ПРОБЛЕМЫ ГИДРОДИНАМИКИ И МЕХАНИКИ СПЛОШНОЙ СРЕДЫ

(к 60-летию академика Л. И. Седова)

PROBLEMS OF HYDRODYNAMICS AND CONTINUUM MECHANICS

МОСКВА—1969

THE THEORETICAL DESIGN LAWS OF WARPING-FREE MULTICELLULAR BOX BEAMS

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A variational method is applied to derive the theoretical laws for skin thicknesses and boom areas under which a multicellular box beam becomes free from warping effects under either torsion or bending. For total freedom under arbitrary combinations of bending and torsion the laws take a strikingly simple form. Their practical implementation is related to the geometrical convexity of a set of elementary cells, each of which contains the shear centre in its interior. In such a case the shear centre, flexure-torsion centre, and centroid are coincident, as are the principal axes of inertia and principal shear axes. The Bernoulli-Navier and de Saint-Venant theories are identical and exact solutions under arbitrary transverse loads.

§ 1. BOX-BEAM GEOMETRY

The box beams considered consist of an arbitrarily complex arrangement of skin panels connected by stringers or spar flanges parallel to z-axis (Fig. 1). In a given cross-section, the traces of skin panels are "arcs" separated by nodes of co-ordinates (x_m, y_m) which are the geometrical centres of the stringers or spar flanges. The bending and torsional rigidity of the stringers and flanges is neglected; they are uniaxially stressed (tension or compression).

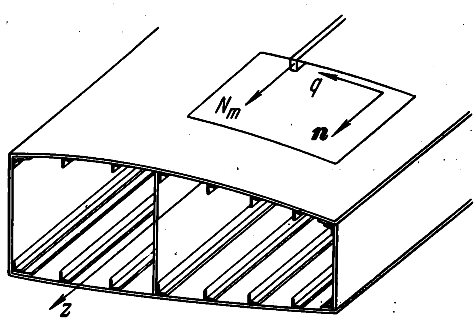

Fig. 1. Box-beam geometry
(n — axial stress flow in skin, q — shear flow in skin, N_m — axial stringer load).

The skin thickness, effective in carrying the shear flow q, is supposed to vary according to an affinity law

$$t_q = h(z)\, a(s) \tag{1.1}$$

where s is measured along the arcs. In a cantilevered beam, we can take $h(0)=1$ at the root section, and $a(s)$ is the skin thickness at the root. Similarly, the skin thickness effective in carrying the normal stress flow n and the flange area carrying the normal load N_m at node (m) will be taken to vary as

$$t_n = g(z)\, b(s), \qquad A_m = g(z)\, S_m. \tag{1.2}$$

The difference between the thicknesses allows for an orthotropic skin structure. Further, the assumption of closely spaced diaphragms, maintaining

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the geometrical shape of any cross-section without inducing axial restraints, is adopted (only membrane stresses in the skin are then needed to resist transverse loads). The displacement vector in a point P (x, y, z) of the skin is then resolved into three locally orthogonal components: $u_n(s, z)$ along the normal $\boldsymbol{n}$,

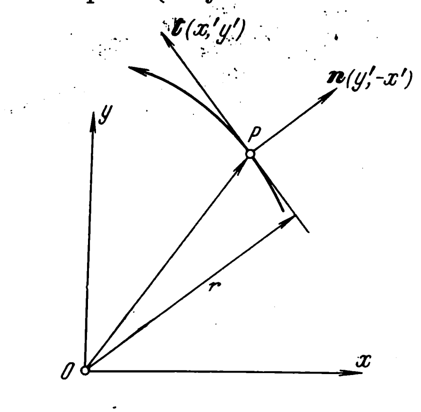

Fig. 2. Definition of r; $(\boldsymbol{n}, \boldsymbol{t})$ is a rotated (Ox, Oy).

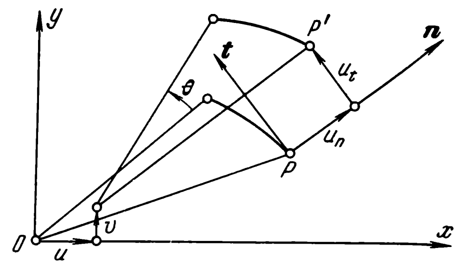

Fig. 3. Diaphragm displacement (u, v, θ) and definitions of u_n and u_t.

u_t (s, z) along the tangent $\boldsymbol{t}$ to the arc, $w(s, z)$ along the generator parallel to z-axis. For the second component

$$u_t = ux' + vy' + \theta r \tag{1.3}$$

where $x' = dx/ds$, $y' = dy/ds$ and

$$r = xy' - yx' \tag{1.4}$$

is the projection of OP on $\boldsymbol{n}$ (Fig. 2). The diaphragm passing through P has been translated by $u(z)$ parallel to Ox, $v(z)$ parallel to Oy, and rotated through an angle $\theta(z)$ about Oz (Fig. 3). The shear strain in the skin at P is

$$\gamma = w' + \dot{u}_t \tag{1.5}$$

where the "prime" denotes differentiation with respect to s, the "dot" differentiation with respect to z.

§ 2. EQUILIBRIUM EQUATIONS

In the absence of axial constraints from the diaphragms and without external axial loads on the skin, the equilibrium of a skin element in the direction of a generator is

$$q' + \dot{n} = 0. \tag{2.1}$$

Similarly for a stringer element at node (m)

$$\dot{N}_m = T_m(q). \tag{2.2}$$

The operator T_m is defined with respect to a conventional sense of description along each arc:

$$T_m(q) = (\Sigma q)_{\text{conv}} - (\Sigma q)_{\text{div}}.$$

The first sum is extended to all the values of shear flows at the node related to arcs convergent to the node, the second, to arcs divergent from the node. Note that the algebraic value of q is itself related to this conventional sense of description.

The T_m operator is extremely useful in connection with integrations extended over the totality of arcs of a cross-section. The following formula for integration by parts is easily established:

$$\int_{\mathscr{C}} \alpha\, d\beta = \sum_m T_m(\alpha\beta) - \int_{\mathscr{C}} \beta\, d\alpha. \tag{2.3}$$

Whenever one of the functions is single-valued, it can be factored out; in the case of a single-valued α, for instance,

$$\int_{\mathscr{C}} \alpha\, d\beta = \sum_m \alpha_m T_m(\beta) - \int_{\mathscr{C}} \beta\, d\alpha. \tag{2.4}$$

Transverse equilibrium of a diaphragm involves the total shear forces acting in a section:

$$S_x(z) = \int_{\mathscr{C}} q\, dx, \qquad S_y(z) = \int_{\mathscr{C}} q\, dy, \tag{2.5}$$

and the torsional moment about Oz:

$$M(z) = \int_{\mathscr{C}} qr\, ds. \tag{2.6}$$

The bending moments are taken with respect to axes parallel to Ox and Oy through the centroid (x_G, y_G) of the section:

$$\begin{cases} M_x = \int_{\mathscr{C}} n(x - x_G)\, ds + \sum_m N_m (x_m - x_G), \\ M_y = \int_{\mathscr{C}} n(y - y_G)\, ds + \sum_m N_m (y_m - y_G). \end{cases} \tag{2.7}$$

Equilibrium equations about these axes are then

$$\dot{M}_x = S_x, \qquad \dot{M}_y = S_y, \tag{2.8}$$

and can be recovered by differentiation of (2. 7), using (2. 1) and (2. 2) and the formula for integration by parts.

Equilibrium with applied transverse loads is expressed through the resultant distributions $p_x(z)$, $p_y(z)$ and $m(z)$ as

$$\dot{S}_x = -p_x, \qquad \dot{S}_y = -p_y, \qquad \dot{M} = -m. \tag{2.9}$$

§ 3. STRESS-STRAIN RELATIONS. BERNOULLI SHEAR CENTRE

The average shear stress $\tau = q/(ha)$ in the skin is related to the shear strain by $\tau = G\gamma$. In terms of the unknown displacements $w(s, z)$, $u(z)$, $v(z)$, and $\theta(z)$ this stress-strain relation takes the form

$$\frac{q}{Gha} = w' + \dot{u}x' + \dot{v}y' + \dot{\theta}r \tag{3.1}$$

stemming from (1.3) and (1.5). The other stress-strain relations are for the axial components in skin and stringers:

$$\frac{n}{gb}=E\dot{w}, \qquad \frac{N_m}{gS_m}=E\dot{w}_m. \tag{3.2}$$

A Bernoulli-Navier type of assumption

$$w=F_0(z)+X_0(z)(x-x_G)+Y_0(z)(y-y_G) \tag{3.3}$$

that keeps the cross-section plane, suppresses in (3.1) the effect of warping and produces a shear flow

$$q_B=Gha[(X_0+\dot{u})\,x'+(Y_0+\dot{v})\,y'+\dot{\theta}r]. \tag{3.4}$$

The corresponding shear centre $B(x_B, y_B)$ is obtained by setting $\dot{\theta}=0$ and expressing that the torsional moment vanishes

$$M_B=\int_{\mathscr{C}} q_B r_B ds=0$$

for arbitrary values of $(X_0+\dot{u})$ and $(Y_0+\dot{v})$. This furnishes the requirements

$$\int_{\mathscr{C}} ar_B dx=0, \qquad \int_{\mathscr{C}} ar_B dy=0; \tag{3.5}$$

when we substitute here the value

$$r_B=(x-x_B)\,y'-(y-y_B)\,x'=r-x_By'+y_Bx', \tag{3.6}$$

we obtain a pair of linear equations to calculate the co-ordinates of the Bernoulli shear centre:

$$\begin{cases} -A_{xx}y_B+A_{xy}x_B=\int_{\mathscr{C}} ar\,dx, \\ -A_{xy}y_B+A_{yy}x_B=\int_{\mathscr{C}} ar\,dy. \end{cases} \tag{3.7}$$

In those formulae appears the symmetrical *reduced area tensor* discovered by Drymael [2]:

$$\begin{cases} A_{xx}=\int_{\mathscr{C}} a\,(x')^2\,ds, \qquad A_{xy}=\int_{\mathscr{C}} ax'y'ds=A_{yx}, \\ A_{yy}=\int_{\mathscr{C}} a\,(y')^2\,ds. \end{cases} \tag{3.8}$$

The contracted tensor is the invariant

$$A_{xx}+A_{yy}=\int_{\mathscr{C}} a\,ds$$

proportional to the total shear resistant area. If the reference axes are translated to the Bernoulli shear centre and the complete shear flow is written as

$$q = Gha\,(w' + \dot{u}_B x' + \dot{v}_B y' + \dot{\theta} r_B),$$

substitution of this into equations (2.5) and (2.6) produces, in view of the properties (3.5), the formulae

$$(3.9)\qquad \begin{cases} \dfrac{S_x}{Gh} = \displaystyle\int_{\mathscr{C}} aw'dx + A_{xx}\dot{u}_B + A_{xy}\dot{v}_B, \\ \dfrac{S_y}{Gh} = \displaystyle\int_{\mathscr{C}} aw'dy + A_{xy}\dot{u}_B + A_{yy}\dot{v}_B, \end{cases}$$

$$(3.10)\qquad \frac{M_B}{Gh} = \int_{\mathscr{C}} ar_B w' ds + I_B\dot{\theta}$$

where

$$(3.11)\qquad I_B = \int_{\mathscr{C}} ar_B^2 ds.$$

Clearly, this quantity is directly related to the torsional stiffness of the box beam under the Bernoulli-Navier assumption. Indeed, the integral in (3.10) vanishes by virtue of (3.5) if we substitute (3.3). The Bernoulli-Navier torsional stiffness I_B is a minimum with respect to the choice of co-ordinates of the centre. For, in the case of another choice, corresponding to

$$r_C = r_B - (x_C - x_B)\,y' + (y_C - y_B)\,x',$$

we obtain, again in view of (3.5),

$$I_C = \int_{\mathscr{C}} ar_C^2 ds = I_B + \int_{\mathscr{C}} a\,(y'\Delta x - x'\Delta y)^2\, ds > I_B.$$

This minimum property was also discovered by Drymael [2].

§ 4. SCALAR PRODUCT. TRUE WARPING

To avoid lengthy formulae we introduce a convenient scalar product notation for two single-valued functions $\alpha(s)$ and $\beta(s)$ defined over the cross-section. By definition

$$(4.1)\qquad (\alpha,\ \beta) = \int_{\mathscr{C}} b\alpha\beta\, ds + \sum_m S_m \alpha_m \beta_m = (\beta,\ \alpha).$$

With this notation (1, 1) is the total root section area resisting to axial loads. The co-ordinates of the centroid are defined by the properties

$$(4.2)\qquad (x - x_G,\ 1) = 0, \quad (y - y_G,\ 1) = 0,$$

and the moments of inertia of the root section are given by

$$(4.3)\qquad I_{xx}=(x-x_G,\ x-x_G),\qquad I_{xy}=(x-x_G,\ y-y_G),$$
$$I_{yy}=(y-y_G,\ y-y_G).$$

We shall denote by $(i_{xx},\ i_{xy},\ i_{yy})$ the elements of the reciprocal tensor, so that

$$i_{xx}I_{xx}+i_{xy}I_{xy}=1,\qquad i_{xx}I_{xy}+i_{xy}I_{yy}=i_{xy}I_{xx}+i_{yy}I_{xy}=0,$$

and

$$i_{xy}I_{xy}+i_{yy}I_{yy}=1.$$

The characteristic properties of a *true warping* can be stated in kinematical form or in terms of stresses. The function $W(s)$ in the affinity law

$$w=f(z)\,W(s)$$

is a true warping if the associated axial loads from (3. 2)

$$n=Egf'bW,\qquad N_m=Egf'S_mW_m$$

are statically equivalent to zero. This requires the vanishing of the total axial load

$$\int_{\mathscr{C}} n\,ds+\sum N_m=Egf'(W,\ 1)=0$$

and of the two bending moments (2. 7), or

$$Egf'(W,\ x-x_G)=0,\qquad Egf'(W,\ y-y_G)=0.$$

A kinematical interpretation of the same conditions

$$(4.4)\qquad (W,\ 1)=0,\qquad (W,\ x-x_G)=0,\qquad (W,\ y-y_G)=0$$

is obtained by looking for the parameters $(\alpha,\ \beta,\ \gamma)$ of an average plane $w^*=\alpha(x-x_G)+\beta(y-y_G)+\gamma$ from which to measure an arbitrary distortion w of the cross-section so as to minimize the squared norm $(w-w^*,\ w-w^*)$. It turns out immediately that, for the best plane, $W=w-w^*$ satisfies conditions (4. 4).

§ 5. DESIGN CONDITIONS FOR ZERO WARPING

The exact solution of the integro-differential equations (2. 1)—(2. 2), (3. 1)—(3. 2) and (2. 5)—(2. 6), with suitable end conditions at $z=0$ and $z=L$, is fairly complicated. With affinity laws such as (1. 1) and (1. 2) the problem can be solved by separation of variables as shown first by von Karman and Chien [4] for symmetrical shapes in torsion and then by Benscoter [1] for the general case. In these approaches w is expanded in a series of eigenwarpings. The first term of the solution is the Bernoulli-Navier approximation. A much faster convergence is obtained by expanding the axial stresses in eigenmodes, as shown by the author [3]. The first-order terms are then made of the de Saint-Venant solutions for bending and torsion. While the first approach can be developed as a variational principle for displacements, the second is really a variational principle for stresses and requires the equilibrium equations to be satisfied beforehand.

Our purpose here is different, we look directly for possible functions $a(s)$, $b(s)$ and S_m, such that the problem becomes freed from warping effects. The net result is that the Bernoulli-Navier and de Saint-Venant theories become identical and exact solutions for arbitrary transverse loading modes. The only loading constraints remaining are in the end sections $z=0$ and $z=L$, if eigenwarping modes are to be completely avoided.

The simplest approach to this goal is variational in nature. Looking at equations (3. 9) and (3. 10) we shall try to obtain the following functional identities:

$$(5.1) \qquad \int_{\mathscr{C}} aW'dx = 0, \qquad \int_{\mathscr{C}} aW'dy = 0,$$

$$(5.2) \qquad \int_{\mathscr{C}} ar_B W'ds = \int_{\mathscr{C}} ar_B dW = 0$$

for any true warping functions $W(s)$.

§ 6. FREEDOM FROM TORSIONAL WARPING

First take condition (5. 2). We may replace in it the true warping function W by any single-valued function w, differentiable along each arc. Indeed, such a function differs from a true warping by a linear form with arbitrary coefficients (α, β, γ)

$$w = W + \alpha(x - x_G) + \beta(y - y_G) + \gamma$$

and, by virtue of properties (3. 5),

$$\int_{\mathscr{C}} ar_B dw = \int_{\mathscr{C}} ar_B dW.$$

The condition

$$\int_{\mathscr{C}} ar_B dw = 0$$

just shown to be equivalent to (5. 2) is now integrated by parts and gives

$$\sum_m w_m T_m(ar_B) - \int_{\mathscr{C}} wd(ar_B) = 0.$$

The necessary conditions following from the arbitrariness of w are

$$(6.1) \qquad dp = 0 \text{ on each arc,}$$

$$(6.2) \qquad T_m(p) = 0 \text{ at each node,}$$

where $p = ar_B$. These conditions are identical to the Kirchhoff conservation laws for currents, the arcs being thought of as the branches of an electric circuit. Their general solution is a superposition of currents circulating in a complete set of independant meshes of the circuit (mesh-currents). Let ω_i denote a mesh-current in the ith mesh with a positive sense of description fixed, for instance, by the rotation of Ox to Oy. Then the general solution of equations (6. 1) and (6. 2) is

$$p = \sum_{i=1}^{k} \omega_i \delta_i . \tag{6.3}$$

The sum is extended to a given choice of k independent meshes. The solution is expressed with reference to the original sense of description adopted for each arc so that

$$\delta_i = \begin{cases} 0 & \text{for those arcs which do not belong to mesh } i; \\ 1 & \text{for the arcs of mesh } i \text{ where mesh sense and original sense coincide;} \\ -1 & \text{for the arcs of mesh } i \text{ where mesh sense and original sense are opposite.} \end{cases}$$

In the design application to the torsion problem, the choice of the Bernoulli shear centre is free. Once fixed, the thickness function is defined by the law

$$a(s) = \frac{1}{r_B} p \tag{6.4}$$

where p, as given by (6. 3), depends on k parameters with obvious limitations in the possible choices to maintain positive thickness everywhere.

It is then easily verified that a simple warping-free solution of the torsion problem is at hand.

For u_B, v_B, and w identically zero but an arbitrary $\theta(z)$, the shear flow

$$q = \dot{\theta} G h p \tag{6.5}$$

and zero axial stresses (one of the semi-inverse assumptions of de Saint-Venant for torsion)

$$n = 0, \qquad N_m = 0 \tag{6.6}$$

satisfy all the equilibrium and compatibility conditions with zero shear loads:

$$\begin{cases} S_x = \dot{\theta} G h \int_{\mathscr{C}} p\, dx = \dot{\theta} G h \int_{\mathscr{C}} a r_B dx = 0, \\ S_y = \dot{\theta} G h \int_{\mathscr{C}} p\, dy = \dot{\theta} G h \int_{\mathscr{C}} a r_B dy = 0, \end{cases} \tag{6.7}$$

but with a couple M_B related to torsion by the Bernoulli torsional rigidity (equal here to that of de Saint-Venant):

$$M_B = \dot{\theta} Gh \int_{\mathscr{C}} a r_B^2 ds = \dot{\theta} Gh I_B. \tag{6.8}$$

The end conditions are compatible with both the built-in case and the free-end case.

§ 7. FREEDOM FROM WARPING IN BENDING

To facilitate subsequent derivations, we shall assume that the orientation of the reference axes is that of the principal axes of the tensor of reduced areas. Then

$$A_{xy} = \int_{\mathscr{C}} a x' dy = \int_{\mathscr{C}} a y' dx = 0. \tag{7.1}$$

Let us examine the possibility of implementing the second of conditions (5. 1). We can replace in it the true warping function by

$$W = w - \alpha (x - x_G) - \beta (y - y_G) - \gamma$$

and, taking (7. 1) into account, obtain the condition

$$\int_{\mathscr{C}} a w' dy - \beta A_{yy} = 0 \tag{7.2}$$

to be satisfied by any single-valued axial displacement function w. We also note that, by virtue of (4. 2), (4. 3), and (4. 4),

$$(w,\ x - x_G) - \alpha I_{xx} - \beta I_{xy} = 0,$$
$$(w,\ y - y_G) - \alpha I_{xy} - \beta I_{yy} = 0.$$

Solving these equations for β and substituting into (7. 2), the freedom condition is finally expressed entirely in terms of the arbitrary w by

$$\int_{\mathscr{C}} a w' dy = \int_{\mathscr{C}} a y' dw = A_{yy} [i_{xy} (w,\ x - x_G) + i_{yy} (w,\ y - y_G)].$$

Integrating by parts and replacing the scalar products by their explicit definitions, we conclude from the arbitrariness of w that we should have

$$(ay')' = [i_{xy} b (x - x_G) + i_{yy} b (y - y_G)] A_{yy}, \tag{7.3}$$

$$T_m (ay') = -[i_{xy} S_m (x_m - x_G) + i_{yy} S_m (y_m - y_G)] A_{yy}. \tag{7.4}$$

If we consider given distributions of $b(s)$ and S_m, these equations provide design laws for $a(s)$ in order to implement the second of conditions (5. 1). They will generally conflict with the laws (6. 1) and (6. 2), and we shall later examine the $b(s)$ and S_m distributions needed to avoid this conflict and obtain freedom of warping effects in both bending and torsion.

Williams and Fine [5] gave and discussed a result equivalent to (7. 3) for the case of a single closed cell with a plane of symmetry (so that $i_{xy}=0$ too).

To solve (7. 3) and (7. 4) for ay' we introduce the concepts of the "open" static moments

$$\begin{cases} X_0(s)=\int_- b(x-x_G)\,ds+\sum_- S_m(x_m-x_G), \\ Y_0(s)=\int_- b(y-y_G)\,ds+\sum_- S_m(y_m-y_G), \end{cases} \tag{7. 5}$$

defined as follows. The closed meshes of the set of arcs are opened by a (in principle arbitrary) choice of k cuts and a node is chosen as origin. On each arc (or segment of cut arc) there is now a unique direction to follow in order to reach the origin and this is conveniently chosen as the positive sense of shear flow. Furthermore, at each point of this tree a new cut separates the graph into two portions. They are distinguished by the convention that, in following the positive flow sense, one passes from the negative side to the positive side. The static moments (7. 5) are then uniquely defined at each point by extending the integral and the sum at the portion on the negative side. By moving of ds in the positive sense, we add a small contribution to the integrals and find

$$dX_0=b(x-x_G)\,ds, \qquad dY_0=b(y-y_G)\,ds. \tag{7. 6}$$

By stepping over a node, coming from several convergent branches and departing along the sole divergent branch, we find that

$$-T_m(X_0)=S_m(x_m-x_G), \qquad -T_m(Y_0)=S_m(y_m-y_G). \tag{7. 7}$$

Naturally, the open static moments are different if a different set of cuts is adopted, but they always satisfy the same equations (7. 6) and (7. 7). Being linear these equations show that the difference between two solutions, that is the difference between two open static moments, is a solution of the corresponding homogeneous equations (6. 1) and (6. 2). Hence the general solution of equations (7. 6) and (7. 7) in the case of X is

$$X=X_0-\sum_{i=1}^{k}\omega_i\delta_i$$

where X_0 corresponds to a fixed set of cuts. The arbitrariness in the solution disappears when one adds the requirements

$$\oint_j\left(\frac{X}{a}\right)\delta_j\,ds=0 \qquad (j=1,\ 2,\ \ldots,\ k), \tag{7. 8}$$

the integrals being taken in the positive circulation sense around each mesh that was opened. Such requirements yield in fact a system of k equations in the k unknowns ω_i:

$$\sum_{i=1}^{k}\omega_i\oint_j\frac{\delta_i\delta_j}{a}ds=\oint_j\frac{X_0}{a}\,ds \qquad (j=1,\ 2,\ \ldots,\ k) \tag{7. 9}$$

and make the solution X unique. This solution and the similar one for Y will be called the *closed static moments.*

Since a closed static moment verifies the same equations

$$(7.10) \qquad dX = b\,(x - x_G)\,ds, \qquad T_m(X) = -S_m(x_m - x_G)$$

as an open moment, it also satisfies the following useful relations:

$$(7.11) \qquad \int_{\mathscr{C}} X\,dx = -I_{xx}, \qquad \int_{\mathscr{C}} X\,dy = -I_{xy}$$

that are immediately established by integration by parts, use of (7.10) and identification with (4.3). Similarly,

$$(7.12) \qquad \int_{\mathscr{C}} Y\,dx = -I_{xy}, \qquad \int_{\mathscr{C}} Y\,dy = -I_{yy}.$$

Coming back to equations (7.3) and (7.4), one possible solution appears to be

$$ay' = (i_{xy}X + i_{yy}Y)\,A_{yy}.$$

It is in fact the only one which ensures that y is single-valued, since the conditions

$$\oint_j \delta_j y' ds = 0 \qquad (j = 1, 2, \ldots, k)$$

are actually implemented by (7.8) and the corresponding property for Y. Furthermore, in view of (7.11) and (7.12), we can check that

$$A_{xy} = \int_{\mathscr{C}} ay'dx = -A_{yy}(i_{xy}I_{xx} + i_{yy}I_{xy}) = 0.$$

The final form of the condition for avoiding a warping effect in bending (in one plane) is thus

$$(7.13) \qquad ay' = -A_{yy}(i_{xy}X + i_{yy}Y).$$

The right-hand side is determined by the $b(s)$ and S_m distributions, except for the free parameter A_{yy}.

Again, if we succeed in satisfying equation (7.13) for a practical design, a simple solution of the bending problem is at hand. The axial stress distribution of this solution is that of the de Saint-Venant theory for a bending moment M_y:

$$(7.14) \qquad \begin{cases} n = M_y b\,[i_{xy}(x - x_G) + i_{yy}(y - y_G)], \\ N_m = M_y S_m\,[i_{xy}(x_m - x_G) + i_{yy}(y_m - y_G)]. \end{cases}$$

The corresponding shear flow distribution in equilibrium with axial stresses is

$$(7.15) \qquad q = -S_y(i_{xy}X + i_{yy}Y)$$

and has, by virtue of (7.13), the alternate expression

$$(7.16) \qquad q = \frac{S_y}{A_{yy}}\,ay'.$$

The bending moment M_x and shear load S_x in the other plane are zero. The deformation equations are satisfied by setting

(7.17) $$\theta = 0,$$

(7.18) $$\ddot{v} = -i_{yy}\frac{M_y}{Eg} + \frac{1}{GA_{yy}}\left(\frac{S_y}{h}\right)^{\cdot},$$

(7.19) $$\ddot{u} = -i_{xy}\frac{M_y}{Eg},$$

(7.20) $$w = \left(\frac{S_y}{GhA_{yy}} - \dot{v}\right)(y - y_G) - \dot{u}\,(x - x_G).$$

Hence, as expected, the cross-section remains plane, and the end conditions are again compatible with both the built-in case and the free-end case. The whole theory, repeated for implementation of the first of conditions (5. 1), would yield a similar solution for bending in the other plane.

§ 8. FREEDOM FROM WARPING IN BENDING AND TORSION

Here we examine the possibility of combining freedom from warping in both torsion and bending in one plane. As we shall see this automatically implies also freedom from warping in bending in the other plane. Suppose we have adopted a law for $a(s)$ that suppresses the de Saint-Venant warping in torsion and let the origin of the axes be placed at the chosen Bernoulli shear centre. We then have the properties

(8.1) $$d(ar) = 0 \quad \text{on each arc},$$

(8.2) $$T_m(ar) = 0 \quad \text{at each node},$$

with

(8.3) $$r = xy' - yx'.$$

To remove also the de Saint-Venant warping due to shear loads S_y we need, according to the results of § 7,

(8.4) $$ay' = -A_{yy}(i_{xy}X + i_{yy}Y).$$

The differential of the left-hand side along an arc is manipulated to incorporate (8. 1) as follows

$$d(ay') = ard\left(\frac{y'}{r}\right) = -\frac{ay}{\rho r}\,ds$$

where

(8.5) $$\frac{1}{\rho} = x'y'' - y'x''$$

is the curvature of the arc, positive when the normal n lies on the convex side. The differential of the right-hand side follows from equations like the first of (7. 10). The final result

$$\frac{ay}{\rho r} = A_{yy}\left[i_{xy}(x - x_G) + i_{yy}(y - y_G)\right] b$$

can serve to specify the $b(s)$ distribution. To avoid infinities for $y=0$ we are led to require furthermore that

(8.6) $$i_{xy}=0 \quad \text{or} \quad I_{xy}=0 \quad \text{and} \quad y_G=0$$

which reduces the law to

(8.7) $$b=\frac{I_{yy}}{A_{yy}}\frac{a}{\rho r}=-\frac{I_{yy}}{A_{yy}}\frac{(ay')'}{y}.$$

From (8.4) and equations like the second of (7.10) we obtain

(8.8) $$S_m=\frac{I_{yy}}{A_{yy}}\frac{T_m(ay')}{y_m}$$

as laws governing the area of stringers.

We now develop relations of symmetry to show that the same laws suppress warping due to S_x shear loads. From the identity

$$ar=x(ay')-y(ax')$$

we obtain, in view of properties (8.1) and (8.2), that

(8.9) $$xd(ay')=yd(ax')$$

and

(8.10) $$x_mT_m(ay')=y_mT_m(ax').$$

Hence the symmetrical result can be proved:

(8.11) $$x_G=0.$$

The value of x_G is found from the property $(x-x_G,\ 1)=0$ or

$$x_G(1,\ 1)=(x,\ 1)=\int_{\mathscr{C}} bx\,ds+\sum_m S_mx_m.$$

But, from (8.7) and (8.8),

$$x_G(1,\ 1)\frac{A_{yy}}{I_{yy}}=-\int_{\mathscr{C}}\frac{x}{y}d(ay')+\sum_m\frac{x_m}{y_m}T_m(ay')$$

and finally, from (8.9) and (8.10),

$$x_G(1,\ 1)\frac{A_{yy}}{I_{yy}}=-\int_{\mathscr{C}}d(ax')+\sum_m T_m(ax')=0$$

as announced. Next calculate

$$A_{xx}=\int_{\mathscr{C}}ax'dx=\sum_m x_mT_m(ax')-\int_{\mathscr{C}}xd(ax').$$

Apply the symmetry relations (8.9) and (8.10) to find

$$A_{xx}=\sum_m\frac{x_m^2}{y_m}T_m(ay')-\int_{\mathscr{C}}\frac{x^2}{y}d(ay')$$

and finally reintroduce b and S_m from (8.7) and (8.8):

$$A_{xx} = \frac{A_{yy}}{I_{yy}} \left(\sum_m S_m x_m^2 + \int_{\mathscr{C}} bx^2 ds \right).$$

This relation, exhibited in the symmetrical form

$$\mu = \frac{A_{xx}}{I_{xx}} = \frac{A_{yy}}{I_{yy}}, \tag{8.12}$$

proves that the laws found for b and S_m can be placed in the form that would have resulted from the suppression of the warping due to S_x instead of the S_y shear loads.

To summarize: the skin thickness carrying shear flow must be distributed to satisfy equations (8. 1) and (8. 2), the nature of this solution was discussed in § 6. The skin thickness carrying normal loads must be distributed according to the law

$$b = \frac{1}{\mu} \frac{a}{r\rho} \tag{8.13}$$

depending on the curvature of the skin and proportional to a desired ratio of inertia moments to reduced areas.

The area of nodal stringers is given by either of the formulae

$$S_m = \frac{1}{\mu} \frac{T_m(ax')}{x_m} = \frac{1}{\mu} \frac{T_m(ay')}{y_m}. \tag{8.14}$$

For such designs the axes Ox and Oy are both principal axes of inertia ($I_{xy}=0$) and principal shear axes ($A_{xy}=0$). The origin is simultaneously the shear centre, flexure-torsion centre and centroid. Under arbitrary transverse loads, and provided the end loads conform with the de Saint-Venant distribution, the section remains plane and the ordinary engineering beam theory is an exact solution.

§ 9. THE PRACTICAL DESIGN OF WARPING-FREE SECTIONS

We still need to deal, in the simplest possible manner, with the limitations of the theory of warping-free sections represented by the requirements for positive thicknesses and areas for skin and stringers. The single closed cell section will be considered to be the simplest building block for more complex structures.

In a single cell we take

$$a = \frac{k}{r} \tag{9.1}$$

where k is a constant connected with the desired torsional rigidity

$$J = \oint ar^2 ds = k \oint r\, ds = 2\Omega k \tag{9.2}$$

Ω being the area of the cell. Hence k should be positive and r likewise. This limits the choice of the shear centre (origin of the co-ordinate axes) to the interior of the cell, which, as will be seen next, must be convex. The skin thickness for normal loads is fixed by

(9.3) $$b=\frac{a}{\mu\rho r}=\frac{k}{\mu\rho r^2}.$$

The parameter μ is chosen according to (8.12) to obtain a desired moment of inertia and is positive. Hence ρ must also be positive: the section must be convex. At the limit, zero curvature could be tolerated, if the property $b=0$ can be represented by a corrugated sheet. At angular points a concentrated stringer area is needed and can be calculated by either one of equations (8.14). It is however more instructive to obtain it by a limiting process.

The total area of a curved skin piece is

$$\int_{M_1}^{M_2} b\,ds=\frac{k}{\mu}\int_{M_1}^{M_2}\frac{ds}{\rho r^2}=\frac{k}{\mu}\int_{\varphi_1}^{\varphi_2}\frac{d\varphi}{r^2}$$

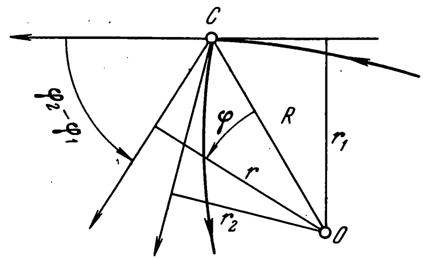

Fig. 4. Calculating the boom area at an angular point (corner) C.

where $\varphi_2-\varphi_1$ is the angle through which the tangent to the arc has turned. In the limit of strong curvature this formula should hold for the stringer area where $\varphi_2-\varphi_1$ is the turning angle of the tangent at the angular point. In this case r varies according to the law $r=R\cos\varphi$ (Fig. 4) and we find

(9.4) $$S=\frac{k}{\mu}\,\frac{\tan\varphi_2-\tan\varphi_1}{R^2}=\frac{k}{\mu}\,\frac{\sin(\varphi_2-\varphi_1)}{r_1r_2}$$

where r_1 and r_2 are the values of r when entering and leaving the angular point. This result is easily proven to be identical with (8.14). For multicellular sections the simplest method is to superimpose the thickness

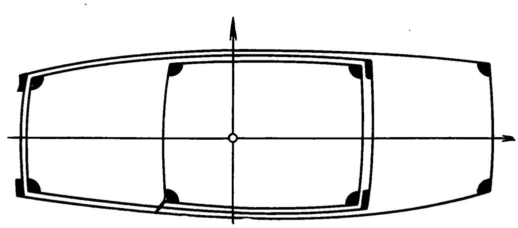

Fig. 5. Principle of superposition of skin thicknesses and boom areas of component single cells.

and area distributions of component single cells (Fig. 5). All component cells should be convex and contain the chosen shear centre in their interior. In addition to the freedom in the choice of a shear centre and the choice of μ (which must be the same for all components), there are as many independent parameters k_i as component cells. The overall torsional stiffness of the multicellular case is readily calculated to be

(9.5) $$J=2\sum_i k_i\Omega_i.$$

The consideration that warping-free beams can resist rapidly varying or even discontinuous transverse loads without stress concentrations due to shear lag or torsion induced bending stresses warrants a careful design study. Admittedly this ideal can in some cases conflict with a more efficient use of the material.

REFERENCES

1. Benscoter S. U., Secondary Stresses in Thin-Walled Beams with Closed Cross-Sections. — NACA Techn. Notes, 1951, n° 2529.
2. Drymael J., Some Theoretical Considerations on Box-Beams. — J. Roy. Aeronaut. Soc., 1941, 45 : 361, 5—28.
3. Fraeijs de Veubeke B., Théorie des coques prismatiques minces renforcées par lisses et diaphragmes. — Mém. Acad. roy. Belgique, Cl. sci., 1961, 32 : 8, n° 1729.
4. von Karman T., Chien W. Z., Torsion with Variable Twist. — J. Aeronaut. Sci., 1946, 13 : 10, 503—510.
5. Williams D., Fine M., Stress Distribution in Reinforced Flat Sheet, Cylindrical Shells and Cambered Box-Beams under Bending. — ARC Repts and Memor., 1940, n° 2099.