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LARGE DISPLACEMENT FORMULATIONS FOR ELASTIC BODIES

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ABSTRACT.

The report discusses the strain measures for large displacements in both lagrangian and eulerian variables. Similarly, relations are established between lagrangian and eulerian variations of fields with a view to construct eulerian variational principles from the known lagrangian principle of (large) displacements. Various definitions of stress tensors are also discussed and related.

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LIST OF ABREVIATIONS AND SYMBOLS.

a_i lagrangian coordinates

x_i eulerian coordinates

$D_i = \partial/\partial a_i$

$\partial_j = \partial/\partial x_j$

$u_i = x_i - a_i$ displacement coordinates

Jacobian matrices

$J = \{ D_j x_i \}$ $A = \{ D_j u_i \}$

$M = \{ \partial_j a_i \}$ $X = \{ \partial_j u_i \}$

Strain tensors

$\{ \alpha_{mn} \} = G_a = \frac{1}{2} (A + A' + A'A)$ (Green)

$\{ \hat{\alpha}_{mn} \} = \hat{G}_a = \frac{1}{2} (A + A' + AA')$ (rotated Green)

$\{ \xi_{mn} \} = G_x = \frac{1}{2} (X + X' - X'X)$ (Almansi)

$\{ \hat{\xi}_{mn} \} = \hat{G}_x = \frac{1}{2} (X + X' - XX')$ (rotated Almansi)

Variations

δ_a lagrangian

δ_x eulerian

Stresses

$\{ t_{ij} \} = T_a$ lagrangian

$\{ s_{im} \} = S_a$ Kirchhoff-Trefftz

$\{ \tau_{ij} \} = T_x$ true stresses

$\{ \sigma_{im} \} = S_x$

I. Lagrangian measure of strain.

The physical concepts will be more transparent if the additional complications introduced by curvilinear coordinates are avoided. Consequently a single cartesian reference frame will be used for positioning the points P of the (elastic) body in any of its configurations, in particular the initial or "reference" configuration and the final configuration.

The initial coordinates a_i $(i = 1, 2, 3)$ of a given point will become x_i in the final configuration. The displacement vector components will then be

$$u_i = x_i - a_i \, .$$

In the so-called lagrangian point of view the final coordinates are considered to be functions of the initial :

$$x_i = x_i(a_j) = a_i + u_i\,(a_j) \qquad \text{(I.I)}$$

and the geometrical relationship between a neighborhood (da_i) of P in the reference state and (dx_i) in the final state

$$dx_i = D_j\, x_i\, da_j = da_i + D_j\, u_i\, da_j \qquad \text{(I.2)}$$

involves either one of the jacobian matrices

$$J = \{\ D_j\, x_i\ \} \qquad \text{or} \qquad A = \{\ D_j\, u_i\ \}$$

Introducing the column matrices

$$da = \{\ da_i\ \} \qquad \text{and} \qquad dx = \{\ dx_i\ \}$$

equations (I.2) are equivalent to the matrix equations

$$dx = J\, da = (E + A)\, da \qquad \text{(I.3)}$$

with E , the 3 x 3 identity matrix.
The differential base vectors of the neighborhood, in its reference state, are represented by the column matrices

$$da_1 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad , \quad da_2 \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad , \quad da_3 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

and, according to (I.3), become the final state vectors represented by

$$dx_{(i)} = J_{(i)}\, da_i \qquad (i = 1, 2, 3)$$

where $J_{(i)}$ denotes the i-th column of the jacobian matrix J .
The initial volume element $da_1\, da_2\, da_3$ becomes the determinant

$$|\, dx_{(1)}\, dx_{(2)}\, dx_{(3)} \,| = |\, J_{(1)}\, J_{(2)}\, J_{(3)} \,|\, da_1\, da_2\, da_3 = |\, J \,|\, da_1\, da_2\, da_3 \tag{I.4}$$

At any stage of a physically possible transition from reference to final state, the volume element cannot vanish, hence we have the property

$$|\, J \,| > 0 \tag{I.5}$$

The square of the distance between neighboring points undergoes the following change

$$dx'dx - da'da = da'\,(J'J - E)\, da = 2\, da'\, G_a\, da \tag{I.6}$$

Expressed as a quadratic form in the da_i , it introduces the cartesian representation of Green's strain tensor

$$G_a = \frac{1}{2}\,(J'J - E) = \frac{1}{2}\,(A + A' + A'A) = G_a' \tag{I.7}$$

which is the most commonly used lagrangian measure of local deformation when one wishes to deal with large displacements. The justification of G_a as a measure of the strain is that if the neighborhood transformation is of a rigid body type, the distance between any two points of the neighborhood is not altered and the elements of G_a are necessarily zero. Conversely, if the elements of G_a vanish, the distances are unaltered and the neighborhood transformation is either a rigid body type of displacement or a rigid body displacement followed (or preceded) by a "reflection".

The last case is however ruled out by condition (I.5). This is perhaps made clearer by considering the polar decomposition of (I.3).

If we consider the vector da to be first transformed into a vector dy by a pure strain

$$dy = (E + H) da \qquad H = H' \tag{I.8}$$

with the physical limitation

$$|E + H| > 0 \tag{I.9}$$

and subsequently dy transformed into dx by a true rotation

$$dy = U\, dx$$

by orthogonal matrix U

$$UU' = U'U = E \tag{I.10}$$

$$|U| = 1 \tag{I.11}$$

(The case $|U| = -1$ is known to represent a rotation followed or preceded by a reflection); then we have

$$dx = U (E + H) da$$

and the question arises whether the polar decomposition

$$J = U\ (E + H) \tag{I.12}$$

is always possible and unique. We note first that, because of (I.11), (I.5) and (I.9) imply each other. Further, in view of (I.8) and (I.10)

$$J'J = (E + H)\ U'U\ (E + H) = (E + H)^2$$

and H is related to Green's tensor by

$$G_a = H + \frac{1}{2} H^2 \tag{I.13}$$

Wathever be the rotation U, H determines uniquely G_a. The converse is also true as can be shown if we reduce the symmetrical matrix H to diagonal form by a principal axes transformation.

Let $\begin{vmatrix} h_1 & o & o \\ o & h_2 & o \\ o & o & h_3 \end{vmatrix}$ be the diagonal form of H in principal axes.

Then H^2 reduces to the diagonal form $\begin{vmatrix} h_1^2 & o & o \\ o & h_2^2 & o \\ o & o & h_3^2 \end{vmatrix}$

and (I.13) shows that G_a reduces to a diagonal form with elements

$$\gamma_i = h_i + \frac{1}{2} h_i^2 \qquad (i = 1, 2, 3) \tag{I.14}$$

From (I.7) we have

$$2\ G_a + E = J'J$$

hence also

$$| 2 G_a + E | = (2 \gamma_1 + 1)(2 \gamma_2 + 1)(2 \gamma_3 + 1) = | J |^2 > 0 \quad (I.15)$$

We conclude that

$$2 \gamma_i + 1 > 0 \qquad (i = 1, 2, 3) \qquad (I.16)$$

because if we consider the transformation from initial to final state to take place continuously (I.15) must be true at any intermediate stage and none of the factors $(2 \gamma_i + 1)$, which are equal to 1 at the beginning, can become negative without passing through zero. In such a case however the volume element would become zero at some intermediate stage, which is physically impossible.

The solution of (I.14)

$$h_i = - 1 + \sqrt{1 + 2 \gamma_i} \qquad (i = 1, 2, 3) \qquad (I.17)$$

is then always real, while the positive sign before the radical must be chosen in order that

$$1 + h_i > 0 \qquad (i = 1, 2, 3) \qquad (I.18)$$

at any stage of the transformation. This is necessary and sufficient, in order to satisfy at all stages the requirement (I.9). Hence for any physically possible G_a matrix, equation (I.13)has a unique real physically acceptable solution for H. Moreover from (I.9) and (I.12) we can then calculate

$$U = J (E + H)^{-1}$$

and verify easily that U satisfies (I.10) and (I.11).

The matrix H is a cartesian representation of another lagrangian strain tensor, which can also serve as a measure of the local deformation. It is however more difficult to use, since, in the case of large displacements it is not so directly expressible in terms of displacement gradients as G_a.

In fact it requires the solution of the principal axes problem to be determined. For very small strains however both measures are identical, since the elements of H^2 become negligible before those of H.
If we add the assumption of small rotations to that of small strains, the rotation matrix U can be written as

$$U = E + K$$

where the elements of K are very small. Then

$$U'U = E \qquad \text{becomes} \qquad (E + K')(E + K) = E$$

or, neglecting the second order elements in $K'K$, $K' + K = 0$. In other words K is an antisymmetrical matrix. We then have from (I.12)

$$J = E + A = (E + K)(E + H) = E + K + H + KH$$

Again, neglecting the second order elements in KH,

$$A = K + H$$

Hence

$$H = \frac{1}{2}(A + A')$$

which is the usual approximation of linear elasticity theory.
The polar decomposition gives a further important information concerning the physical significance of G_a and H. The order in which the straining and rotation operators is taken shows that G_a and H are the representations of tensors in the cartesian reference frame if everything is described in the reference or initial configuration. In the final configuration they are the representations of the same tensors in axes which have been locally turned by the material rotation. Should we desire to have their representations in the final configuration for the cartesian reference frame, we should invert the order of the operators in the polar decomposition :

$$J = U\,(E + H)\,U'U = (E + UHU')\,U = (E + \hat{H})\,U \qquad (I.19)$$

and it appears immediatly that

$$\hat{H} = UHU' \qquad \hat{G}_a = U\,G_a\,U' \qquad (I.20)$$

are the required answers. Also, from (I.19) it follows that

$$JJ' = (E + A)(E + A') = (E + \hat{H})^2$$

and

$$\hat{G}_a = \hat{H} + \frac{1}{2}\hat{H}^2 = \frac{1}{2}\,(A + A' + AA') \qquad (I.21)$$

In other words the "rotated" representation of the Green's tensor is as easily expressed in terms of the displacement gradients as G_a itself.

2. Eulerian measure of strain.

In the so-called eulerian point of view, the reference coordinates are considered to be functions of the final coordinates

$$a_i = a_i\,(x_j) = x_i - u_i\,(x_j) \qquad (i = 1, 2, 3) \qquad (2.1)$$

Between neighborhoods we have the relation

$$da_i = \partial_j\,a_i\,dx_j = dx_i - \partial_j\,u_i\,dx_j \qquad (2.2)$$

or, in matrix form

$$da = M\,dx = (E - X)\,dx \qquad (2.3)$$

with the inverse jacobian matrices

$$M = \{\,\partial_j\,a_i\,\} \qquad \text{and} \qquad X = \{\,\partial_j\,u_i\,\}$$

$$MJ = JM = E \tag{2.4}$$

or
$$A - X = AX = XA \tag{2.5}$$

Investigation of the alteration of distances in a neighborhood, expressed as a quadratic form in the dx :

$$dx'dx - da'da = dx' \ (E - M'M) \ dx = 2 \ dx' \ G_x \ dx \tag{2.6}$$

introduces an eulerian measure of the local deformation known as the strain tensor of ALMANSI :

$$G_x = \frac{1}{2} (E - M'M) = \frac{1}{2} (X + X' - X'X) \tag{2.7}$$

More correctly, G_x is the representation of an eulerian tensor for a giver reference frame in each of the initial or final configurations. From its preceding derivation we must expect the reference frame to be the cartesian frame in the final configuration. In other words G_x must be in one-one correspondence with $\hat{G}_a$. Indeed, from

$$(E + 2 \ \hat{G}_a) = JJ' \tag{2.8}$$

follows

$$(E + 2 \ \hat{G}_a)^{-1} = (J^{-1})' \ J^{-1} = M'M = E - 2 \ G_x \tag{2.9}$$

The order of the operators is the inverse one, described by (I.I8), or, expressed in terms of a polar decomposition of M

$$M = J^{-1} = U' \ (E + \hat{H})^{-1} \tag{2.I0}$$

Conversely, the direct order of operators

$$M = (E + H)^{-1} U' \tag{2.11}$$

becomes related to

$$\hat{G}_x = U' G_x U = \frac{1}{2}\left(E - (MU)' MU\right) = \frac{1}{2}\left(E - (E + H)^{-2}\right) \tag{2.12}$$

Hence $\hat{G}_x$ describes the eulerian strain tensor in the cartesian frame for the reference configuration and is in a one-one correspondence with G_a

$$E - 2\hat{G}_x = (E + H)^{-2} = (E + 2G_a)^{-1} \tag{2.13}$$

Since from (2.11) we have

$$MM' = (E + H)^{-2}$$

then it follows that

$$\hat{G}_x = \frac{1}{2}(E - MM') = \frac{1}{2}(X + X' - XX') \tag{2.14}$$

3. Some useful formulas for the strain tensors.

From the definitions (1.7), (1.20), (2.7) and (2.14) the elements of the lagrangian and eulerian strain tensors can be written as follows

$$G_a = \{\alpha_{mn}\} \qquad 2\alpha_{mn} = D_m x_i D_n x_i - \delta_{mn} = D_m u_n + D_n u_m + D_m u_i D_n u_i \tag{3.1}$$

$$\hat{G}_a = \{\hat{\alpha}_{mn}\} \qquad 2\hat{\alpha}_{mn} = D_i x_m D_i x_n - \delta_{mn} = D_m u_n + D_n u_m + D_i u_m D_i u_n \tag{3.2}$$

$$G_x = \{\xi_{mn}\} \qquad 2\xi_{mn} = \delta_{mn} - \partial_m a_i \partial_n a_i = \partial_m u_n + \partial_n u_m - \partial_m u_i \partial_n u_i \tag{3.3}$$

$$\hat{G}_x = \{\hat{\xi}_{mn}\} \qquad 2\hat{\xi}_{mn} = \delta_{mn} - \partial_i a_m \partial_i a_n = \partial_m u_n + \partial_n u_m - \partial_i u_m \partial_i u_n \tag{3.4}$$

The reciprocity between the jacobian matrices J and M is expressed by

$$JM = E \qquad \text{or} \qquad D_j\, x_i\, \partial_r\, a_j = \delta_{ir} \tag{3.5}$$

$$MJ = E \qquad \text{or} \qquad \partial_j\, a_i\, D_r\, x_j = \delta_{ir} \tag{3.6}$$

Between the Green and Almansi tensors we have the relations

$$M' G_a\, M = G_x \qquad \text{or} \qquad \alpha_{mn}\, \partial_r\, a_m\, \partial_j\, a_n = \xi_{rj} \tag{3.7}$$

$$J' G_x\, J = G_a \qquad \text{or} \qquad \xi_{rj}\, D_m\, x_r\, D_n\, x_j = \alpha_{mn} \tag{3.8}$$

$$J' G_x = G_a\, M \qquad \text{or} \qquad G_x\, J = M' G_a \qquad \text{or} \qquad \alpha_{ms}\, \partial_r\, a_m = \xi_{rj}\, D_s\, x_j \tag{3.9}$$

Formulas (2.9) and (2.13) are equivalent to

$$\hat{G}_a - G_x = 2\, \hat{G}_a\, G_x = 2\, G_x\, \hat{G}_a$$

$$G_a - \hat{G}_x = 2\, G_a\, \hat{G}_x = 2\, \hat{G}_x\, G_a$$

and show that for very small strains $\hat{G}_a$ and G_x are equivalent, G_a and $\hat{G}_x$ likewise. If we add the assumption of small rotations all the gradients $D_m\, u_i$ and $\partial_m\, u_i$ become small and

$$\hat{\alpha}_{mn} = \alpha_{mn} = \frac{1}{2}\,(D_m\, u_n + D_n\, u_m)$$

become equivalent to

$$\hat{\xi}_{mn} = \xi_{mn} = \frac{1}{2}\,(\partial_m\, u_n + \partial_n\, u_m)$$

4. Lagrangian and eulerian variations.

Consider a familly of configuration changes which depends on a parameter ε :

$$x_i = x_i (a_j , \varepsilon)$$

A lagrangian variation of the configuration that takes place for $\varepsilon = 0$ is defined by

$$\delta_a x_i = d\varepsilon \left(\frac{\partial x_i}{\partial \varepsilon} \right)_{\varepsilon = 0} \tag{4.I}$$

As the notation suggests, the lagrangian coordinates a_j are kept constant and the variation represents the change in coordinates of a given point P, or particle, originally located at (a_1, a_2, a_3) when passing from the final configuration for $\varepsilon = 0$ to the one where ε is given a small increment $d\varepsilon$. It is therefore also called "material variation". Since

$$x_i = a_i + u_i (a_j , \varepsilon)$$

we note that

$$\delta_a x_i = \delta_a u_i \tag{4.2}$$

If the familly of configuration changes is presented in the form

$$a_i = a_i (x_j , \varepsilon)$$

an eulerian variation is defined by

$$\delta_x a_i = d\varepsilon \left(\frac{\partial a_i}{\partial \varepsilon} \right)_{\varepsilon = 0} \tag{4.3}$$

Here, the final coordinates are kept unchanged and the variation represents the change in label of the particle that will finally occupy the same position. With respect to the final configuration it is therefore also called a "local variation". Since

$$a_i = x_i - u_i\ (a_j\ ,\ \varepsilon\ )$$

we note that

$$\delta_x\ a_i = -\ \delta_x\ u_i \tag{4.4}$$

Similarly, any function which is attached to a particle but also depends on the parameter ε, can be subject to a material variation

$$\delta_a\ f = d\varepsilon\ \left(\frac{\partial f}{\partial \varepsilon}\right)_a \tag{4.5}$$

where the particle label (a_1, a_2, a_3) is unchanged. If, on the contrary, we keep the final coordinates unchanged

$$\delta_x\ f = \delta_a\ f + D_i\ f\ \delta_x\ a_i \tag{4.6}$$

Conversely, considering f to be a function of the x_i and ε

$$\delta_a\ f = \delta_x\ f + \partial_i\ f\ \delta_a\ x_i \tag{4.7}$$

Formulas (4.6) and (4.7) applied respectively to x_j and a_j, yield

$$\delta_a\ x_j + D_i\ x_j\ \delta_x\ a_i = \delta_x\ x_j = 0 \tag{4.8}$$

$$\delta_x\ a_j + \partial_i\ a_j\ \delta_a\ x_i = \delta_a\ a_j = 0 \tag{4.9}$$

They relate the material and local variations of the coordinates. In matrix form they are

$$\delta_a\ x + J\ \delta_x\ a = 0 \qquad\qquad \delta_x\ a + M\ \delta_a\ x = 0 \tag{4.IO}$$

In view of (4.2) and (4.4) they can also be written

$$\delta_a u_j = D_i x_j \delta_x u_i \qquad \text{or} \qquad \delta_a u = J \delta_x u \tag{4.11}$$

$$\delta_x u_j = \partial_i a_j \delta_a u_i \qquad \text{or} \qquad \delta_x u = M \delta_a u \tag{4.12}$$

It is important to observe that, since the application of the operator δ_a does not change the a_j, it may be commuted with the application of the operators D_i :

$$\delta_a D_i f = D_i \delta_a f \tag{4.13}$$

Similarly the operator δ_x is taken with constant x_j and commutes with the operators ∂_i :

$$\delta_x \partial_i f = \partial_i \delta_x f \tag{4.14}$$

We still need to investigate variations of the volume elements. Let

$$d R_a = da_1 \, da_2 \, da_3$$

denote the volume element in the reference configuration. According to (I.4) the volume element in the final configuration is

$$d R_x = | J | \, d R_a \tag{4.15}$$

and we proceed to calculate its material variation, that is the change in the volume occupied by the same collection of particles when the final configuration is slightly altered. Obviously $\delta_a d R_a = 0$ and

$$\delta_a d R_x = d R_a \delta_a | J | \tag{4.16}$$

To calculate the material variation of the jacobian determinant, introduce c_{rm}, the cofactor of $D_r x_m$ in $| J |$; the expansion rules of a determinant allow us to write

$$c_{rm}\, D_r\, x_j = |\, J \,|\, \delta_{mj}$$

Multiply this by $\partial_j\, a_i$ and use (3.6) to find

$$c_{rm}\, \delta_{ir} = |\, J \,|\, \delta_{mj}\, \partial_j\, a_i \qquad \text{or} \qquad c_{im} = |\, J \,|\, \partial_m\, a_i \tag{4.17}$$

We now expand

$$\delta_a\, |\, J \,| = c_{im}\, \delta_a\, (D_i\, x_m) = c_{im}\, D_i\, (\delta_a\, x_m) = |\, J \,|\, \partial_m\, a_i\, D_i\, (\delta_a\, x_m)$$

or, finally,

$$\delta_a\, |\, J \,| = |\, J \,|\, \partial_m\, (\delta_a\, x_m) \tag{4.18}$$

This result and (4.15) substituted into (4.16) gives

$$\delta_a\, d\, R_x = d\, R_x\, \partial_m\, (\delta_a\, x_m) \tag{4.19}$$

From completely similar calculations applied to

$$d\, R_a = |\, M \,|\, d\, R_x \tag{4.20}$$

we find that

$$\delta_x\, |\, M \,| = |\, M \,|\, D_m\, (\delta_x\, a_m) \tag{4.21}$$

and

$$\delta_x\, d\, R_a = d\, R_a\, D_m\, (\delta_x\, a_m) \tag{4.22}$$

The variation of volume integrals does not present special difficulties. Consider the integral

$$I_a = \int_{R_a} f_a\, d\, R_a \tag{4.23}$$

Its lagrangian variation is obviously

$$\delta_a I_a = \int_{R_a} (\delta_a f_a) \, d R_a \tag{4.24}$$

Its eulerian variation can be processed as follows, using (4.6) and (4.22),

$$\delta_x I_a = \int_{R_a} \delta_x (f_a \, d R_a) = \int_{R_a} (\delta_a f_a + D_i f_a \delta_x a_i + f_a D_i \delta_x a_i) \, d R_a$$

$$= \int_{R_a} (\delta_a f_a) \, d R_a + \int_{R_a} D_i (f_a \delta_x a_i) \, d R_a$$

The last term is in the form of a divergence integral and is converted to a surface integral over the boundary surface ∂R_a that limits the reference configuration. If (ℓ_i) denotes the direction cosines of the outward normal on ∂R_a :

$$\delta_x I_a = \delta_a I_a + \int_{\partial R_a} f_a (\ell_i \delta_x a_i) \, d \partial R_a \tag{4.25}$$

A similar transformation can be applied to volume integral extended to the final configuration

$$I_x = \int_{R_x} f_x \, d R_x \tag{4.26}$$

on the one hand

$$\delta_x I_x = \int_{R_x} (\delta_x f_x) \, d R_x \tag{4.27}$$

on the other hand, using (4.7) and (4.19),

$$\delta_a I_x = \int_{R_x} \delta_a (f_x \, d R_x) = \int_{R_x} (\delta_x f_x + \partial_i f_x \delta_a x_i + f_x \partial_i \delta_a x_i) \, d R_x$$

$$= \int_{R_x} (\delta_x f_x) \, d R_x + \int_{R_x} \partial_i (f_x \delta_a x_i) \, d R_x$$

or, finally, with (n_i) denoting the direction cosines of the outward normal to ∂R_x

$$\delta_a I_x = \delta_x I_x + \int_{\partial R_x} f_x (n_i \delta_a x_i) \, d \, \partial R_x \qquad (4.28)$$

Note that $I_a = I_x$ if $f_x = | M | f_a$ or $f_a = | J | f_x$. Hence, by comparing (4.25) and (4.28),

$$\int_{\partial R_a} f_a (\ell_i \delta_x a_i) \, d \, \partial R_a + \int_{\partial R_x} | M | f_a (n_i \delta_a x_i) \, d \, \partial R_x = 0 \qquad (4.29)$$

or, in view of (4.2) and (4.4)

$$\int_{\partial R_a} f_a (\ell_i \delta_x u_i) \, d \, \partial R_a = \int_{\partial R_x} | M | f_a (n_i \delta_a u_i) \, d \, \partial R_x \qquad (4.30)$$

5. Stress tensors.

Conceptually, the simplest stress definition is by reference to the cartesian metric in the reference state of the elastic body. The force $\vec{F}_i$ on a facet which was originally perpendicular to the cartesian axis of subscript i, is divided by the original area of the facet and the resulting cartesian components denoted by t_{ij}. Thus, for a facet whose outward normal in the reference configuration has the direction cosines (1, 0, 0), the surface traction has the components $(t_{11} \, da_2 \, da_3 \, , \, t_{12} \, da_2 \, da_3 \, , \, t_{13} \, da_2 \, da_3)$. The virtual work produced by this surface traction in a material variation of the displacement field is $(t_{1j} \delta_a u_j) \, da_2 \, da_3$. The virtual work of all surface tractions on some element, occupying the volume of a parallelopiped in the reference configuration, is then equal to

$$D_i (t_{ij} \delta_a u_j) \, da_1 \, da_2 \, da_3 = D_i (t_{ij} \delta_a u_j) \, d R_a$$

External forces distributed in the volume of the body, defined per unit volume in the reference configuration, will have cartesian components denoted by A_j and will produce a virtual work

$$A_j \delta_a u_j \, d R_a \, .$$

If W_a denotes the internal energy per unit volume in the reference configuration, the statement of conservation of energy for a hyperelastic body will be

$$\begin{aligned} \delta_a W_a &= D_i (t_{ij} \delta_a u_j) + A_j \delta_a u_j \\ &= \delta_a u_j (D_i t_{ij} + A_j) + t_{ij} D_i (\delta_a u_j) \end{aligned} \tag{5.I}$$

In translation modes, where each $\delta_a u_j$ is a constant, there is no energy increase and consequently

$$D_i t_{ij} + A_j = 0 \qquad (j = 1, 2, 3) \tag{5.2}$$

When these translational equilibrium equations, satisfied by the "lagrangian" stresses t_{ij}, are taken into account together with the commutativity properties between the operators D_i and δ_a, equation (5.I) becomes

$$\delta_a W_a = t_{ij} \delta_a (D_i u_j) \tag{5.3}$$

Hence the right hand side is a total differential and suggests that the energy per unit reference volume should be considered a function of the nine displacement gradients $D_i u_j$ and the lagrangian stresses as the partial derivatives of the energy with respect to such variables.

$$t_{ij} = \partial W_a / \partial D_i u_j \tag{5.4}$$

The nature of W_a as a function of the gradients can be made more precise by considering that $\delta_a W_a = 0$ also when the $\delta_a u_j$ field is one of a rigid rotation. In expressing the rigid body rotation field due consideration must be given to the fact that it must be superimposed on a configuration where the particles are already displaced from their reference configuration. Hence

$$\begin{aligned} &\delta_a u_1 = x_3 \delta \omega_2 - x_2 \delta \omega_3 \quad \delta_a u_2 = x_1 \delta \omega_3 - x_3 \delta \omega_1 \\ &\delta_a u_3 = x_2 \delta \omega_1 - x_1 \delta \omega_2 \end{aligned} \tag{5.5}$$

where $$x_i = a_i + u_i\,(a_j)$$

Equating to zero the coefficients of $\delta\omega_1$, $\delta\omega_2$ and $\delta\omega_3$ in the right-hand side of (5.3) produces the rotational equilibrium equations

$$t_{i3}\,D_i\,x_2 - t_{i2}\,D_i\,x_3 = 0 \qquad t_{i1}\,D_i\,x_3 - t_{i3}\,D_i\,x_1 = 0 \,,$$
$$t_{i2}\,D_i\,x_1 - t_{i1}\,D_i\,x_2 = 0 \tag{5.6}$$

which must be satisfied by the lagrangian stresses. They show that those stresses are not symmetrical and, in the matrix T_a which represents the lagrangian stresses in the cartesian reference frame, we must distinguish the row index i and the column index j

$$T_a = \{\ t_{ij}\ \}$$

Equations (5.6) are then equivalent to a statement of symmetry of the matrix $J\,T_a$

$$(J\,T_a)' = T'_a\,J' = J\,T_a \tag{5.7}$$

They can also be interpreted as a statement that the energy W_a is a function of the displacement gradients through the elements α_{mn} of the Green's strain matrix. If, for instance, one substitutes (5.4) into (5.6), the three resulting partial differential equations of first order for W_a are easily solved by the method of characteristics and indicate that $W_a = W_a\,(\alpha_{mn})$. More directly, introduce a new stress matrix $S_a = \{\ s_{im}\ \}$ by

$$T_a = S_a\,J' \qquad \text{or} \qquad t_{ij} = s_{im}\,D_m\,x_j \tag{5.8}$$

so that, by virtue of (5.7) S_a is symmetrical. Substitute into (5.3)

$$\delta_a\,W_a = s_{im}\,D_m\,x_j\,\delta_a\,(D_i\,u_j) = s_{im}\,D_m\,x_j\,\delta_a\,(D_i\,x_j)$$

Exchange the summation indices i and m

$$\delta_a W_a = s_{mi} D_i x_j \delta_a (D_m x_j)$$

and add the two formulas using the symmetry property of s_{im} :

$$\delta_a W_a = \frac{1}{2} s_{im} \delta_a (D_m x_j D_i x_j) = s_{im} \delta_a \alpha_{im} \qquad (5.9)$$

It is now clear that W_a is a function of the six elements

$$\alpha_{ii} \qquad (i = 1, 2, 3)$$

$$\alpha_{im} + \alpha_{mi} \qquad (i \neq m)$$

and that the Kirchhoff-Trefftz stresses s_{im} are the partial derivatives

$$\begin{aligned} s_{ii} &= \partial W_a / \partial \alpha_{ii} && (i = 1, 2, 3) \\ s_{im} &= \partial W_a / \partial (\alpha_{im} + \alpha_{mi}) = s_{mi} && (m \neq i) \end{aligned} \qquad (5.10)$$

The symmetrical general notation

$$s_{im} = \partial W_a / \partial \alpha_{im} \qquad (5.11)$$

is not ambiguous if we specify that α_{im} and α_{mi} , although equal, should be treated as independant variables. The equilibrium equations satisfied by the symmetrical Kirchhoff-Trefftz stresses were given by SIGNORINI; from (5.2) and (5.4) they are

$$D_i (s_{im} D_m x_j) + A_j = 0 \qquad (j = 1, 2, 3) \qquad (5.12)$$

The Kirchhoff-Trefftz stresses can be interpreted geometrically as defined by unit surface in the reference configuration but decomposed in the metric induced by the deformation. In the initial configuration a set of unit vectors parallel to the cartesian axes in P becomes, in the final configuration, a set of local base vectors $\vec{g}_j$ of cartesian components $D_j\ x_i$, each representing a column of the jacobian matrix J as already observed in section I . The force $\vec{F}_1$ on the facet of initial outward normal (1, 0, 0) is then resolved as follows

$$\vec{F}_1 = (s_{11}\ \vec{g}_1 + s_{12}\ \vec{g}_2 + s_{13}\ \vec{g}_3)\ da_2\ da_3$$

and similarly for the other facets. Projecting this vectorial relation on the cartesian axes and remembering the definition of the lagrangian stresses : one obtains :

$$t_{11} = s_{11}\ D_1\ x_1 + s_{12}\ D_2\ x_1 + s_{13}\ D_3\ x_1$$

$$t_{12} = s_{11}\ D_1\ x_2 + s_{12}\ D_2\ x_2 + s_{13}\ D_3\ x_2$$

$$t_{13} = s_{11}\ D_1\ x_3 + s_{12}\ D_2\ x_3 + s_{13}\ D_3\ x_3$$

This agrees with the previously introduced general definition (5.8).
The two stress tensors encountered so far are fundamentally of lagrangian nature. Stress tensors of eulerian nature can also be used. The most obvious one is the "true stresses tensor" in which we consider an elementary parallelopiped with facets perpendicular to the cartesian axes in the final configuration. The true stresses are then defined per unit final area and resolved along the cartesian axes; we denote them by τ_{ij} . The energy conservation principle is then formulated as follows

$$\frac{1}{dR_x}\ \delta_a\ (W_a\ d\ R_a) = \frac{dR_a}{dR_x}\ \delta_a\ W_a = |\ M\ |\ \delta_a\ W_a = \partial_i\ (\tau_{ij}\ \delta_a\ u_j) + X_j\ \delta_a\ u_j \qquad (5.13)$$

where X_j are the body forces per unit final volume

$$X_j\ d\ R_x = A_j\ d\ R_a \qquad \text{or} \qquad X_j = |\ M\ |\ A_j \qquad (5.14)$$

The equilibrium equations implicit in (5.13) are obtained by setting $\delta_a W_a = 0$ when the $\delta_a u_j$ reduce to translations

$$\partial_i \tau_{ij} + X_j = 0 \qquad (j = 1, 2, 3) \qquad (5.15)$$

or to rigid body rotations as expressed by (5.5). In this case, because the partial derivatives of the variational field are taken with respect to the final coordinates, the rotational equilibrium equations reduce to a statement of symmetry of the tensor of true stresses :

$$\tau_{ji} = \tau_{ij} \qquad (5.16)$$

The conservation of energy, taking (5.15) into account, is now

$$| M | \delta_a W_a = \tau_{ij} \partial_i (\delta_a u_j) \qquad (5.17)$$

Because the operators ∂_i and δ_a do not commute, it becomes necessary to manipulate the expression; first $\partial_i = \partial_i a_m D_m$

$$| M | \delta_a W_a = \tau_{ij} \partial_i a_m D_m (\delta_a u_j) = \tau_{ij} \partial_i a_m \delta_a (D_m x_j) \qquad (5.18)$$

Next we separate the dummy subscript j by including a Kronecker symbol δ_{pj}

$$| M | \delta_a W_a = \tau_{ij} \partial_i a_m \delta_a (D_m x_p) \delta_{pj}$$

From (3.5) we can replace $\delta_{pj} = D_n x_p \partial_j a_n$

$$| M | \delta_a W_a = (\tau_{ij} \partial_i a_m \partial_j a_n) D_n x_p \delta_a (D_m x_p)$$

We now make use of the symmetry property of the true stresses; exchanging the dummy subscripts i and j and also m and n :

$$| M | \delta_a W_a = (\tau_{ij} \partial_i a_m \partial_j a_n) D_m x_p \delta_a (D_n x_p)$$

Addition of the two formulas and division by 2 yields finally

$$|M|\ \delta_a W_a = (\tau_{ij}\ \partial_i a_m\ \partial_j a_n)\ \frac{1}{2}\ \delta_a\ (D_m x_p\ D_n x_p)$$

$$= (\tau_{ij}\ \partial_i a_m\ \partial_j a_n)\ \delta_a\ \alpha_{mn} \qquad (5.19)$$

A direct comparison with (5.9) is now possible and furnishes the relations between true stresses and the Kirchhoff-Trefftz stresses :

$$|M|\ s_{mn} = \tau_{ij}\ \partial_i a_m\ \partial_j a_n \qquad (5.20)$$

In matrix formulation, with $T_x = \{ \tau_{ij} \}$,

$$|M|\ S_a = M\ T_x\ M' \qquad \text{or} \qquad |J|\ T_x = J\ S_a\ J' \qquad (5.21)$$

We have seen in section 2 that there is a one-one correspondence between the elements α_{mn} of G_a and $\hat{\xi}_{mn}$ of $\hat{G}_x$. Consequently the energy density W_a , which has been recognized to be a function of the α_{mn} must also be a function of the $\hat{\xi}_{mn}$ and, as such, must generate another system of stresses as partial derivatives. To illustrate this we can start from (3.5) in the form

$$D_m x_j\ \partial_i a_m = \delta_{ji}$$

and, applying the operator δ_a to it, substitute

$$\partial_i a_m\ \delta_a\ (D_m x_j) = -\ D_m x_j\ \delta_a\ (\partial_i a_m) \qquad (5.22)$$

into (5.18) to obtain

$$|M|\ \delta_a W_a = -\ \tau_{ij}\ D_m x_j\ \delta_a\ (\partial_i a_m)$$

Next split the dummy subscript i by use of a Kronecker symbol δ_{ip}

$$|M|\ \delta_a W_a = -\ \tau_{ij}\ D_m x_j\ \delta_a\ (\partial_p a_m)\ \delta_{ip}$$

and replace

$$\delta_{ip} = D_n x_i \partial_p a_n$$

$$| M | \delta_a W_a = - \tau_{ij} D_m x_j D_n x_i \partial_p a_n \delta_a (\partial_p a_m)$$

Again, relying on the symmetry of τ_{ij}, we exchange simultaneously the pairs of indices (i, j) and (m, n)

$$| M | \delta_a W_a = - \tau_{ij} D_m x_j D_n x_i \partial_p a_m \delta_a (\partial_p a_n)$$

and after adding and dividing by 2

$$| M | \delta_a W_a = \tau_{ij} D_m x_j D_n x_i \delta_a (- \frac{1}{2} \partial_p a_m \partial_p a_n)$$

or, in view of (3.4),

$$| M | \delta_a W_a = \tau_{ij} D_m x_j D_n x_i \delta_a \hat{\xi}_{mn} \tag{5.23}$$

The stresses σ_{mn} defined by

$$\delta_a W_a = \sigma_{mn} \delta_a \hat{\xi}_{mn} \tag{5.24}$$

as partial derivatives of the energy density W_a with respect to the elements of $\hat{G}_x$, are thus related to the true stresses by

$$| M | \sigma_{mn} = \tau_{ij} D_m x_j D_n x_i \tag{5.25}$$

In the notation $S_x = \{ \sigma_{mn} \}$

$$| M | S_x = J' T_x J \quad \text{or} \quad | J | T_x = M' S_x M \tag{5.26}$$

The next matrix equations summarize the connexion between the various stress systems

$$J\, T_a = J\, S_a\, J' = |\, J\,|\, T_x = M'\, S_x\, M \qquad (5.27)$$

It is useful to recall that S_a and S_x are generated by the partial derivatives of W_a respectively with respect to the elements of G_a and of $\hat{G}_x$.

6. The partial derivatives of W_x.

From the equality $W_a\, d\, R_a = W_x\, d\, R_x$ follows

$$W_a = |\, J\,|\, W_x \qquad (6.1)$$

and, in view of (4.18),

$$\delta_a\, W_a = |\, J\,| \{ \delta_a\, W_x + W_x\, \partial_m\, (\delta_a\, x_m) \} \qquad (6.2)$$

Any total differential expression obtained for $\delta_a\, W_a$ can consequently be transformed into one for $\delta_a\, W_x$, provided we can put $\partial_m\, (\delta_a\, x_m)$ in the form of a total differential. This can be done as follows :

$$\partial_m\, (\delta_a\, x_m) = \partial_m\, a_j\, D_j\, (\delta_a\, x_m) = \partial_m\, a_j\, \delta_a\, (D_j\, x_m) = \partial_m\, a_j\, \delta_a\, (D_j\, x_p)\, \delta_{mp}$$

$$= \partial_m\, a_j\, \partial_m\, a_i\, D_i\, x_p\, \delta_a\, (D_j\, x_p)$$

exchanging the subscripts i and j, adding and dividing by 2,

$$\partial_m\, (\delta_a\, x_m) = \partial_m\, a_i\, \partial_m\, a_j\, \delta_a\, (\tfrac{1}{2}\, D_i\, x_p\, D_j\, x_p) = \partial_m\, a_i\, \partial_m\, a_j\, \delta_a\, \alpha_{ij} \qquad (6.3)$$

For instance if we substitute into (6.2), (6.3) and (5.9) and solve for $\delta_a\, W_x$:

$$\delta_a\, W_x = (\, |\, M\,|\, s_{ij} - W_x\, \partial_m\, a_i\, \partial_m\, a_j)\, \delta_a\, \alpha_{ij} \qquad (6.4)$$

Hence we can consider W_x, the energy density in the final configuration, to be also a function of the elements of G_a , however its partial derivatives are the quantities between brackets in (6.4).

Similarly, using relations like (5.22),

$$\partial_m (\delta_a x_m) = - D_j x_m \delta_a (\partial_m a_j) = - D_j x_m \delta_a (\partial_p a_j) \delta_{mp}$$

$$= - D_j x_m D_i x_m \partial_p a_i \delta_a (\partial_p a_j) = D_j x_m D_i x_m \delta_a$$

$$(- \frac{1}{2} \partial_p a_i \partial_p a_j)$$

or $$\partial_m (\delta_a x_m) = D_i x_m D_j x_m \delta_a \hat{\xi}_{ij} \qquad (6.5)$$

If we substitute this and (5.24) into (6.2) and solve for $\delta_a W_x$

$$\delta_a W_x = (| M | \sigma_{ij} - W_x D_i x_m D_j x_m) \delta_a \hat{\xi}_{ij} \qquad (6.6)$$

showing what are the partial derivatives of W_x considered as a function of the elements of $\hat{G}_x$. Finally, in view of (5.25), it is clear that the corresponding modifications to apply to the true stresses are simply $\tau_{ij} - W_x \delta_{ij}$.

7. Lagrangian variational principles.

In lagrangian coordinates the principle of variation of displacements stating that the total energy, strain energy plus potential energy of dead loads, is stationary with respect to arbitrary admissible variations of the displacement field is not restricted to the linear elasticity case but is applicable to the large displacement case.

$$\int_{R_a} (W_a - A_j u_j) \, d R_a - \int_{\partial R_a} p_j u_j \, d \partial R_a \quad \text{station.} \qquad (7.1)$$

In this formulation, the region R_a is supposedly simply connected and traction surfaces p_j are imposed on the whole boundary ∂R_a . Furthermore, the energy density W_a is considered to be a known function of the elements

of the Green's strain tensor G_a as defined by (3.I).
From (5.II), considered as the equations defining the Kirchhoff-Trefftz stresses, we obtain

$$\delta_a W_a = s_{im} \delta_a \left(\frac{1}{2} D_i x_j D_m x_j - \frac{1}{2} \delta_{im} \right)$$

$$= s_{im} \frac{1}{2} (D_i x_j \delta_a D_m x_j + D_m x_j \delta_a D_i x_j)$$

and, in view of the symmetry of the stresses,

$$\delta_a W_a = s_{im} D_m x_j \delta_a (D_i x_j) = s_{im} D_m x_j D_i (\delta_a u_j)$$

An integration by parts gives

$$\delta_a \int_{R_a} W_a \, d R_a = \int_{R_a} (\delta_a W_a) \, d R_a = \int_{\partial R_a} \ell_i s_{im} D_m x_j \delta_a u_j \, d \partial R_a$$

$$- \int_{R_a} D_i (s_{im} D_m x_j) \delta_a u_j \, d R_a$$

Hence the material variation of (7.I) produces the Euler-Lagrange variational equations :

$$D_i (s_{im} D_m x_j) + A_j = 0 \qquad (j = 1, 2, 3) \qquad (5.I2)$$

obtained by equating to zero the coefficients of $\delta_a u_j$ in the volume integral. They are the Signorini equilibrium equations.
The natural boundary conditions obtained by equating to zero the coefficients of $\delta_a u_j$ in the surface integral are

$$P_j = \ell_i s_{im} D_m x_j \qquad (j = 1, 2, 3) \qquad (7.2)$$

They furnish of course the definition of surface traction components in terms of the Kirchhoff-Trefftz stresses; they reduce as they should to lagrangian stress components (5.8) when the facet becomes perpendicular to one of the cartesian axes.

A generalization of the principle of variation of displacements to a two-field principle is possible by the method of Friedrichs. We consider W_a to be the same function of the α_{im} but incorporate the constraints

$$\frac{1}{2} (D_i x_j D_m x_j - \delta_{im}) - \alpha_{im} = 0 \tag{7.3}$$

into the variational principle by lagrangian multipliers s_{im} (the similarity in notation with the Kirchhoff-Trefftz stresses is introduced on purpose and justified later). The generalized principle is then

$$\int_{R_a} (W_a (\alpha_{im}) + s_{im} (\frac{1}{2} (D_i x_j D_m x_j - \delta_{im}) - \alpha_{im}) - A_j u_j) \; d R_a$$

$$- \int_{\partial R_a} p_j u_j \; d \partial R_a \qquad \text{station} \tag{7.4}$$

The variations on the multipliers s_{im} restore the constraints (7.3) as Euler-Lagrange equations; the variations on the displacements yield, as before, the equilibrium equations (5.I2) and (7.2); finally the variations on the α_{im} yield as Euler-Lagrange equations

$$s_{im} = \partial W_a / \partial \alpha_{im} \tag{5.II}$$

Hence our multipliers are identified as the Kirchhoff-Trefftz stresses related through W_a to the Green's tensor strains. By incorporating directly this result into the principle (7.4), the α_{im} field can be completely eliminated. It is necessary to this effect to express as a function of the s_{im} the quantity

$$s_{im} \alpha_{im} - W_a (\alpha_{im}) = \Phi_a (s_{im}) \tag{7.5}$$

and this is possible if, as will be postulated, the stress-strain relations (5.II) can be solved for the strains.

Equation (7.5) is then a Legendre contact transformation which defines the lagrangian complementary energy density $\Phi_a (s_{im})$. Formal differentiation of (7.5) produces

$$\delta_a \alpha_{im} (s_{im} - \partial W_a / \partial \alpha_{im}) + \alpha_{im} \delta_a s_{im} = \delta_a \Phi_a .$$

Hence, in view of (5.II) we conclude that

$$\alpha_{im} = \partial \Phi_a / \partial s_{im} \qquad (7.6)$$

In this general formulation of the dual stress-strain relations (7.6) we have implicitly distinguished s_{im} from s_{mi} (for $i \neq m$) and varied them independantly. In practice this distinction is not made and Φ_a is considered to be a function of the six variables s_{ii} and s_{im} $(i < m)$. In this case the stress-strain relations (7.6) must be written

$$\alpha_{ii} = \partial \Phi_a / \partial s_{ii} \qquad (i = 1, 2, 3)$$

$$\alpha_{im} + \alpha_{mi} = 2\, \alpha_{im} = \partial \Phi_a / \partial s_{im} \qquad (i < m)$$

The variational principle (7.4) takes now the form of a two-field variational principle, namely that

$$\int_{R_a} \{ s_{im} \frac{1}{2} (D_i x_j D_m x_j - \delta_{im}) - \Phi_a (s_{im}) - A_j u_j \} \, d R_a - \int_{\partial R_a} P_j u_j \, d \partial R_a$$

is stationary (7.7)

where a displacement field and a stress field may be varied independantly. The Euler-Lagrange equations are (5.I2) for the displacement variations;

$$\frac{1}{2} (D_i x_j D_m x_j - \delta_{im}) = \partial \Phi_a / \partial s_{im} \qquad (7.8)$$

for the stress variations. The natural boundary conditions are again represented by (7.2).

If, instead of applying specified surface tractions p_j on the boundary ∂R_a, specified displacements u_j^* were imposed there, the potential energy

$$- \int_{\partial R_a} p_j \, u_j \, d \, \partial \, R_a$$

would have to be removed but could be replaced by the dislocation potential

$$\int_{\partial R_a} \lambda_j \, (u_j^* - u_j) \, d \, \partial \, R_a \qquad (7.9)$$

where the variation of the lagrangian multipliers λ_j permits to include the boundary conditions $u_j = u_j^*$ as natural conditions (i.e. provided by the principle itself). In this case the natural boundary conditions provided by $\delta_a \, u_j$ on ∂R_a are

$$\ell_i \, s_{im} \, D_m \, x_j - \lambda_j = 0 \qquad (7.10)$$

and can serve to identify the lagrangian multipliers. With this identification introduced the modified principle (7.7) is

$$\int_{R_a} \{ s_{im} \frac{1}{2} (D_i \, x_j \, D_m \, x_j - \delta_{im}) - \Phi_a \, (s_{im}) - A_j \, u_j \} \, d \, R_a$$

$$+ \int_{\partial R_a} \ell_i \, s_{im} \, D_m \, x_j \, (u_j^* - u_j) \, d \, \partial \, R_a \quad \text{station} \qquad (7.11)$$

It can be used as a two-field principle by introducing independant simplifying assumptions on the displacements u_j and on the Kirchhoff-Trefftz stresses s_{im}. From it we try to deduce a dual of the principle (7.I) which would contain only a field of statically admissible stresses. To this end we transform in (7.II)

$$\frac{1}{2} s_{im} \, (D_i \, x_j \, D_m \, x_j - \delta_{im})$$

$$= \frac{1}{2} (s_{im} \, D_m \, x_j) \, D_i \, x_j - \frac{1}{2} s_{im} \, \delta_{im}$$

$$= (s_{im} \, D_m \, x_j) \, D_i \, x_j - \frac{1}{2} s_{im} \, (\delta_{im} + D_m \, x_j \, D_i \, x_j)$$

Or, finally

$$\frac{1}{2} s_{im} (D_i x_j D_m x_j - \delta_{im}) = (s_{im} D_m x_j) D_i u_j - s_{im} \alpha_{im} + s_{im} D_m u_i \quad (7.12)$$

The first term is the one that, after integration by parts, will equilibrate the volume forces A_j and the surface terms associated with u_j. Hence, assuming a statically admissible stress field, we have

$$- \int_{R_a} \{ \Phi_a (s_{im}) + s_{im} \frac{\partial}{\partial s_{im}} \Phi_a (s_{im}) - s_{im} D_m u_i \} dR_a$$

$$+ \int_{\partial R_a} \ell_i s_{im} u_j^* D_m x_j \, d\, \partial R_a \quad \text{stationary} \quad (7.13)$$

In contrast to the linear case, it is observed that the kernel of this principle is no more the simple complementary energy density. Moreover it still contains explicitly the displacement field in R_a.
Another objection to its practical use is the difficulty of solving the Signorini equilibrium equations which also contain explicitly the displacement field.

8. Eulerian variational principles.

From the general formula (4.28) we can write

$$\delta_a \int_{R_a} (W_a - A_j u_j) dR_a = \delta_x \int_{R_x} (W_x - X_j u_j) dR_x + \int_{\partial R_x} (W_x - X_j u_j) n_i \delta_a u_i \, d\partial R_x$$

This can be used to transform the variation of the first term of (7.1). For the second,

$$\int_{\partial R_a} p_j \delta_a u_j \, d\partial R_a = \int_{\partial R_x} q_j \delta_a u_j \, d\partial R_x$$

where q_j are the surface tractions on ∂R_x.
Hence the eulerian principle

$$\delta_x \int_{R_x} (W_x - X_j u_j) dR_x + \int_{\partial R_x} \{ (W_x - X_j u_j) n_i \delta_a u_i - q_j \delta_a u_j \} \, d\partial R_x = 0 \quad (8.1)$$

in which we should still substitute

$$\delta_a u_i = D_j x_i \delta_x u_j \quad (8.2)$$

In verifying the truth of the principle we make the assumption that the material is homogeneous; so that from (6.6)

$$\delta_x W_x = (| M | \sigma_{ij} - W_x D_i x_m D_j x_m) \delta_x \hat{\xi}_{ij}$$

Using (5.25), this becomes

$$\delta_x W_x = (\tau_{mn} D_j x_n - W_x D_j x_m) D_i x_m \delta_x \hat{\xi}_{ij}$$

Them, in view of (3.4) and the symmetry in the subscripts i and j

$$\delta_x W_x = (\tau_{mn} D_j x_n - W_x D_j x_m) D_i x_m \partial_r a_i \partial_r \delta_x u_j$$

$$= (\tau_{mn} D_j x_n - W_x D_j x_m) \partial_m \delta_x u_j$$

The Euler equation stemming from $\delta_x u_j$ in R_x is then

$$\partial_m (\tau_{mn} D_j x_n - W_x D_j x_m) + X_j = 0 \tag{8.3}$$

or

$$\partial_m (\tau_{mn} D_j u_n + \tau_{mj} - W_x D_j x_m) + X_j = 0$$

and, since (5.I5) applies, we should have

$$\partial_m (\tau_{mn} D_j u_n - W_x D_j x_m) = 0 \tag{8.4}$$

The surface terms are now given by

$$\int_{\partial R_x} \left[\{ (W_x - X_r u_r) n_i - q_i \} D_j n_i + n_m (\tau_{mn} D_j x_n - W_x D_j x_m) \right] \delta_x u_j \, d\partial R_x = 0$$

The W_x terms are seen to cancel; there remain as natural boundary conditions on ∂R_x

$$\left(n_m \tau_{mi} - (X_r u_r) n_i - q_i \right) D_j x_i = 0$$

which imply, more simply

$$n_m \tau_{mi} - (X_r u_r) n_i = q_i \tag{8.5}$$

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13. ABSTRACT

The report discusses the strain measures for large displacements in both lagrangian and eulerian variables. Similarly, relations are established between lagrangian and eulerian variations of fields with a view to construct eulerian variational principles from the known lagrangian principle of (large) displacements. Various definitions of stress tensors are also discussed and related.

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