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REPORT OA-10

"A MAXIMUM-MINIMUM PRINCIPLE FOR BANG-BANG SYSTEMS"

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A MAXIMUM-MINIMUM PRINCIPLE FOR BANG-BANG SYSTEMS

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Summary.

Consider the problem of a rocket in vertical flight in a uniform gravitational field, the aerodynamic drag being neglected. The thrust is provided by several chemical rocket engines working in parallel that can be separated and dropped according to some optimal sequence in order to provide a maximum payload for a given total thrust at departure and a prescribed velocity gain.

The mathematical formulation provides the possibility of a continuous reduction in thrust, that is for the limiting case of an infinity of infinitesimal propulsion units. In this case it is known that, if the velocity performance is set high enough, the optimal sequence consists of a constant thrust arc during which no engines are dropped, followed by a continuous reduction in thrust that keeps the acceleration constant. There is however another type of extremal representing the separation from a finite amount of thrust. The real technical problem involves only this type of extremal and the constant thrust extremal. The optimization problem is then of the bang-bang type, the continuous acceleration type of extremal representing a "chattering" of the control.

It is remarkable that optimal bang-bang solutions, each corresponding to a prescribed number of engine separations, are found by applying a minimum principle for the Hamiltonian (instead of the usual maximum principle) during a portion of the trajectory. More precisely the optimal bang-bang trajectories imply the use of the maximum principle up to the first reversal in the sign of the switching function, then of the minimum principle with a finite number of sign reversals, then of the maximum principle again to the end. Eventually the first or last part (or both) are missing.

The optimality of such bang-bang solutions is established by the analysis of the second variation.

I. THE OPTIMUM STAGING OF ROCKETS IN PARALLEL

I.I. Basic differential equations.

Consider a cluster of chemical rockets, whose instantaneous mass can be conceptually subdivided as follows :

$$M = M_u + \sigma M_1 + M_p + \frac{F}{Kg} \tag{I.I}$$

M_u is the payload mass or useful mass

σM_1 is the structural mass considered to represent a given fraction of the total mass M_1 at departure.

These two are fixed quantities, the other ones are variables :

M_p the instantaneous mass of propellants,

$M_e = \frac{F}{Kg}$ the mass of propulsion equipment, based on the assumption that the thrust F it delivers is proportional (factor K) to its weight gM_e .

If c denotes a fixed effective exhaust velocity of burnt gasses, the thrust is also given by

$$F = - c \frac{dM_p}{dt} \tag{I.2}$$

By elimination of the mass of propellant between (I.I) and (I.2) follows one of the basic differential equations :

$$\frac{dM}{dt} = - \frac{F}{c} + \frac{1}{Kg} \frac{dF}{dt} \tag{I.3}$$

It assumes that the thrust can be continuously reduced by separation of infinitesimal propulsion units. As will appear later, this idealized formulation does not only furnish a method for assessing the optimal performance ceiling that can be reached by the principle of parallel staging of propulsion equipment but also provides a scientific approach to the real problem of discrete staging.
A control variable α is now introduced to govern the programming of engine separation by expressing that the thrust can only decrease :

$$\frac{dF}{dt} = - \alpha^2 \tag{I.4}$$

Finally a performance equation is needed to pose a maningful problem of optimal staging. The simplest one that offers a complete analytical solution is the equation of motion for vertical flight in a uniform gravity field, the aerodynamic drag being neglected :

$$M \frac{dV}{dt} = F - Mg \tag{I.5}$$

The basic differential system consists of the equations (I.3), (I.4) and (I.5).

I.2. Dimensionless form of the basic system.

Introduce the dimensionless variables

$\omega = V/c$ for the velocity

$\tau = tg/c$ for the time

$\phi = \ln \frac{M_1}{M}$ for the instantaneous mass

$\beta = \frac{F}{gM}$ an instantaneous acceleration factor.

It is important to note that the acceleration factor has from equation (I.I) an upper limit K when the propellants are used up ($M_p = o$) and M_u and σ approach zero :

$$o < \beta < K \tag{I.6}$$

In the new variables the basic differential system takes the form

$$\frac{d\phi}{d\tau} = \beta + \gamma^2$$

$$\frac{d\beta}{d\tau} = \beta^2 + \gamma^2(\beta - K)$$

$$\frac{d\omega}{d\tau} = \beta - 1$$

where the control variable α has been changed to γ by

$$\gamma^2 = \frac{c}{K\, g^2\, M_1}\, e^{\phi}\, \alpha^2$$

A further simplification can be introduced, provided no constraints be introduced on the duration of the flight. The time τ is then an ignorable variable and ϕ , which is strictly monotonically increasing, can serve as independant variable. Hence, dividing the last two equations of the basic system by the first, and changing once more the control variable to

$$v = \frac{\beta}{\beta + \gamma^2} \qquad v \in (0,1)$$

we obtain

$$\frac{d\beta}{d\phi} = \beta - K + K\, v \tag{I.7}$$

$$\frac{d\omega}{d\phi} = \left(1 - \frac{1}{\beta}\right) v \tag{I.8}$$

I.3. The optimization problem.

We set up the following optimization problem : the initial velocity being zero and a prescribed terminal velocity having to be reached at burnout $(M_p = o)$, maximize the payload gM_u for a given thrust F_1 available at departure. This is equivalent to minimize the functional

$$J = - \frac{g\, M_u}{F_1}$$

or, taking M_u from equation (I.I) at burnout

$$M_u = M_2 - \sigma\, M_1 - \frac{F_2}{Kg}$$

and substituting

$$J = \frac{1}{\beta_1} \left(\sigma + \left(\frac{\beta_2}{K} - 1\right) e^{-\phi_2} \right) \quad \text{minimum} \tag{I.9}$$

This is an example of a functional depending on the initial and final values of the independant and state variables.

The Hamiltonian of the problem is, from equations (I.7) and (I.8),

$$H = \lambda_\beta \ (\beta - K) + v\, S \tag{I.I0}$$

where the switching function S , which decides on the choice of the control variable v , is

$$S = K \ \lambda_\beta + \left(1 - \frac{1}{\beta} \right) \lambda_\omega \tag{I.II}$$

The adjoint differential system is

$$\frac{d\lambda_\beta}{d\phi} = -\frac{\partial H}{\partial \beta} = -\lambda_\beta - \frac{v}{\beta^2}\lambda_\omega$$

$$\frac{d\lambda_\omega}{d\phi} = -\frac{\partial H}{\partial \omega} = 0$$

and has an immediate first integral

$$\lambda_\omega = \text{constant} \tag{I.I2}$$

A second first integral is provided by the equation

$$\frac{dH}{d\phi} = \frac{\partial H}{\partial \phi} = 0 \qquad \text{whence}$$

$$H = \text{constant}$$

This avoids the necessity of integrating the first equation of the adjoint system. In fact, if λ_β is eliminated between equations (I.I0) and (I.II), the switching function can be expressed entirely in terms of the state variable β, the control variable v and the constants H and λ_ω

$$S = \frac{K\,H + (\beta-K)(1-\beta^{-1})\,\lambda_\omega}{\beta - K + K\,v} \tag{I.I3}$$

The discussion of the maximum principle will be further simplified if we introduce the new constant ε defined by

$$\frac{H\,K}{\lambda_\omega} = 1 + K - 2\,\varepsilon\,\sqrt{K} \tag{I.I4}$$

and put the switching function in the form

$$\frac{S}{\lambda_\omega} = \frac{(\beta-\theta_1)(\beta-\theta_2)}{\beta(\beta-K+Kv)} \tag{I.I5}$$

where

$$\theta_1 = \sqrt{K}\,(\varepsilon - \sqrt{\varepsilon^2 - 1})$$
$$\theta_2 = \sqrt{K}\,(\varepsilon + \sqrt{\varepsilon^2 - 1}) \tag{I.I6}$$

In addition to the prescribed end values

$$\phi_1 = o \qquad \omega_1 = o \qquad \omega_2 = \bar{\omega}_2$$

we shall need the transversality conditions based on the fact that ϕ_2 , β_1 and β_2 are not prescribed. They are

$$H = H_2 = \frac{\partial J}{\partial \phi_2} = \frac{1}{\beta_1} \left(1 - \frac{\beta_2}{K} \right) e^{-\phi_2} \tag{I.17}$$

$$(\lambda_\beta)_1 = \frac{\partial J}{\partial \beta_1} = - \frac{1}{\beta_1^2} \left[\sigma + \left(\frac{\beta_2}{K} - 1\right) e^{-\phi_2} \right] \tag{I.18}$$

$$(\lambda_\beta)_2 = - \frac{\partial J}{\partial \beta_2} = - \frac{1}{K\beta_1} e^{-\phi_2} \tag{I.19}$$

A first important conclusion stemming from (I.17) and (I.6) is that the constant of the Hamiltonian is positive

$$H > o \tag{I.20}$$

A second is obtained from equation (I.10) at the end of the trajectory

$$H_2 = (\lambda_\beta)_2 \, (\beta_2 - K) + v_2 \, S_2$$

when we substitute (I.17) and (I.19); there comes

$$v_2 \, S_2 = o \tag{I.21}$$

Hence at the end of the trajectory we must have that either the control variable or the switching function vanish.

I.4. Nature of the extremals.

The problem is regular in the sense that the manoeuvrability domain (hodograph space) is convex. It is the straight line segment of figure I.

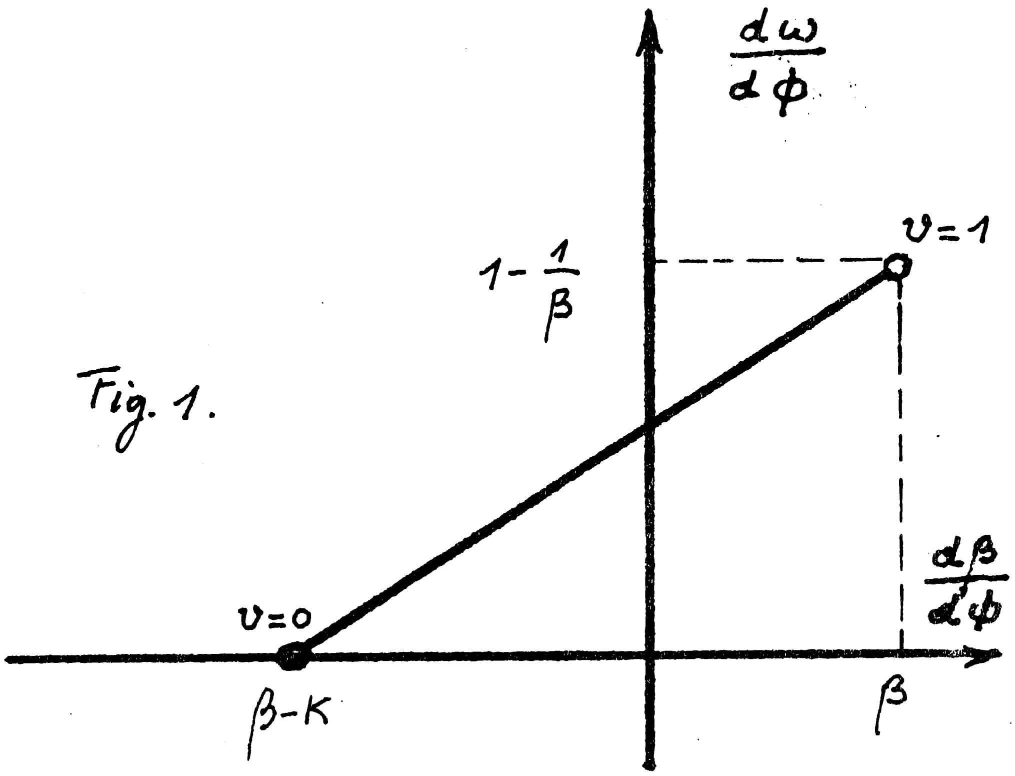

Fig. 1.

The extremals will be characterized by $v = 1$, $v = o$ or possibly some intermediate value corresponding to the persistence of the switching function to vanish.

a. The constant thrust extremal.

It corresponds to $v = 1$ and, in accordance with the maximum principle, to positive values of the switching function. The old control variables γ and α are zero so that, from (I.4), F is indeed constant. There is no engine separation and ϕ increases only through consumption of propellant. From (I.7)

$$\frac{d\beta}{d\phi} = \beta > o$$

and the acceleration is increasing along this type of arc.

b. The constant time extremal.

It corresponds to $v = o$ and negative values of the switching function. Since the old control variable α tends to infinity, we find, by returning to the original differential equations, that

$$dt = o \qquad dM_p = o \qquad dV = o \qquad dF = KgdM$$

No propellant is used, no velocity gained, the reduction in mass corresponds solely to the separation of propulsion equipment. However, as the time also stands still, we can consider that a finite portion of such an extremal arc corresponds to the instantaneous separation of a finite thrust unit, which is technically meaningful.

Naturally the acceleration decreases, as further indicated by

$$\frac{d\beta}{d\phi} = \beta - K < o$$

c. The constant acceleration extremal.

Differentiating (I.II) and substituting the derivatives of the state variable β and the multipliers, we find in general that

$$\frac{dS}{d\phi} = - S + \lambda_\omega (1 - \frac{K}{\beta^2}) \tag{I.22}$$

Hence if S remains zero for some finite interval of ϕ, we must have either $\lambda_\omega = o$ or $\beta = \sqrt{K}$. The first possibility is ruled out by the consequence that, from (I.II), λ_β should also have to be zero and both multipliers would then vanish along the whole trajectory, together with the Hamiltonian. The second possibility, the constant acceleration one

$$\beta = \sqrt{K} \tag{I.23}$$

gives, when substituted into (I.7), the control value

$$v = 1 - \frac{1}{\sqrt{K}} \tag{I.24}$$

which lies in the possible range. Furthermore, there follows from (I.II) and $S = o$, that

$$\lambda_\beta = - \frac{1}{K} (1 - \frac{1}{\sqrt{K}}) \lambda_\omega \tag{I.25}$$

and, from (I.IO), that

$$H = (1 - \frac{1}{\sqrt{K}})^2 \lambda_\omega \tag{I.26}$$

Equation (I.I4) then gives

$$\varepsilon = 1 \tag{I.27}$$

so that the roots θ_1 and θ_2 of (I.16) are confluent

$$\theta_1 = \theta_2 = \sqrt{K} \qquad (I.28)$$

The velocity gain is given by

$$\frac{d\omega}{d\phi} = \left(1 - \frac{1}{\sqrt{K}} \right)^2 \qquad (I.29)$$

Technically speaking, this arc is a limiting case. It implies a continuous separation of infinitesimal thrust units as the rocket is gaining velocity so that the acceleration can be kept constant, despite the reduction in mass due to propellant consumption.

I.5. Synthesis of optimal trajectories.

The general composition of optimal trajectories for any type of functional and boundary conditions is easily obtained from a (β, S) graph based on equation (I.I5). This graph depends only on the sign of the constant λ_ω and on the nature and position of the roots (θ_1, θ_2). Here we shall restrict ourselves to the particular problem at hand.

We can first observe that the end condition (I.2I) really reduces to

$$S_2 = o \qquad \text{(I.30)}$$

Indeed, since a velocity gain is imposed, the trajectory must contain at least one arc of the constant thrust or constant acceleration type. If the trajectory ends on a constant thrust arc $(v = 1)$, condition (I.30) is needed to implement (I.2I). If it ends on a constant acceleration arc, (I.30) is actually satisfied. In both cases we can switch to $v = o$ and, provided the switching function becomes negative, add a final constant time arc. However this terminal arc does not change the terminal velocity nor the value of the functional; in fact, as shown by equation (I.7) for $v = o$

$$(\beta/K - 1)\, e^{-\phi} \qquad \text{remains constant.}$$

Hence the only thing that can be achieved by such an extension of the trajectory is a separation of the payload and structure from a part or from the total of the remaining propulsion units. This new solution is not essentially different and there is no loss in generality in ending the trajectory on the constant thrust or constant acceleration arc.

Then from (I.30) follows

$$\lambda_\omega = - \frac{\beta_2}{\beta_2 - 1} K (\lambda_\beta)_2$$

or, taking (I.I9) into account

$$\lambda_\omega = \frac{\beta_2}{\beta_2 - 1} \; \frac{e^{-\phi_2}}{\beta_1} \qquad \text{(I.3I)}$$

It is also obvious that we must have

$$\beta_2 > o \tag{I.32}$$

For, if this were not true, the final arc would be a constant thrust one with continuous reduction in velocity and continuous increase in the value of the functional; the required terminal velocity would already have been reached earlier with a smaller value of the functional. From (I.3I) and (I.32) it follows that

$$\lambda_\omega > o \tag{I.33}$$

The nature of the initial arc can be fixed by considering a physical limitation : the optimal payload for given F_1 must still be positive or, stated otherwise, the optimal value of the functional must be negative. Comparing (I.9) and (I.I8) this holds only if

$$(\lambda_\beta)_1 > o \tag{I.34}$$

This condition in turn is compatible with (I.20) and $\beta_1 < K$ only when $v_1\ \beta_1 > o$. Hence, from the maximum principle,

$$v_1 = 1 \tag{I.35}$$

and the initial arc is of constant thrust type. This conclusion also permits to write

$$H = (\lambda_\beta)_1\ (\beta_1 - K) + \beta_1 = (\lambda_\beta)_1\ (\beta_1 - K) + (1 - \frac{1}{\beta_1})\ \lambda_\omega$$

When in this relation we substitute H from (I.I7), $(\lambda_\beta)_1$ from (I.I8) and λ_ω from (I.3I), we find

$$\sigma \frac{\beta_1}{\beta_1 - 1} = e^{-\phi_2} \frac{\beta_2}{\beta_2 - 1} \tag{I.36}$$

This relation completes with $\omega_1 = o$, $\omega_2 = \overline{\omega}_2$, $\phi_1 = o$, the set of boundary conditions needed for the two basic differential equations (I.7) and (I.8). It also shows in conjunction with (I.32) that

$$\beta_1 > 1 \tag{I.37}$$

This inequality was otherwise necessary to obtain a lift-of capability.
The correct (β,S) graph can now be constructed with this information. The sign of λ_ω is known from (I.33).
Further we have

$$1 < \varepsilon < \frac{K+1}{2\sqrt{K}} \tag{I.38}$$

The lower bound is justified by the fact that the trajectory ends with $S_2 = o$ and from (I.15) this can only occur for $\beta = \theta_1$ or θ_2. Hence the roots cannot be complex conjugate nor negative. The upper bound is justified from the definition (I.14) of ε and the fact that, according to (I.20) the Hamiltonian is, like λ_ω, positive.
In the range (I.38) of ε values we have

$$1 < \theta_1 < \sqrt{K} < \theta_2 < K \tag{I.39}$$

and the (β,S) graph based on (I.15) is as depicted on figure 2.

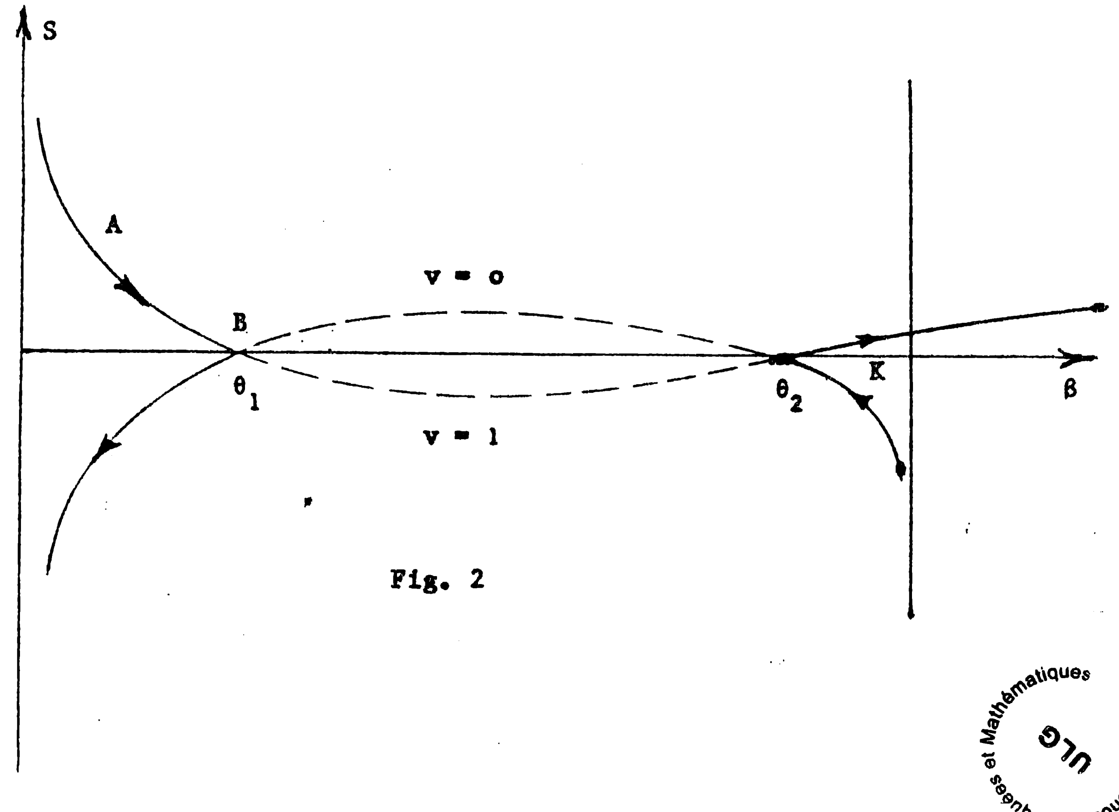

Fig. 2

The part of the branches $v = o$ and $v = 1$ which violate the maximum principle are drawn interrupted. From the sense of description of the branches it is clear that an optimal trajectory can only consist of a single constant thrust arc, like the one represented by the segment AB . The technical variant consisting in adding a constant time arc is possible.
For such a solution it is found that

$$\beta_2 = \beta_1 e^{\phi_2} = \theta_1 < \sqrt{K} \qquad \beta_1 = 1 + \sigma(\beta_2 - 1)$$

$$\omega_2 = \phi_2 + \frac{1}{\beta_2} - \frac{1}{\beta_1}$$

$$J = \frac{\sigma}{\beta_1} + \frac{1}{K} + \frac{1}{\beta_2}$$

The functional is negative (payload positive) if

$$\sigma < \frac{K - \beta_2}{\beta_2(\beta_2 - 1) + K}$$

This solution is optimal until β_2 reaches $\sqrt{K}$ and

$$\omega_2 = \ln \frac{\sqrt{K}}{1 + \sigma(\sqrt{K} - 1)} + \frac{1}{\sqrt{K}} - \frac{1}{1 + \sigma\sqrt{K}}$$

For higher values of ω_2 we must consider the limiting case $\varepsilon = 1$, for which $\theta_1 = \theta_2 = \sqrt{K}$. The corresponding (β, S) graph is shown on figure 3, where it can be seen that the optimal trajectory consists of an initial constant thrust arc until the tangency point where $S = o$ is reached. By switching then to $v = 1 - \frac{1}{\sqrt{K}}$ we stay at the tangency point which represents a constant acceleration arc, until the required velocity is reached.
Again a constant time arc can be added afterwards to separate the payload. In this case we find

$$\frac{1}{\beta_1} = 1 - \sigma\left(1 - \frac{1}{\sqrt{K}}\right) e^{\phi_2} \qquad \beta_2 = \sqrt{K}$$

$$\omega_2 = \frac{1}{\sqrt{K}}\left(2 - \frac{1}{\sqrt{K}}\right) \ln\left(\frac{\sqrt{K}}{\beta_1}\right) + \frac{1}{\sqrt{K}} - \frac{1}{\beta_1} + \phi_2\left(1 - \frac{1}{\sqrt{K}}\right)^2$$

The solution becomes meaningless when the velocity required is so large that the functional J becomes positive.

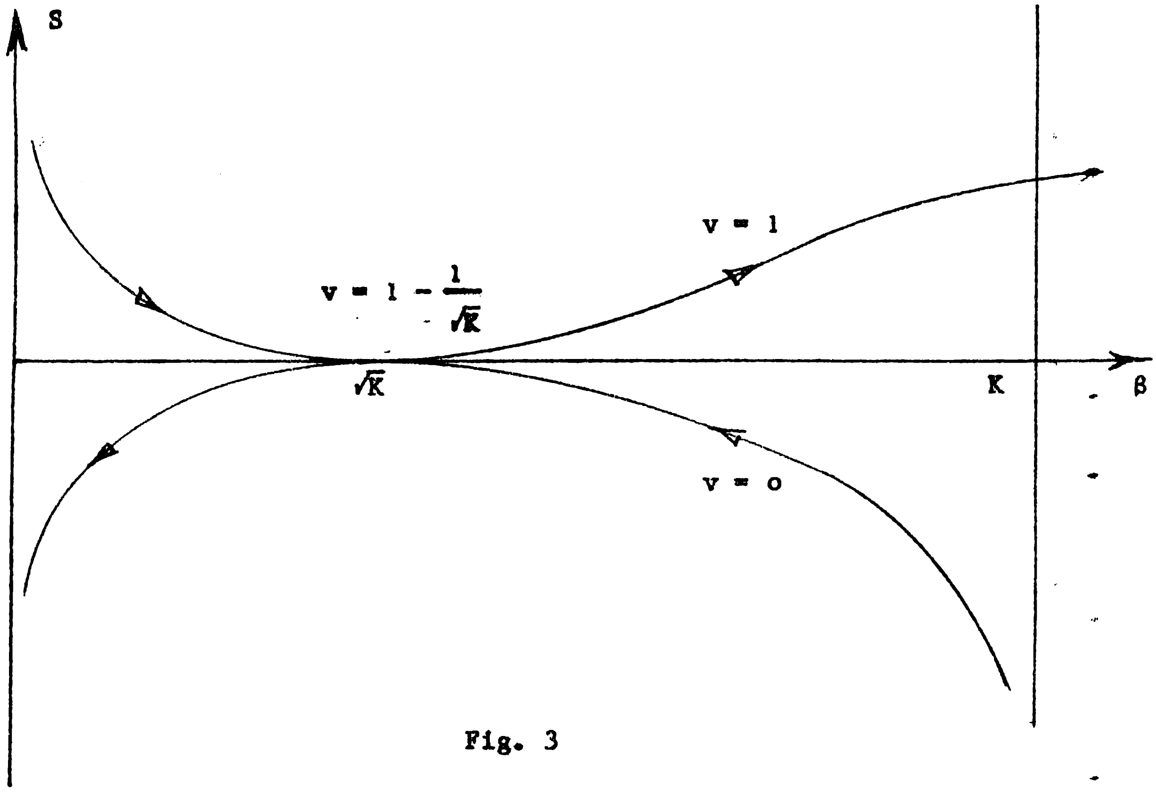

Fig. 3

Figure 4 gives an example of optimal values as functions of the terminal velocity for $\sigma = 0.I$ and $K = 25$.

2. THE MAXIMUM-MINIMUM PRINCIPLE

2.I. Chattering-free solutions of the staging problem.

Any optimal solution involving chattering is of theoretical interest only. In practice it can perhaps be approximated closely if the physical implication of chattering is a high frequency commutation of an electronic relay switch.
In the present case, a too large number of small propulsion units would bring about weight increases and loss of reliability which could only be introduced in the formulation at the price of considerable mathematical complications.
On the other hand, if the number of units into which the propulsion equipment is subdivided has been specified beforehand, physical intuition suggests that an optimal programming for the sequence of separation and the size of separating units must still exist. Such an optimal solution would consist only on

constant thrust arcs separated by a finite number of constant time arcs. It has still to satisfy the requirements for the vanishing of the first variation but not necessarily the stronger requirement of the maximum principle. This means that on the (β, S) graph of figure 2, the parts of the $v = o$ and $v = 1$ branches which violate the maximum principle can be used. We then obtain a solution based on the first variation which is depicted on figure 5. The first

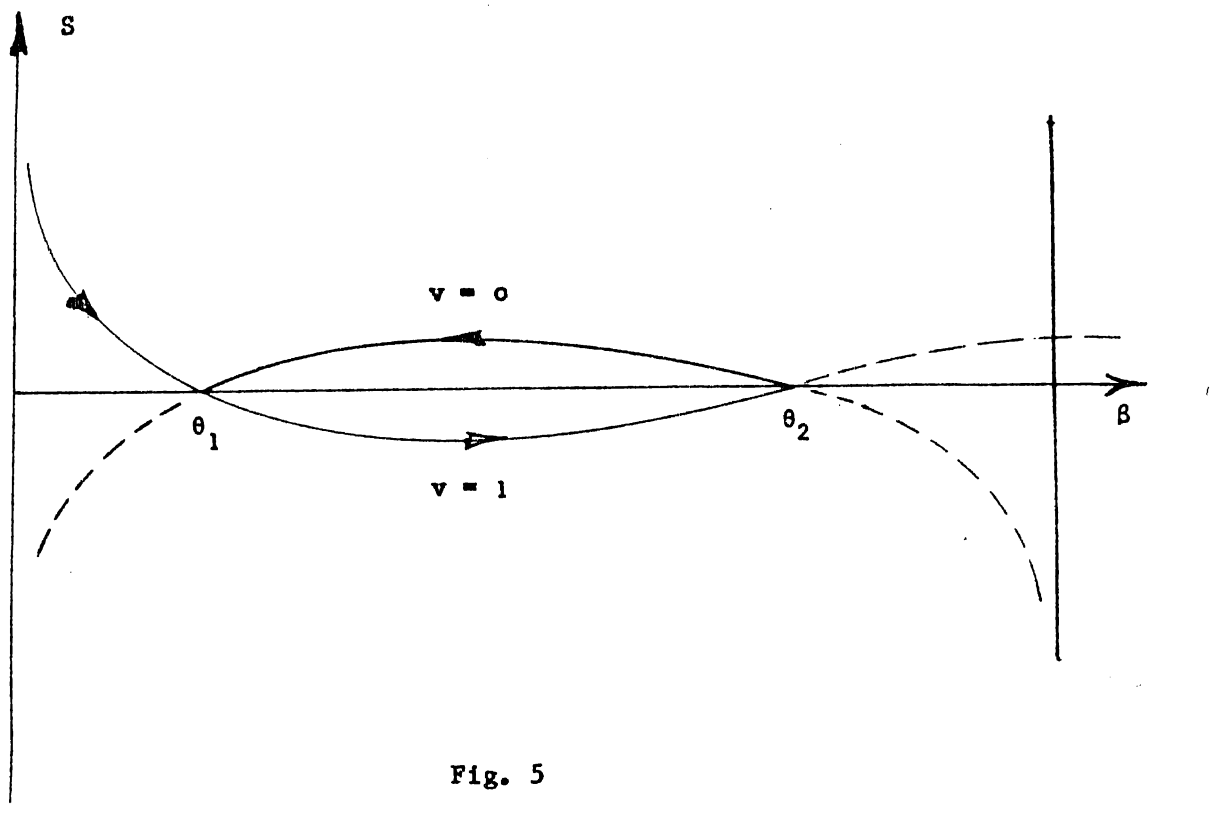

Fig. 5

constant thrust arc is driven across the first switching point in the negative S region until β reaches θ_2. There we can return to θ_1 by a constant time arc, go back to θ_2 via a new constant thrust arc and continue this game until the prescribed number of constant time arcs (separations) is reached. The choice of ε, or distance between θ_1 and θ_2, regulates the terminal velocity. The nature of this solution is described on the (ϕ, β) graph of figure 6.

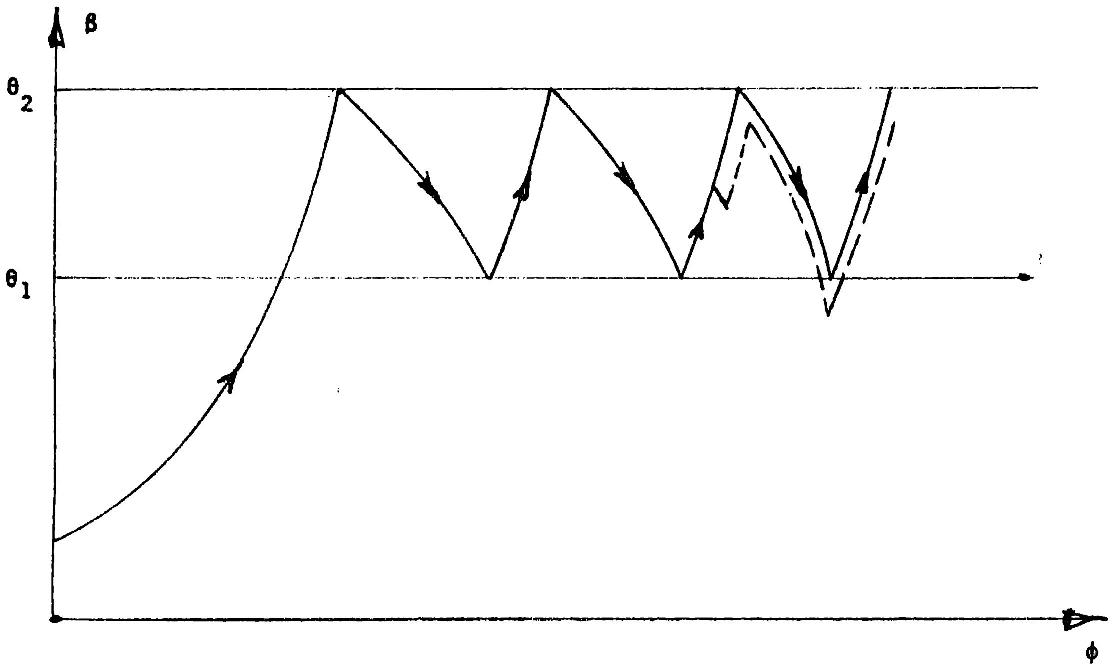

Fig. 6

It will be observed that, up to the first switching point, the choice of controls follows the maximum principle, after that a minimum principle (for the Hamiltonian). The reason for this is also clear when one remembers that the maximum principle follows from the strong variation criterion.
A strong variation applied before the first switching point actually increases the value of the functional. Applied after, as shown in dotted lines on figure 6, it reduces the value of the functional and so theoretically provides a better trajectory. However the number of constant time arcs has been thereby increased by one unit and does no more correspond to the prescribed number.
It is physically obvious that the use of this maximum-minimum principle provides the optimal solution in the case of a prescribed number of constant time arcs.
A mathematical proof is given in the next section.

2,2. The second variation test for the discrete staging problem.

Because all differential equations of this problem have elementary integrals, the optimization of the discrete staging can be reduced to an algebraic problem of constrained minimum. The prescribed number of constant time arcs is denoted by n . The $n+1$ constant thrust arcs have, as yet unknown, initial and final β values denoted by the sequence (β_1,γ_0) , (α_1,γ_1) ... (α_n,γ_n) with $\gamma_n \equiv \beta_2$.

The terminal velocity is easily found to be

$$\omega_2 = \sum_{o}^{n} m(\gamma_i) - \sum_{1}^{n} m(\alpha_i) - m(\beta_1) \qquad (2.I)$$

where $m(x) = \ln x + \frac{1}{x}$ (2.2)

The terminal value of ϕ

$$\phi_2 = \ln \frac{\gamma_o}{\beta_1} + \sum_{1}^{n-1} \ln \frac{\gamma_i}{\alpha_i} + \sum_{1}^{n} \ln \frac{K - \alpha_i}{K - \gamma_{i-1}}$$

inserted into the functional (I.9) gives

$$J = \frac{\sigma}{\beta_1} - \frac{1}{K} L(\alpha, \gamma) \qquad (2.3)$$

where $\ln L = \sum_{1}^{n} p(\alpha_i) - \sum_{o}^{n} p(\gamma_i)$ (2.4)

$$p(x) = \ln \frac{x}{K - x} \qquad (2.5)$$

Equating to zero the partial derivatives of the augmented function $f = J + \mu\omega_2$ with respect to the unknowns α_i and γ_i, we find that each of them satisfies the same algebraic equation

$$- \lambda x^2 + x (\lambda + \lambda K + K) - K \lambda = o \qquad (2.6)$$

where $\lambda = \frac{K \mu}{L}$

Denoting by θ_1 and θ_2 the roots of (2.6) we have

$$K = \theta_1 \theta_2$$

$$\frac{K}{\lambda} = \frac{L}{\mu} = - (\theta_1 - 1)(\theta_2 - 1) \qquad (2.7)$$

Since $\gamma_i > \alpha_i$, we must chose

$\alpha_i = \theta_1$ the smallest root

$\gamma_i = \theta_2 = \frac{K}{\theta_1}$ the largest root and we can compute

$$L = \frac{(\theta_1-1)^{n+1}}{(\theta_2-1)^n} > 0 \qquad (2.8)$$

and

$$\mu = -\frac{L}{(\theta_1-1)(\theta_2-1)} = -\frac{(\theta_1-1)^n}{(\theta_2-1)^{n+1}} < 0 \qquad (2.9)$$

The partial derivative of the augmented function with respect to β_1, which plays a special role, gives

$$\frac{\partial f}{\partial \beta_1} = -\frac{\sigma}{\beta_1^2} - \mu\, m'(\beta_1) = 0 \qquad \text{or}$$

$$\sigma + \mu(\beta_1 - 1) = 0 \qquad (2.10)$$

and finally, in view of (2.9)

$$\beta_1 = 1 + \sigma \frac{(\theta_2-1)^{n+1}}{(\theta_1-1)^n} \qquad (2.11)$$

and

$$J = \frac{\sigma}{\beta_1} - \frac{1}{K} \frac{(\theta_1-1)^{n+1}}{(\theta_2-1)^n} \qquad (2.12)$$

This solution coincides with the one obtained by the maximum-minimum principle. To prove its optimality we apply the second variation test.
The constraint $\delta\omega_2 = 0$ on the first variations is

$$\sum_0^n m'(\gamma_i)\delta\gamma_i - \sum_1^n m'(\alpha_i)\delta\alpha_i - m'(\beta_1)\delta\beta_1 = 0$$

Because of the first variation conditions $\partial f/\partial x = 0$, which require

$$L\, p'(x) + K \mu\, m'(x) = 0 \qquad x = \alpha_i, \gamma_i$$

it can be placed in the more convenient form

$$\frac{L}{K}\sum_0^n p'(\gamma_j)\delta\gamma_j - \frac{L}{K}\sum_1^n p'(\alpha_i)\delta\alpha_i - \frac{\sigma}{\beta_1^2}\delta\beta_1 = 0 \qquad (2.13)$$

The second variation of the augmented function is

$$\delta^2 f = \frac{1}{2}\ \frac{\partial^2 f}{\partial \beta_1^2}\ (\delta\beta_1)^2 - \frac{1}{2}\ \sum_1^n \left(\frac{L}{K}\ p''(\alpha_i) + \mu\ m''(\alpha_i)\right)\ (\delta\alpha_i)^2$$

$$+ \frac{1}{2}\ \sum_o^n \left(\frac{L}{K}\ p''(\gamma_i) + \mu\ m''(\gamma_i)\right)\ (\delta\gamma_i)^2$$

$$- \frac{1}{2}\ \frac{L}{K}\ \left(\sum_o^n\ p'(\gamma_j)\delta\gamma_j - \sum_1^n\ p'(\alpha_i)\delta\gamma_i\right)^2$$

Its last term can be simplified in view of (2.I3) and the resulting form of the second variation contains only squares of variations :

$$\delta^2 f = \frac{1}{2}\ B(\delta\beta_1)^2 + \frac{1}{2}\ A \sum_1^n\ (\delta\alpha_i)^2 + \frac{1}{2}\ C \sum_o^n\ (\delta\gamma_i)^2 \qquad (2.I4)$$

where

$$A = -\frac{L}{K}\ p''(\alpha_i) - \mu\ m''(\alpha_i) = -\frac{L}{K}\ p''(\theta_1) - \mu\ m''(\theta_1)$$

which, after computation is seen to be

$$A = \frac{L\ \theta_2(\theta_2 - \theta_1)}{K\ \theta_1^2(\theta_1-1)(\theta_2-1)^2} > o$$

$$C = \frac{L}{K}\ p''(\gamma_i) + \mu\ m''(\gamma_i) = \frac{L}{K}\ p''(\theta_2) + \mu\ m''(\theta_2)$$

or

$$C = \frac{L\ \theta_1(\theta_2 - \theta_1)}{K\ \theta_2^2(\theta_1-1)^2(\theta_2-1)} > o$$

$$B = \frac{\partial^2 f}{\partial \beta_1^2} - \frac{K\ \sigma^2}{L\ \beta_1^4} = \frac{1}{\beta_1^3}\ \left(2\ \sigma - 2\ \mu + \mu\ \beta_1\ - \frac{K\ \sigma^2}{L\ \beta_1}\right)$$

To show that this coefficient is also positive we use (2.3) and (2.I0) to find

$$B = \beta_1^{-3}\ \left(-\ \mu - \frac{\sigma K}{L}\ J\right)$$

But from (2.9) μ is negative, from (2.8) L is positive and, to have physical significance, J must be negative (If it is positive the velocity performance was set too high for a solution with positive payload to exist).
Hence B is also positive and it is clear without further calculations that the constraint (2.I3) cannot prevent the second variation (2.I4) from being positive definite.
Although it is unnecessary in this case, the constraint can be applied to transform the second variation test in an eigenvalue problem [3,4]. Substracting the eigenvalue parameter ζ to the (diagonal) matrix of second derivatives of f and bordering by the coefficients of the constraint, we obtain the eigenvalues from the determinant

$$\begin{vmatrix} B-\zeta & & & & & & m'(\beta_1) \\ & A-\zeta & & & & & m'(\theta_1) \\ & & \ddots & & & & \vdots \\ & & & A-\zeta & & & m'(\theta_1) \\ & & & & C-\zeta & & -m'(\theta_2) \\ & & & & & \ddots & \vdots \\ & & & & & C-\zeta & -m'(\theta_2) \\ m'(\beta_1) & m'(\theta_1) & \cdots & m'(\theta_1) & -m'(\theta_2) & \cdots\; -m'(\theta_2) & 0 \end{vmatrix} = 0$$

This is equivalent to the equation

$$\frac{\left(m'(\beta_1)\right)^2}{B-\zeta} + \frac{n\left(m'(\theta_1)\right)^2}{A-\zeta} + \frac{(n+1)\left(m'(\theta_2)\right)^2}{C-\zeta} = 0$$

which shows that there are exactly two eigenvalues, one in each of the intervals defined by the numbers (A, B, C) . They are certainly positive if A, B and C are positive, hence the second variation is positive definite.

2.3. The maximum-minimum principle in general.

There are many other binary systems, with a bang-bang control, which, for certain functionals and boundary conditions, exhibit optimal trajectories involving chattering. One of the simplest examples is

$$\dot{q}_1 = u \qquad \dot{q}_2 = q_1^2 \qquad u = \pm 1$$

$$q_1(o) = a \qquad q_2(o) = o \qquad q_1(T) = b \qquad T > |a| + |b|$$

$$q\ (T)\ \text{minimum}$$

A common characteristic of the chattering condition of such systems is that it takes place in some algebraic variety of the state space ($\beta = \sqrt{K}$ for the staging problem, $q_1 = o$ for the example above). A general theory of such binary systems is possible and shows that any optimal trajectory involving a chattering arc can be approximated by a bang-bang law of control with a prescribed number of switching points. This law is optimal if the control is chosen according to the maximum-minimum principle. The maximum principle is applied up to the first zero of the switching function.

After that the control minimizes the Hamiltonian and the switching function exhibits an oscillatory behavior. When the prescribed number of switching points is reached the trajectory is either at an end or one must revert to the maximum principle after the last switching point.

Physical intuition suggests that the maximum-minimum principle applies to bang-bang controlled systems of higher order. In such cases the situation gets complicated from the fact that chattering is generally no more confined to a variety of the state space. For instance, if we complicate the rocket staging problem by imposing a terminal total energy (kinetic plus potential) per unit mass instead of a terminal velocity, the differential equation

$$\frac{de}{d\mu} = \omega\, v$$

governing the total energy e must be added to the system.
The condition of persistance of a zero value of the switching function does no more result in some holonomic constraint between state variables but gives directly the control as a function of the state variables :

$$v = f(\beta, \omega)$$

Chattering can in principle take place so long as this function has a value between zero and one.

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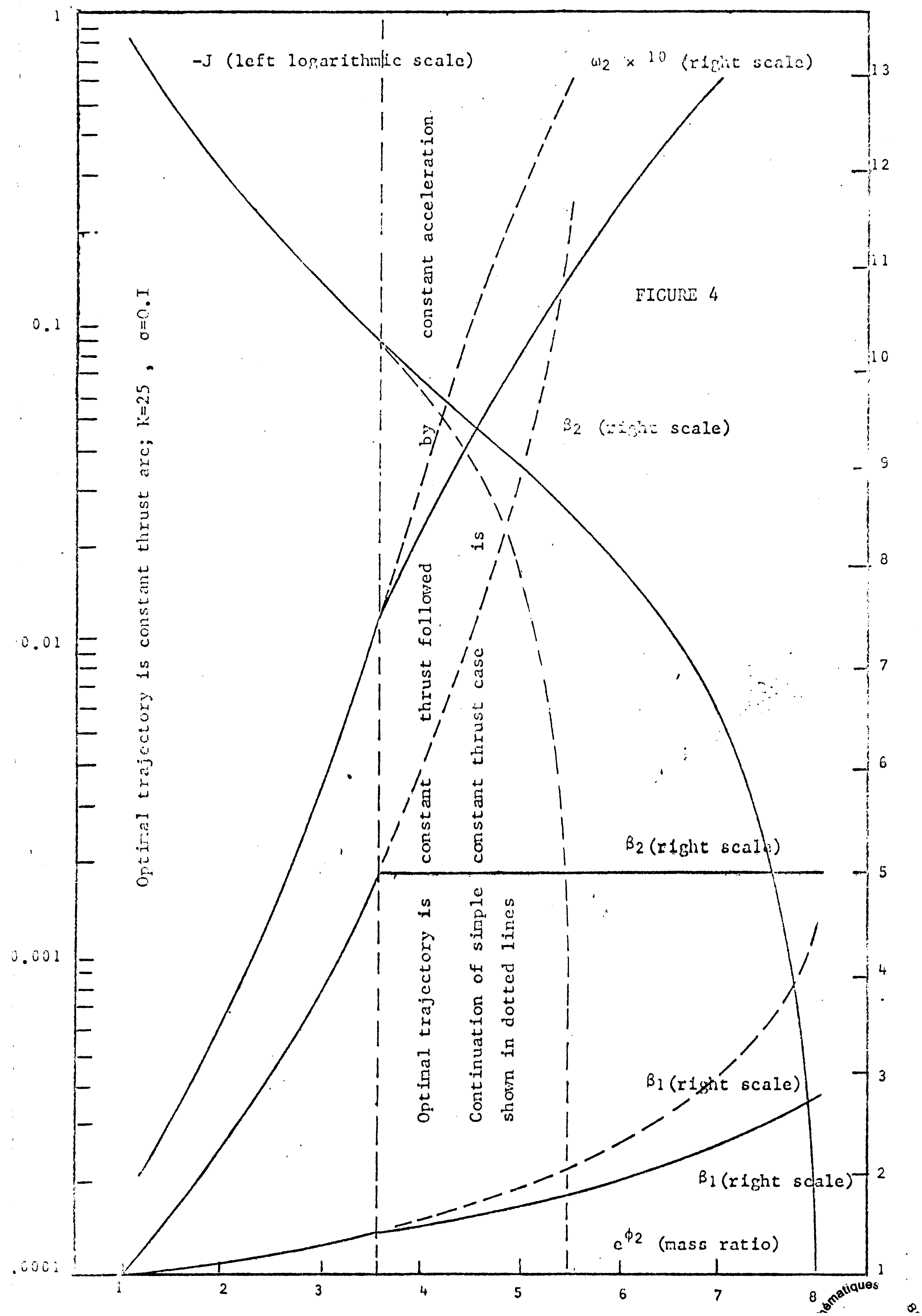
-J (left logarithmic scale)
ω2 × 10 (right scale)
constant acceleration
Optimal trajectory is constant thrust arc; k=25 , σ=0.1
β2 (right scale)
β2 (right scale)
constant thrust followed by
constant thrust case is
Optimal trajectory is
Continuation of simple
shown in dotted lines
β1 (right scale)
β1 (right scale)
eφ2 (mass ratio)
1
0.1
0.01
0.001
.0001
1
2
3
4
5
6
7
8
13
12
11
10
9
8
7
6
5
4
3
2
1

FIGURE 4