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NUMERICAL INTEGRATION OF PLANE ORBITAL TRANSFERS WITH MULTIPLE POWERED ARCS

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ABSTRACT

The purpose of this numerical investigation has been to evaluate the penalty on propellant expenditure in a transfer using a chemical rocket of small thrust. The thrust acceleration at departure was taken to be 0·03 times the gravitational acceleration. Results show that the excess expenditure over that of a bi-impulsive transfer can be reduced by a factor 4 when passing from a TCT optimal solution to a TCTCT one.

POSITION OF THE PROBLEM

The problem is a planar one. The time and the angle are open. The orbit of arrival is defined by its total energy and its angular momentum. Optimization is for minimum propellant expenditure of a rocket of limited thrust with respect to both thrust orientation and cutoff-relight capability. All state variables are made non-dimensional by using the radius of the circular departure orbit, the corresponding orbital velocity and the initial mass as units. The 4 state variables are

z: inverse of the angular momentum

A and B: related to eccentricity e and argument of pericenter θ_0 by:

$$A = ze \cos \theta_0$$
$$B = ze \sin \theta_0$$

θ: the polar angle with respect to a fixed direction

z, A, B are osculating variables, they remain constant during a coasting arc. The independent variable is the characteristic velocity

$$\phi = c \ln \mu$$

where μ is the (reduced) reciprocal of instantaneous mass.

The two dimensionless parameters of the problem (excluding the geometrical characteristics of the orbits of departure and of arrival) are

c: the reduced effective exhaust velocity,

a: the ratio of thrust acceleration to gravitational acceleration at departure.

The numerical values adopted for the parameters were

$$c = 0{\cdot}3228 \qquad a = 0{\cdot}03$$

The method of Runge–Kutta–Gill was used to integrate the system of 8 differential equations[2] (4 equations for the system of state variable and 4 equations for the adjoint system). According to the maximum principle, it is possible to compute at each step the optimal thrust orientation ψ, while the sequence of powered and coasting arcs is ruled by a switching function also calculated step by step.

For open time transfers, the Lagrangian multipliers are orbital constants just like the osculating variables. The only variables during a coasting arc are the polar angle θ and the time (which is here a separable variable anyway). Hence, if θ_c denotes the polar angle at a cutoff, the only unknown will be the polar angle θ_r at the next relight condition. From the analytical results[1, 2] there are two types of orbital jump, a symmetrical one and an asymmetrical one and the type of jump can be decided by tests. There are two unknown initial values: ψ_a and θ_a; their choice determines a discrete set of orbits of arrival. The choice of θ_a and ψ_a is limited by the condition that the initial rate of growth of "the switching function" be positive; this condition can be expressed analytically.[2]

THRUST-COAST-THRUST TRANSFERS (TCT)

With a program, named ORVAL, written in FORTRAN IV for IBM7040 it has been possible to draw Fig. 1. It is a region of initial values allowing a TCT transfer, starting from a circular orbit. This region is very narrow and therefore difficult to delimitate. Some points, corresponding to different r_b (radius of a circular orbit of arrival) are plotted on this figure. Another program (named CATI) determines ψ_a and θ_a for an imposed transfer. The values of ψ_a and θ_a are automatically improved by a convergent process to satisfy the end conditions. This is achieved, classically, by experimental measurement of the sensitivities of end conditions to small increments in initial values, followed by linear corrections. After several iterations, it is possible to obtain the initial values with sufficient accuracy. A typical co-circular transfer obtained with the CATI program will be illustrated. The radius of departure is 1·0 and the radius of the orbit of arrival is 1·5. The following figures show the evolution of some of the variables.

Figure 2 presents μ and r against θ (r reduced radial distance).

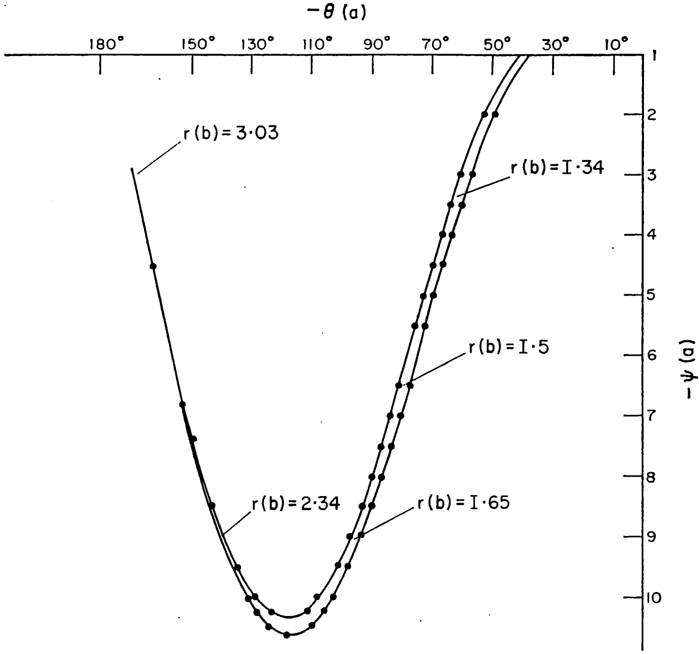

FIG. 1. $\theta(a)$ (in degrees) against $\psi(a)$ (in degrees multiplied by 10).

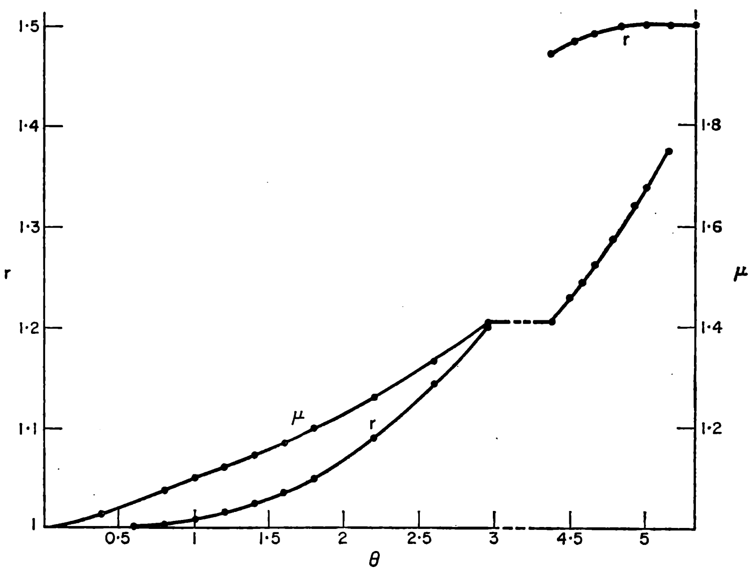

FIG. 2. r and μ against θ (in radians).

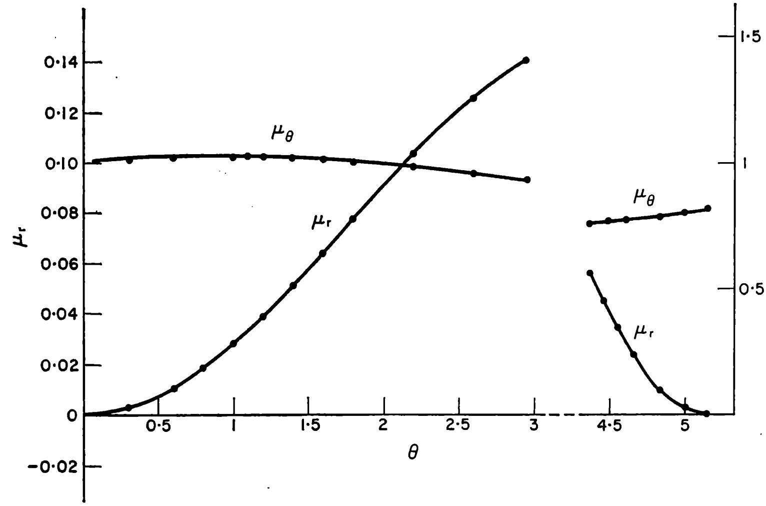

FIG. 3. u_θ and u_r against θ (in radians).

Figure 3 presents, against θ, the tangential velocity u_θ and the radial velocity u_r.

Figure 4 presents the optimal thrust orientation ψ against θ.

Figure 5 is a picture of the co-circular transfer in polar coordinates. The thrusted arcs are the solid curves, the coasting arc is dashed.

Figure 6 also depicts a TCT transfer but the radius of the final orbit is raised to 3·03.

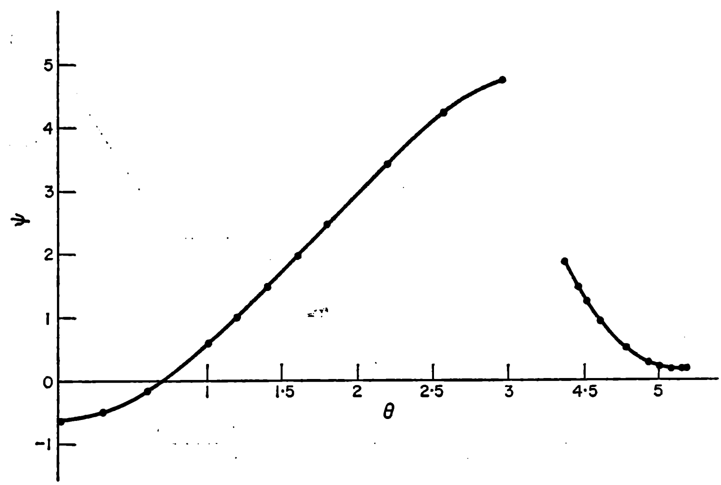

FIG. 4. ψ (in degrees) against θ (in radians).

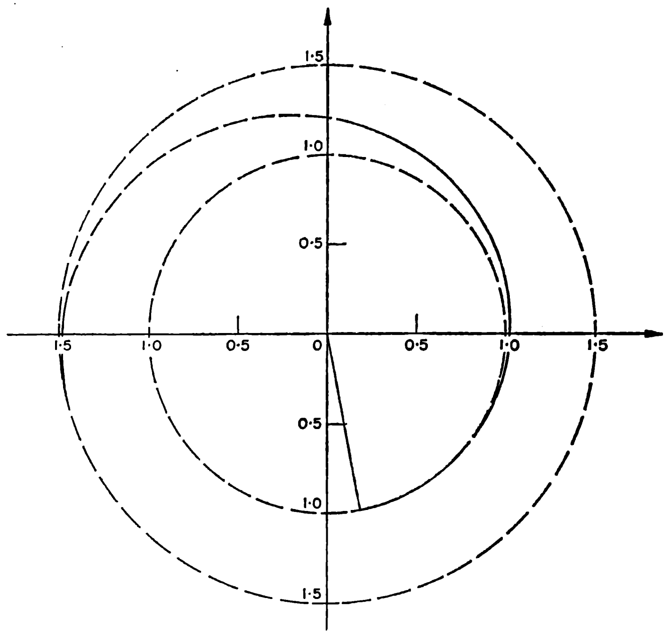

FIG. 5. Co-circular transfer with $r_b = 1{\cdot}5$.

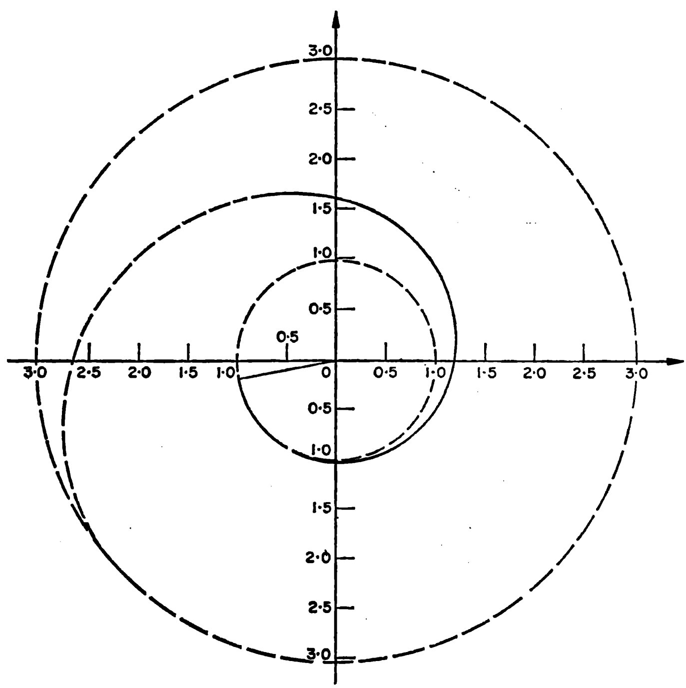

FIG. 6. Co-circular transfer with $r_b = 3{\cdot}03$.

COMPARISON BETWEEN TCT TRANSFERS, TCTCT TRANSFERS AND HOHMANN BI-IMPULSE TRANSFERS

If m_{bH} is the residual mass at arrival for a Hohmann bi-impulsive transfer and m_b the residual for a TCT transfer:

$$\varepsilon = 1 - \frac{m_b}{m_{bH}}$$

is the relative penalty of propellant expenditure.

Figure 7 shows how this penalty increases in a co-circular transfer with the increase in the ratio of radii.

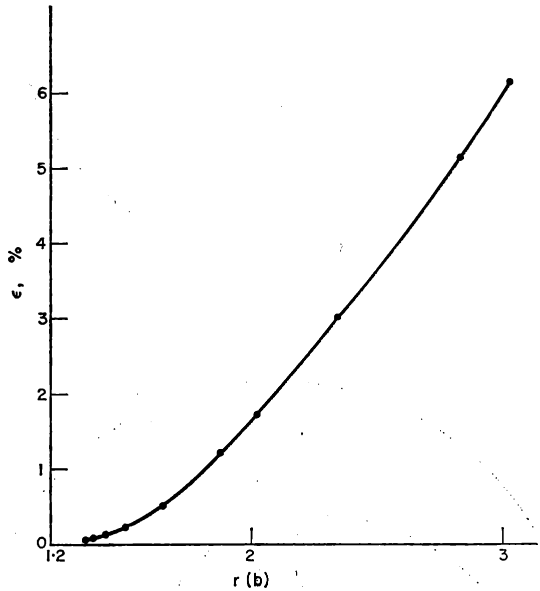

FIG. 7. ε against $r(b)$.

A comparison was also made between several types of co-circular transfers by varying the number of coasting arcs. The ratio of radii was held fixed at 1·5. A simple TCT transfer was compared with two TCTCT transfers (containing two coasting arcs). The best one of those, which has a first symmetrical jump followed by an antisymmetrical, reduced the penalty by a factor of 4.

The residual masses at arrival orbit (for $r_b = 1{\cdot}5$) were found to be respectively

$m_{bH} = 0{\cdot}5696$ (reference Hohmann value)
$m_{TCT} = 0{\cdot}5684$
$m_{TCTCT} = 0{\cdot}5693$

CONCLUSIONS

The characteristics of the orbit of arrival are extremely sensitive to the initial parameters θ_a and ψ_a.

The penalty in propellant expenditure due to finite thrust is either small or can be made so, even for large energy changes, by introducing a second coasting arc.

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