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THE INERTIA TENSOR OF AN INCOMPRESSIBLE FLUID BOUNDED BY WALLS IN RIGID BODY MOTION

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Abstract—The rigid body motion of a cavity determines an irrotational motion of the liquid filling its inside. From the point of view of external dynamics, the fluid behaves as an equivalent solid. The purpose of the paper is to examine general properties of the equivalent inertia tensor and to calculate it explicitly in the case of a cylindrical fuel tank.

LET (x, y, z) be cartesian co-ordinates referred to body axes of a cavity entirely filled with an inviscid incompressible fluid. The motion of the cavity is described by the velocity components (U, V, W) of the origin along, and the angular velocity components (P, Q, R) about the body axes. The resulting rigid body velocity field is then given by

$$\begin{aligned} u &= U + Qz - Ry \\ v &= V + Rx - Pz \\ w &= W + Py - Qx \ . \end{aligned} \tag{1}$$

The absolute velocity field of the liquid is governed by a potential ϕ, satisfying the Laplace equation

$$\nabla^2 \phi = 0 \tag{2}$$

and the boundary conditions

$$\frac{\mathrm{d}\phi}{\mathrm{d}\nu} = lu + mv + nw \quad \text{on the cavity surface } S \tag{3}$$

where (l, m, n) are the direction cosines of the outward normal ν. The unicity of the solution of this Neumann problem is well established and the velocity field of the liquid is, at each moment, completely determined by the instantaneous values of the six parameters (U, V, W, P, Q, R). The equivalent inertia tensor can therefore be obtained directly from the expression of the kinetic energy of the liquid

$$T = \tfrac{1}{2}\rho \int_v (\text{grad}\ \phi, \text{grad}\ \phi)\, \mathrm{d}\tau \ . \tag{4}$$

Use of the Green formula

$$\int_v [(\text{grad} f, \text{grad}\ g) + f\nabla^2 g]\, \mathrm{d}\tau = \int_S f \frac{\mathrm{d}g}{\mathrm{d}\nu}\, \mathrm{d}S \tag{5}$$

with $f = g = \phi$ and of equation (2) transforms equation (4) in the well known expression

$$T = \tfrac{1}{2}\rho \int_S \phi \frac{\mathrm{d}\phi}{\mathrm{d}\nu}\, \mathrm{d}S \ . \tag{6}$$

Setting

$$\phi = U\phi_u + V\phi_v + W\phi_w + P\phi_p + Q\phi_q + R\phi_r \tag{7}$$

and substituting in the left-hand side of equation (3) while substituting equations (1) in the right-hand side, there follows on equating the coefficients of the parameters

$$\frac{d\phi_u}{d\nu} = l\,, \qquad \frac{d\phi_v}{d\nu} = m\,, \qquad \frac{d\phi_w}{d\nu} = n\,,$$
$$\frac{d\phi_p}{d\nu} = yn - zm\,, \qquad \frac{d\phi_q}{d\nu} = zl - xn\,, \qquad \frac{d\phi_r}{d\nu} = xn - yl\,. \tag{8}$$

Those are the boundary conditions of the component potentials, each of which satisfying the Laplace equation

$$\Delta^2 \phi_k = 0 \qquad (k = U, V, W, P, Q, R)\,. \tag{9}$$

When equation (7) is substituted into equation (6) the kinetic energy is expressed as a quadratic form in the parameters

$$T = \tfrac{1}{2} \sum_K \sum_H KHA_{kh} \tag{10}$$

the coefficients of which are the components of the equivalent inertia tensor and are expressed by

$$A_{kh} = \rho \int_S \phi_k \frac{d\phi_h}{d\nu}\, dS\,. \tag{11}$$

The second Green formula

$$\int_v (f\nabla^2 g - g\nabla^2 f)\, d\tau = \int_s \left(f\frac{dg}{d\nu} - g\frac{df}{d\nu}\right) dS$$

applied with $f = \phi_k$ and $g = \phi_h$, establishes the symmetry

$$A_{kh} = A_{hk} \tag{12}$$

of the inertia tensor.

THE TRANSLATION POTENTIALS. REDUCTION OF THE PROBLEM TO THAT OF ROTATIONS

In contrast to the problem of the motion of a solid through a liquid [1], the potentials ϕ_u, ϕ_v and ϕ_w of the interior problem are trivial solutions

$$\phi_u = x\,, \qquad \phi_v = y\,, \qquad \phi_w = z \tag{13}$$

independent of the shape of the cavity.
Hence we find immediately

$$A_{uu} = \rho \int_S x \frac{dx}{d\nu}\, dS = \rho \int_S xl\, dS = \rho\tau$$

where τ is the total volume of the cavity. Similarly for the equivalent masses in the other two directions, so that

$$A_{uu} = A_{vv} = A_{ww} = \rho\tau\,. \tag{14}$$

Calculating

$$A_{uv} = \rho \int_S x \frac{\mathrm{d}y}{\mathrm{d}\nu} \mathrm{d}S = \rho \int_S xm \, \mathrm{d}S$$

we find, transforming back to a volume integral by means of the formula (5) with $f = x$ and $g = y$, that $A_{uv} = 0$. The absence of mass coupling is also established by similar means between the other pair of directions, so that

$$A_{uv} = A_{vw} = A_{wu} = 0 \ . \tag{15}$$

It is still possible to evaluate explicitly such cross coefficients as

$$A_{kh} \quad \text{where} \quad k = u, v, w \quad \text{and} \quad h = p, q, r. \tag{16}$$

Take first

$$A_{up} = \rho \int_S \phi_u \frac{\mathrm{d}\phi_p}{\mathrm{d}\nu} \mathrm{d}S = \rho \int_S x(yn - zm) \, \mathrm{d}S.$$

With $f = xy$ and $g = z$, the Green formula (5) gives

$$0 = \int_S xy \frac{\mathrm{d}z}{\mathrm{d}\nu} \mathrm{d}S = \int_S xyn \, \mathrm{d}S$$

and for $f = xz$ and $g = y$ it furnishes

$$0 = \int_S xz \frac{\mathrm{d}y}{\mathrm{d}\nu} \mathrm{d}S = \int_S xzm \, \mathrm{d}S$$

Whence $A_{up} = 0$ and by similar calculations

$$A_{up} = A_{vq} = A_{wr} = 0 \ . \tag{17}$$

Take next

$$A_{vp} = \rho \int_S \phi_v \frac{\mathrm{d}\phi_p}{\mathrm{d}\nu} \mathrm{d}S = \int_S y(yn - zm) \, \mathrm{d}S \ .$$

In this case the Green formula is applied once with $f = y^2$ and $g = z$, then with $f = yz$ and $g = y$, giving

$$A_{vp} = -\rho \int_v z \, \mathrm{d}\tau \ .$$

The results of similar calculations can be summarized as follows

$$\begin{aligned} -A_{vp} &= A_{uq} = \rho \int_v z \, \mathrm{d}\tau \\ -A_{wq} &= A_{vr} = \rho \int_v x \, \mathrm{d}\tau \\ -A_{ur} &= A_{wp} = \rho \int_v y \, \mathrm{d}\tau \ . \end{aligned} \tag{18}$$

It can be concluded that all cross inertia coefficients of type (16) can be made to vanish by placing the origin of the axes at the centre of mass of the fluid. In that case the quadratic form of the kinetic energy is reduced to the following

$$T = \tfrac{1}{2}\rho\tau(U^2 + V^2 + W^2) + \tfrac{1}{2}(A_{pp}P^2 + A_{qq}Q^2 + A_{rr}R^2 + 2A_{pq}PQ + 2A_{qr}QR + 2A_{rp}RP) \,. \quad (19)$$

At this stage it does not seem possible to derive further results of general character. The evaluation of the remaining six inertia components requires explicit calculation of the potentials ϕ_p, ϕ_q and ϕ_r in relation to the shape of the cavity.

THE VELOCITY POTENTIAL ϕ_q FOR A CYLINDRICAL CAVITY

An important technical case is that of the cylindrical tank with flat ends. It applies to missile reservoirs where the propellant is slowly displaced and maintained between moving diaphragms or pistons. The geometry of the problem is shown in Fig. 1. Because of the symmetry of revolution about Ox, $\phi_p \equiv 0$ and consequently

$$A_{pp} = A_{pq} = A_{pr} = 0$$

and also

$$A_{qr} = 0 \quad \text{and} \quad A_{rr} = A_{qq} \,.$$

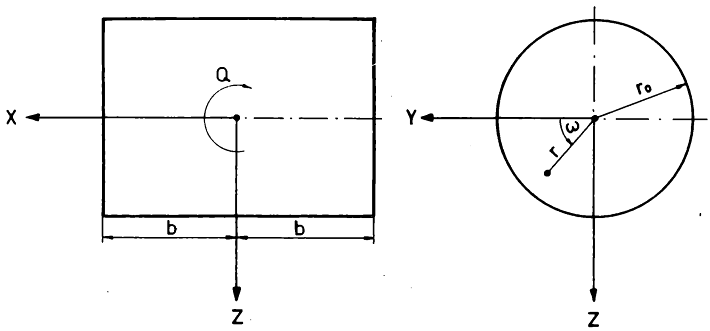

FIG. 1. Geometry of cylinder tank.

The problem is thus reduced to the determination of this last coefficient. It will be necessary first to determine the velocity potential ϕ_q.

Dropping for convenience the subscript q, the potential will have to obey the following boundary conditions:

on the cylindrical surface where $l = 0$, $m = \cos\omega$, and $n = \sin\omega$;

$$\frac{\partial\phi}{\partial r} = -x\sin\omega \,, \qquad r = r_0 \,,$$

on the flat ends where $l = \pm 1$, $m = 0$, and $n = 0$;

$$\frac{\partial\phi}{\partial x} = r\sin\omega \,, \qquad x = \pm b \,.$$

Those conditions suggest that we should set

$$\phi = \sin \omega[-rx + \psi(r, x)] . \qquad (20)$$

Since the potential satisfies the Laplace equation it is easily found that ψ has therefore to satisfy

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\psi = 0 \qquad (21)$$

with boundary conditions

$$\partial\psi/\partial r = 0 , \quad \text{for} \quad r = r_0 \qquad (22)$$

$$\partial\psi/\partial x = 2r , \text{ for } \quad x = \pm b . \qquad (23)$$

The first boundary condition was intentionally rendered homogeneous in order to obtain an eigenvalue problem by separation of the variables x and r. Denoting by k^2 the constant of separation and noting the type of symmetry required in x, there comes

$$\psi = \sinh kxY(r)$$

as typical solution with Y obeying the ordinary differential equation

$$Y'' + \frac{1}{r}Y' + \left(k^2 - \frac{1}{r^2}\right)Y = 0 \qquad (24)$$

and the boundary condition

$$Y'(r_0) = 0 . \qquad (25)$$

The appropriate solution of equation (24) is the Bessel function

$$Y = J_1(kr)$$

and the eigenvalues $\mu_n = k_n r_0$ are given by the roots of

$$J'(\mu_n) = 0 , \qquad (26)$$

$$\mu_1 = 1{\cdot}84 , \quad \mu_2 = 5{\cdot}33 , \quad \mu_3 = 8{\cdot}53 , \quad \mu_4 = 11{\cdot}1 \quad \dots .$$

The general solution is

$$\psi = \sum_1^\infty A_n \sinh(\mu_n x/r_0) J_1(\mu_n r/r_0) . \qquad (27)$$

The coefficients A_n are to be determined from the remaining boundary condition equation (23) that now appears in the form

$$\frac{1}{r_0}\sum_1^\infty \mu_n A_n \cosh(\mu_n \beta) J_1(\mu_n r/r_0) = 2r \qquad (28)$$

where the ratio of the tank length to diameter was denoted

$$\beta = b/r_0 .$$

From equation (28) it is possible to establish the values of the A_n 'à la Fourier' by using the Lommel integrals of the theory of Bessel functions. For the sake of completeness we give a derivation of those integrals below.

From the differential equations

$$\frac{d^2 Y_n}{d\gamma^2} + \frac{1}{\gamma}\frac{dY_n}{d\gamma} + \left(\mu_n^2 - \frac{1}{\gamma^2}\right) Y_n = 0 \tag{29}$$

satisfied by the functions $Y_n = J_1(\mu_n \gamma)$, we deduce

$$\gamma Y_n Y_m (\mu_n^2 - \mu_m^2) = \frac{d}{d\gamma}\left[\gamma\left(Y_n \frac{dY_m}{d\gamma} - Y_m \frac{dY_n}{d\gamma}\right)\right]$$

and

$$(\mu_n^2 - \mu_m^2)\int_0^1 Y_n Y_m \gamma \, d\gamma = \mu_m J_1(\mu_n) J'(\mu_m) - \mu_n J_1(\mu_m) J'(\mu_n) \tag{30}$$

or in our case, by virtue of equations (26)

$$\int_0^1 J_1(\mu_n \gamma) J_1(\mu_m \gamma)\gamma \, d\gamma = 0 \qquad \text{if } n \neq m . \tag{31}$$

For the case $m = n$, equation (30) is first divided by $(\mu_n^2 - \mu_m^2)$ and the true value in the limit $\mu_n = \mu_m$ is found by the rule of l'Hospital. There comes

$$\int_0^1 Y_n^2 \gamma \, d\gamma = \tfrac{1}{2}(J_1'^2(\mu_n) - J_1(\mu_n) J_1''(\mu_n)) - \frac{1}{2\mu_n} J_1(\mu_n) J_1'(\mu_n)$$

and, since

$$J_1''(\mu_n) = \frac{1}{\mu_n^2}\left(\frac{d^2 Y_n}{d\gamma^2}\right)_{\gamma=1}$$

we can use the differential equation (29) to eliminate the second derivative, whereby, in general

$$\int_0^1 J_1^2(\mu_n \gamma)\gamma \, d\gamma = \frac{1}{2}\left[J_1'^2(\mu_n) + \left(1 - \frac{1}{\mu_n^2}\right) J_1^2(\mu_n)\right]$$

or, again, in our case where equation (26) applies

$$\int_0^1 J_1^2(\mu_n \gamma)\gamma \, d\gamma = \frac{1}{2}\left(1 - \frac{1}{\mu_n^2}\right) J_1^2(\mu_n) . \tag{32}$$

Equations (31) and (32) are now applied to secure the values of the coefficients in expansion (28). To this end both sides are multiplied by $rJ_1(\mu_m r/r_0)\,dr$ and integrated between $r = 0$ and $r = r_0$. Changing variables by $\gamma = r/r_0$ and using equations (31) and (32) there follows

$$\frac{\mu_m^2 - 1}{\mu_m} A_m \cosh(\mu_m \beta) = 4r_0^2 \int_0^1 \gamma^2 J_1(\mu_m \gamma) \, d\gamma . \tag{33}$$

To evaluate the last integral we recast equation (29) in the form

$$-\mu_n^2 \gamma^2 Y_n = \gamma^2 \frac{d^2 Y_n}{d\gamma^2} + \gamma \frac{dY_n}{d\gamma} - Y_n = \frac{d}{d\gamma}\left(\gamma^2 \frac{dY_n}{d\gamma} - \gamma Y_n\right)$$

whence, by integration

$$-\mu_n^2\int_0^1 \gamma^2 Y_n\,\mathrm{d}\gamma = \mu_n J_1^{10}(\mu_n) - J_1(\mu_n)$$

or again in view of equation (26)

$$\int_0^1 \gamma^2 J_1(\mu_n\gamma)\,\mathrm{d}\gamma = \frac{1}{\mu_n^2}\,J_1(\mu_n)\,. \tag{34}$$

After substitution in equation (33) we finally obtain

$$A_m = \frac{4r_0^2}{\mu_m(\mu_m^2-1)J_1(\mu_m)\cosh(\mu_m\beta)}\,. \tag{35}$$

The velocity potential ϕ_q is now determined by insertion of these values in equation (27) and (27) in equation (20).

THE EQUIVALENT MOMENT OF INERTIA A_{qq} FOR THE CYLINDRICAL CAVITY

According to equations (8) and (11)

$$A_{qq} = \rho\int_S \phi(zl - xn)\,\mathrm{d}S = (A_{qq})_1 + (A_{qq})_2\,.$$

The first part is the contribution of the boundary $r = r_0$ where

$$(zl - xn)\,\mathrm{d}S = -r_0 x\sin\omega\,\mathrm{d}\omega\,\mathrm{d}x$$

and

$$\phi = \sin\omega\left[-r_0x + 4r_0^2\sum_1^\infty\frac{\sinh(\mu_n x/r_0)}{\mu_n(\mu_n^2-1)\cosh(\mu_n\beta)}\right].$$

Both the integrals in ω between 0 and 2π and in x between $-b$ and $+b$ are elementary. There is found

$$(A_{qq})_1 = \rho\pi r_0^5\left[\tfrac{2}{3}\beta^3 - 8\sum_1^\infty\frac{\beta\mu_n - \tanh(\beta\mu_n)}{\mu_n^3(\mu_n^2-1)}\right].$$

The second part is the contribution of the boundaries $x = \pm b$. Symmetry allows simply to double the contribution on the side $x = b$, where $(zl - xn)\,\mathrm{d}S = r^2\sin\omega\,\mathrm{d}\omega\,\mathrm{d}r$ and

$$\phi = \sin\omega\left[-rb + 4r_0^2\sum_1^\infty\frac{\tanh(\mu_n\beta)}{\mu_n(\mu_n^2-1)}\,\frac{J_1(\mu_n r/r_0)}{J_1(\mu_n)}\right]$$

The integral over ω is trivial, the one over r, when transformed over γ, involves only the result equation (34). There is found

$$(A_{qq})_2 = \rho\pi r_0^5\left[-\tfrac{1}{2}\beta + 8\sum_1^\infty\frac{\tanh(\beta\mu_n)}{\mu_n^3(\mu_n^2-1)}\right]$$

Before adding the two contributions, advantage can be taken of the formula

$$8\sum_1^\infty\frac{1}{\mu_n^2(\mu_n^2-1)} = 1\,. \tag{36}$$

A proof of this is obtained by multiplying both sides of equation (28) by $r^2\,dr$ and integrating between 0 and r_0. On the left-hand side the integrals reduce again to equations (34); and equation (36) follows once the A_n are replaced by their values (35).

The final result can then be placed in the form

$$A_{qq} = \pi\rho r_0^5(\tfrac{2}{3}\beta^3 - \tfrac{3}{2}\beta + H) \tag{37}$$

where

$$H(\beta) = 16 \sum_1^\infty \frac{\tanh(\beta\mu_n)}{\mu_n^3(\mu_n^2 - 1)}. \tag{38}$$

A short table of the values of $H(\beta)$ is appended.

TABLE

β	$H(\beta)$
0	0
0·2	0·383
0·4	0·679
0·6	0·868
0·8	0·973
0·866	0·995
0·9	1·005
1·0	1·028
1·4	1·068
2·0	1·079
3·0	1·081
∞	1·082

In comparing the results with the corresponding ones for a solid body, we may note that in the latter case the total moment of inertia I_{qq} is the sum of two contributions

$$I_{qq} = \rho \int_v (z^2 + x^2)\,d\tau = \pi\rho r_0^5 \tfrac{2}{3}\beta^3 + \pi\rho r_0^5 \tfrac{1}{2}\beta .$$

The first is the moment of inertia of a rod, the mass of each normal cross section being concentrated on the x-axis. The second takes into account the rotation of each cross section about an axis parallel to oy in its own plane.

For elongated tanks (β large) one would expect the fluid to behave approximately as a solid rod; for very flat tanks as a solid plate. In between, the ratio of the moment of inertia of the liquid to that of the solid would go through a minimum when the length is comparable to the diameter. In that case the tank shape approaches that of a sphere for which obviously the entrainment of the liquid by the walls vanishes. The ratio in question is plotted as curve 1 of Fig. 2 and furnishes a confirmation of this reasoning. The sharp minimum seems to occur for the value $\beta = \sqrt{(3)}/2$ where the two parts of I_{qq} are equal. The fractional value of each of those parts are drawn for comparison as curves 2 and 3.

To estimate A_{qq} practically in the whole range of β values one can proceed as follows. For $\beta > 1$ take it to be the moment of inertia of a rod affected by a reduction factor C_r:

$$A_{qq} = \tfrac{2}{3}\pi\rho r_0^2 b^3 C_r \qquad C_r\left(\frac{1}{\beta}\right) = 1 - \frac{9}{4/\beta^2} + \tfrac{3}{2}H .$$

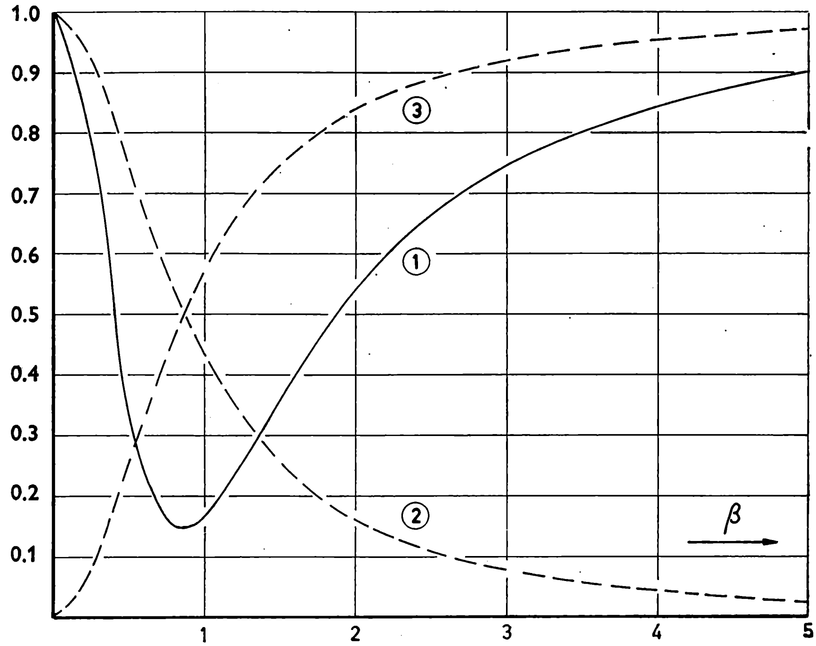

FIG. 2. Curve 1—ratio A_{qq}/I_{qq} of moment of inertia of liquid to that of solid of same specific mass. Curves 2 and 3—ratios to I_{qq} of moment of inertia of circular plate and homogeneous rod of same total mass.

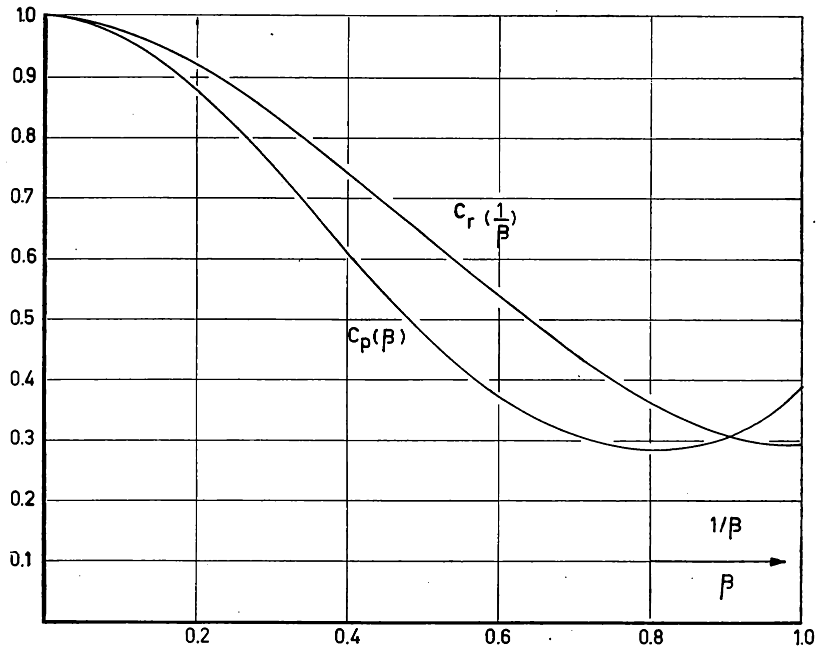

FIG. 3. Correction factors to calculate A_{qq} as an equivalent rod or an equivalent plate.

For $\beta < 1$ take it to be the moment of inertia of a plate affected by a reduction factor C_p:

$$A_{qq} = \tfrac{1}{2}\pi\rho r_0^4 b C_p \qquad C_p(\beta) = \tfrac{4}{3}\beta^2 - 3 + \frac{2}{\beta}H$$

The two correction factors are displayed in Fig. 3.

REFERENCE

[1] H. LAMB, *Hydrodynamics*, Chap. 6, Cambridge University Press (1926).

Résumé—Le mouvement en corps solide d'une cavité détermine un écoulement irrotationel du liquide au sein de celle-ci. Du point de vue de la dynamique externe le fluide se comporte comme un solide équivalent. L'objet de cette étude est d'examiner les propriétés générales du tenseur d'inertie équivalent et de le calculer explicitement dans le cas d'un réservoir à combustible cylindrique.

Zusammenfassung—Die Bewegung eines Hohlraums als ein starrer Körper bestimmt eine wirbelfreie Strömung der Flüssigkeit die den Hohlraum ausfüllt. Vom Standpunkt der Aussendynamik verhält sich die Flüssigkeit wie ein gleichwertiger Körper. Der Zweck dieses Berichts ist es, die allgemeinen Eigenschaften des gleichwertigen Trägheitstensors zu untersuchen und diesen explizit für den Fall eines zylindrischen Brennstofftanks zu berechnen.

Sommario—Il movimento in corpo solido di una cavità determina un movimento irrotazionale del liquido che la riempie. Dal punto di vista della dinamica esterna, il combustibile si comporta come un corpo solido equivalente. Lo scopo di questo sutdio è di esaminare le proprietà generali del tensore d'inerzia equivalente e di calcolarlo, esplicitamente nel caso di un serbatoio di combustibile cilindrico.

Аннотация — Движение жесткого тела полости определяет невихревое движение жидкости наполняжщей его внутренность. С точки зрения внешней динамики жидкость ведёт себя как соответствующее твёрдое тело.

Целью статьи является изучать общие свойства эквивалента тензора инерции и вычислять его в частности в случае цилиндрического бака для горучего.