0263	0·0165	0·00953				
Argyris 4	0·0468	0·0299	0·0177	0·00982				
Fict. spars	0·0588	0·0340	0·0194	0·01048	0·109	0·085	0·0619	0·0401

Table 2

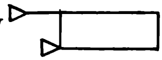
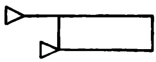
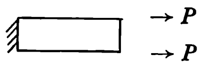
Methods	*Case III*  ← *P* → *P*							
Tension 4	0·1136	0·0454	0·0227	0·0113	0·5681	0·227	0·113	0·0568
Tension 16	0·0976	0·0425	0·0220	0·01117	0·4201	0·1973	0·1054	0·05466
Turner 16	0·0570	0·0325	0·0189	0·0103	0·117	0·0895	0·0642	0·0410
Turner 4	0·0468	0·0291	0·0176	0·0099	0·0409	0·0370	0·0317	0·0248
Argyris 1	0·0508	0·0304	0·0182	0·0101				
Argyris 4	0·0632	0·0321	0·0186	0·0102				
Fict. spars	0·0588	0·0340	0·0194	0·0104	0·109	0·0852	0·0619	0·0401
	Case IV  → *P* → *P*							
Tension 4	0·113	0·0454	0·0227	0·0113	0·568	0·227	0·113	0·0568
Tension 16	0·0885	0·0406	0·0214	0·0110	0·310	0·162	0·0935	0·0511
Turner 16	0·0520	0·0305	0·0182	0·0100	0·169	0·116	0·0768	0·0458
Turner 4	0·0318	0·0222	0·0148	0·0089	0·161	0·112	0·0750	0·0450
Argyris 1	0·0318	0·0222	0·0148	0·00895				
Argyris 4	0·0399	0·0267	0·0166	0·00953				
Fict. spars	0·0588	0·0340	0·0194	0·0104	0·109	0·0852	0·0619	0·0401

Table 3

$r = \frac{a}{b}$	$r = 1$				$r = 5$			
$R = \frac{S}{bt}$	0·4	1·0	2·0	4·0	0·4	1·0	2·0	4·0
Methods	*Case I* P							
Tension 4	0·172	0·104	0·0818	0·0704	9·803	4·098	2·196	1·246
Tension 16	0·148	0·0953	0·0803	0·0701	5·345	3·089	1·903	1·166
Turner 16	0·112	0·0893	0·0767	0·0687	2·131	1·700	1·300	0·937
Turner 4	0·0920	0·0805	0·0753	0·0664	0·800	0·756	0·687	0·601
Argyris 1	0·0942	0·0807	0·0753	0·0665				
Argyris 4	0·0949	0·0823	0·0759	0·0678				
Beam. theory	0·0986	0·0813	0·0719	0·06603	5·239	3·068	1·896	1·1630
Fict. spars	0·1179	0·0931	0·0758	0·0695	1·965	1·544	1·152	0·795
	Case II → P							
Tension 4	0·1136	0·0454	0·0227	0·0113	0·568	0·227	0·113	0·0568
Tension 16	0·0877	0·0404	0·0211	0·0110	0·365	0·0178	0·0999	0·0529
Turner 16	0·0496	0·0296	0·0178	0·00998	0·140	0·101	0·0700	0·0432
Turner 4	0·0390	0·0255	0·0170	0·00944	0·099	0·0740	0·0530	0·0348
Argyris 1	0·0410	0·0261	0·0171	0·00952				
Argyris 4	0·0476	0·0290	0·0175	0·00963				
Fict. spars	0·0588	0·0340	0·0194	0·01048	0·109	0·085	0·0619	0·0401

Table 4

Methods	*Case III* 							
Tension 4	0·1136	0·0454	0·0227	0·0113	0·568	0·227	0·113	0·0568
Tension 16	0·0971	0·0418	0·0216	0·0110	0·420	0·197	0·105	0·0544
Turner 16	0·0569	0·0324	0·0189	0·0103	0·117	0·0896	0·0641	0·0410
Turner 4	0·0469	0·0259	0·0150	0·0099	0·0409	0·0370	0·0317	0·0248
Argyris 1	0·0508	0·0304	0·0182	0·0101				
Argyris 4	0·0631	0·0321	0·0186	0·0102				
Fict. spars	0·0588	0·0340	0·0194	0·01048	0·109	0·085	0·0619	0·0401
	Case IV 							
Tension 4	0·1136	0·0454	0·0227	0·0113	0·568	0·227	0·113	0·0568
Tension 16	0·0783	0·0390	0·0206	0·0110	0·310	0·162	0·0935	0·0511
Turner 16	0·0424	0·0268	0·0167	0·00963	0·163	0·114	0·0759	0·0454
Turner 4	0·0312	0·0219	0·0146	0·0089	0·158	0·111	0·0744	0·0448
Argyris 1	0·0312	0·0219	0·0147	0·00893				
Argyris 4	0·0390	0·0260	0·0165	0·00950				
Fict. spars	0·0588	0·0340	0·0194	0·01048	0·109	0·085	0·0619	0·0401

matrices are obtained by the same procedures as used for the stiffness method applied to triangular fields[8]. The major difference with respect to a subdivision into triangles is that the stress field associated with the displacement assumption is not an equilibrium field. The equilibrium conditions are violated continuously but smaller discontinuities in the stresses are recorded across the common edges. This method, together with Turner 4 and 16, must yield lower bounds. Whenever used, it gave results between the Turner 4 and 16 methods, generally closer to the last one.

(6) *Beam theory.* This refers to a purely compatible Bernoulli type of bending theory, applicable to loading case *I*. It consists in assuming for the panel a field of the type

$$u = yB(x)$$
$$v = V(x)$$

The unknown functions $B(x)$ and $V(x)$ are determined from the principle of variations for displacements. The value obtained for the influence coefficient is classical

$$\frac{d}{P} = \frac{(2a)^3}{3EI} + \frac{2a}{2btG}$$

where

$$I = 2Sb^2 + \frac{2tb^3}{3(1-\nu^2)}$$

is the moment of inertia of the cross-section with an increased web contribution due to the assumption of inextensibility in the y direction. Because of this assumption the result is poor when r and R are small. Because the sections normal to x are assumed to remain plane, the method applies as well to Model 1 as to Model 2.

The method provides an interesting lower bound for large values of r. It is seen to be very close to the Tension 16 method and consequently both are then very accurate.

(7) *Fictitious spar theory.* This is the only hybrid assumption investigated that does not with certainty provide an upper or lower bound. It consists in replacing the direct stress-carrying capacity of the plate by fictitious edge and central spars. When one-sixth of a transverse area of the plate is concentrated as an edge spar and two-thirds as a central spar both the total cross-sectional area and the total moment of inertia are conserved. The stresses transmitted by the plate are then reduced to a system of four shear fields. In practice the assumption works well. The influence coefficients it provides are located between the Turner and Tension 16 methods, which are generally the best upper and lower bounds.

It is, however, recognized that in this idealization both the Poisson's ratio effect and sweep-back effects are lost.

It would therefore be interesting to pursue the investigations on a model incorporating sweep-back and perhaps taper.

Note: The values recorded in the tables and compared in Figs. 17, 18, 19 and 20 are displacements corresponding to specific values:

$$\text{load } P = 1000 \text{ kilos}$$
$$E = 2{\cdot}2\ 10^4 \text{ kilo/mm}^2$$
$$b = 1000 \text{ mm}$$
$$t = 2 \text{ mm}$$

To obtain the non-dimensional quantities

$$d\frac{Et}{P} = f(r,R)$$

where d is the displacement corresponding to the load, the table values should be multiplied by the factor 44. In the figures, heavy continuous lines are for lower bounds, dotted lines for upper bounds. The thin continuous line is for the fictitious spar theory.

3. REDUCTION OF SELF-STRAINING CONDITIONS FROM SYMMETRY CONSIDERATIONS

Sometimes advantage can be gained from the symmetries in reducing the number of self-strainings to consider. Take for example Model 1 with loading case II. Figure 15 shows that this case can be considered as the superposition of loading cases IV and III.

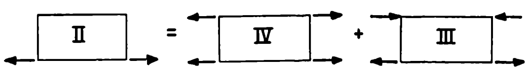


Fig. 15

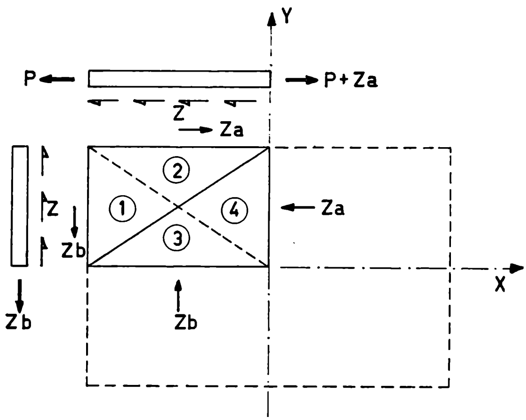


Fig. 16

(a) *Case IV.* It is symmetrical with respect to both the x and y axes. Hence we need only consider a self-straining of type Z when dealing with the Tension 16 method. It is readily found by investigation of the equilibrium conditions of the triangles (Fig. 16) that in triangles 1 and 2 the state of stress is one of pure shear:

$$\sigma_x = 0 \qquad \sigma_y = 0 \qquad \tau_{xy} = Z$$

while in triangles 3 and 4 it is the state

$$\sigma_x = -Z\frac{a}{b} \qquad \sigma_y = -Z\frac{b}{a} \qquad \tau_{xy} = 0$$

The corresponding strain energies (see p. 166) are

$$\frac{1}{2}\frac{ab}{2}\frac{t}{E}\{2(1+\nu)\tau_{xy}^2\} = \frac{abt}{2E}(1+\nu)Z^2$$

$$\frac{1}{2}\frac{ab}{2}\frac{t}{E}\left\{Z^2\frac{b^2}{a^2} + Z^2\frac{a^2}{b^2} - 2\nu Z^2\right\} = \frac{abt}{4E}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2\nu\right)Z^2$$

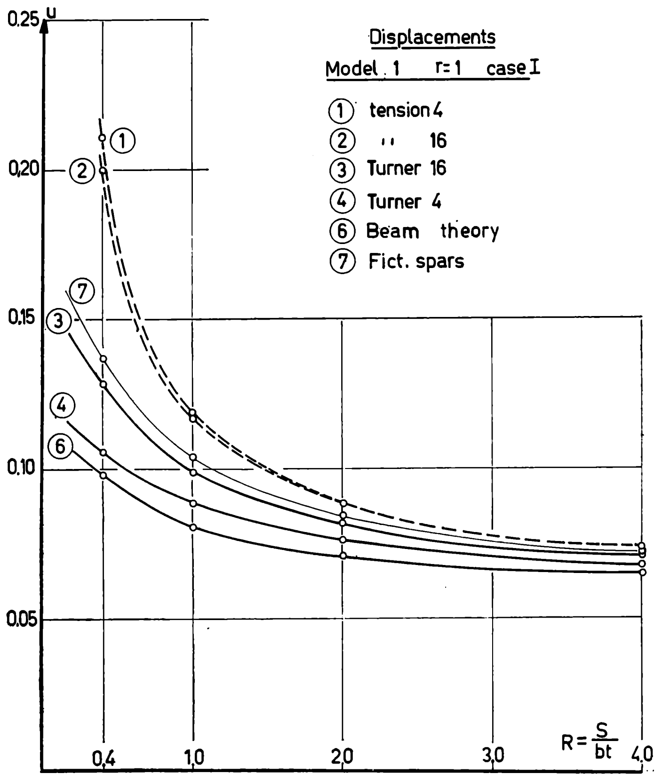


Fig. 17

Adding and multiplying by 4 (the other three rectangular fields have symmetrical states of stress) the panel energy is

$$U_p = \frac{abt}{E}\left\{2 + \frac{a^2}{b^2} + \frac{b^2}{a^2}\right\} Z^2$$

In half of a vertical beam the tension, computed from equilibrium considerations, is

$$T = Zt(b - y)$$

and the corresponding strain energy for both complete verticals

$$U_v = \frac{2}{3}\frac{t^2b^3}{ES} Z^2$$

In half of a horizontal beam the tension is the superposition of the self-straining effect and a particular equilibrium state under the external horizontal load P applied at the end:

$$T = P + Zt(a + x)$$

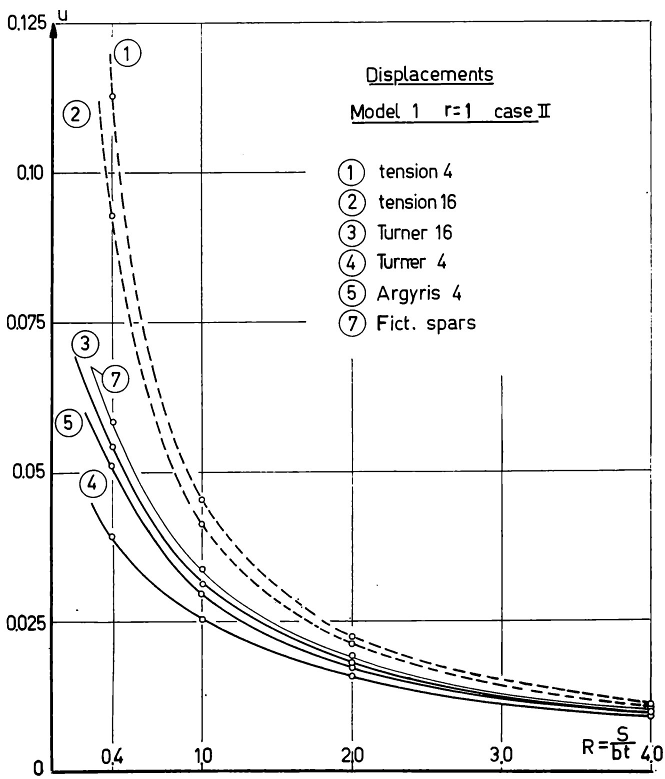


Fig. 18

The corresponding strain energy for both complete horizontals

$$U_h = \frac{2}{ES}\{aP^2 + a^2PZt + \tfrac{1}{3}a^3t^2Z^2\}$$

The total energy of the structure is thus found to be

$$U = \frac{Z^2}{E}\left\{abt\left(2 + r^2 + \frac{1}{r^2}\right) + \frac{2tb^2r^3}{3R}\right\} + 2\,\frac{ZP}{E}\,\frac{r^2b}{R} + \frac{P^2}{E}\,\frac{2r}{Rt}$$

From Menabrea's theorem $\partial U/\partial Z = 0$ follows

$$Z = -\frac{P}{bt}\left\{\frac{3r}{3R(2 + r^2 + r^{-2}) + 2r^2}\right\}$$

and finally from Castigliano's formula

$$d = \frac{1}{2}\frac{\partial U}{\partial P} = \frac{1}{E}\left\{\frac{2r^2b}{R}\,Z + \frac{4r}{Rt}\,P\right\}$$

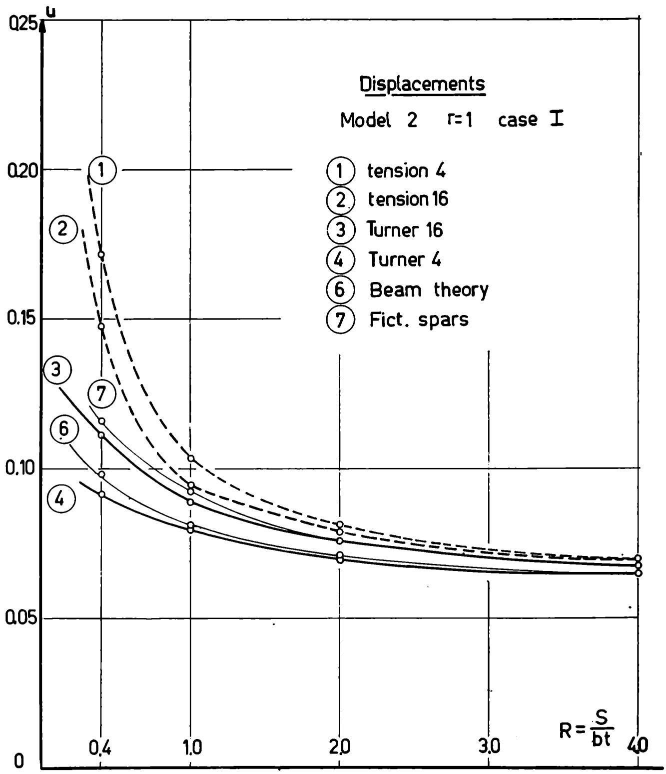


Fig. 19

where d is the deflexion under each P load of Model 1. After substitution of the Z value there comes

$$d = \frac{P}{Et}\frac{r}{R}\left\{2 - \frac{3r^2}{3R(2 + r^2 + r^{-2}) + 2r^2}\right\}$$

Note that in the tension 4 approach no self-strainings exist and we would have $Z = 0$ with a corresponding deflexion

$$\frac{P}{Et}\,2\,\frac{r}{R}$$

This is the first term of the previous formula which represents the elongation of a horizontal beam under the end load when the plate is under no stress and does not participate to the general stiffness. The second term represents the plate contribution in reducing the deflexion.

(b) *Case III.* It is symmetrical with respect to the y axis, antisymmetrical with respect to the x axis. It can be dealt with by con-

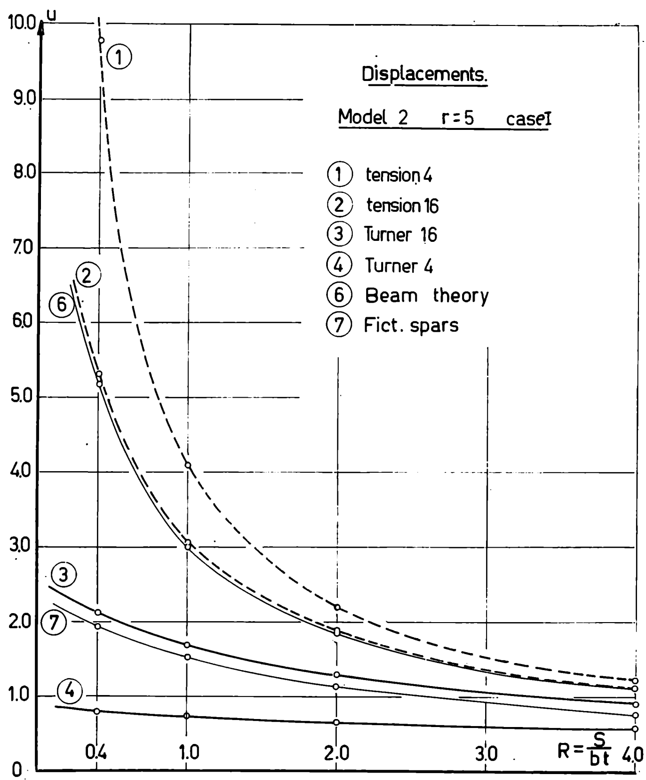


Fig. 20

sidering only the Y type of self-straining. By similar calculations there is found

$$Y = \frac{P}{tR} \frac{r}{\dfrac{2}{3}\dfrac{r}{R} + \dfrac{1}{r^3} + 3r + \dfrac{2(1 + 2\nu)}{r}}$$

$$d = \frac{P}{Et}\frac{r}{R}\left\{2 - \frac{r^4}{\frac{2}{3}r^4 + R + 3Rr^4 + 2(1 + 2\nu)r^2 R}\right\}$$

Again the first term represents the beam elongation in the absence of the plate, as would be found in the tension 4 approach. The second term is the plate effect given by the tension 16 approach.

(c) *Case II.* Returning to case II, it is obvious from the principle of superposition that the deflexion under each load is the average between those of case III and case IV.