SA-21

NON LINEAR THEORY OF SHELLS

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1. Coordinate system.

We begin by considering a shell to derive from a flat plate through a finite geometrical transformation. Let x^i $(i = 1, 2, 3)$ denote cartesian coordinates of a point in the flat plate configuration, $x^3 = 0$ being the middle plane. In the shell configuration the same point has cartesian coordinates $y^j (x^1, x^2, x^3)$. We can conceive the x^i as lagrangian or convective curvilinear coordinates for the shell. From the position vector of a point of the shell

$$\vec{R} = y^i \vec{e}_i$$

where $\vec{e}_i$ are the cartesian base vectors, follows the definition of the base vectors associated with the curvilinear coordinates

$$\vec{g}_j = D_j \vec{R} = (D_j y^i) \vec{e}_i \qquad D_j = \frac{\partial}{\partial x^j}$$

and the corresponding fundamental metric tensor

$$g_{jm} = D_j y^i D_m y^n \vec{e}_i . \vec{e}_n = D_j y^i D_m y^i$$

A considerable simplification of the analysis will be obtained by requiring that the base vector $\vec{g}_3$ remains everywhere orthogonal to $\vec{g}_1$ and $\vec{g}_2$.
This condition, expressed by

$$g_{3\alpha} = D_3 y^i D_\alpha y^i = 0 \qquad (\alpha = 1, 2) \qquad (1.1)$$

can be satisfied by adopting the following structure for the geometrical transformation of coordinates

$$y^i = r^i(x^1, x^2) + x^3 b^i(x^1, x^2) \qquad (1.2)$$

and setting

$$b^i (D_\alpha r^i + x^3 D_\alpha b^i) = 0 \qquad (\alpha = 1, 2) \qquad (1.3)$$

Since this should hold for all x^3 values within the thickness boundaries of the shell, we must have separately

$$b^i D_\alpha r^i = 0 \qquad (\alpha = 1, 2) \qquad (1.4)$$

$$b^i D_\alpha b^i = 0 \qquad (\alpha = 1, 2) \qquad (1.5)$$

The general solution of (1.4), considered as a linear system in the b^i, is

$$b^i = (e_{ijk} D_1 r^j D_2 r^k) c (x^1, x^2) \qquad (1.6)$$

where e_{ijk} is the permutation symbol and $c (x^1, x^2)$ an arbitrary function. Simultaneously (1.5) requires

$$g_{33} = b^i b^i = \text{constant} \qquad (1.7)$$

Without sacrificing generality, the constant can be set equal to unity. Then, substitution of (1.6) into (1.7) determines the function c through the equation

$$\begin{aligned} 1 &= c^2 e_{ijk} e_{ipq} D_1 r^j D_2 r^k D_1 r^p D_2 r^q \\ &= c^2 (D_1 r^p D_1 r^p D_2 r^q D_2 r^q - D_1 r^p D_1 r^q D_2 r^p D_2 r^q) \end{aligned}$$

In conclusion the functions b^i are completely determined by the geometrical transformation $y^i = r^i (x^1, x^2)$ of the middle plane of the plate into the middle surface of the shell. This derivation justifies the classical choice of curvilinear coordinates for shells :

$$\vec{R} = \vec{r} (x^1, x^2) + x^3 \vec{a}_3 (x^1 , x^2) \qquad (1.8)$$

where $\vec{r}$ is the position vector of a middle surface point, and where $\vec{a}_3$ can be chosen suth that

$$\vec{a}_3 \cdot D_\alpha \vec{r} = 0 \qquad (\alpha = 1, 2) \qquad (1.9)$$

$$\vec{a}_3 \cdot \vec{a}_3 = 1 \qquad (1.10)$$

2. Base vectors for surface and space tensors.

In what follows the greek letter indices will always refer to the surface curvilinear coordinates x^1 and x^2, the latin indices to the complete set of space curvilinear coordinates x^1, x^2 and x^3.

The base vectors, tangent to the middle plane $x^3 = 0$, are given by

$$\vec{a}_\alpha = D_\alpha \vec{r} \tag{2.1}$$

They generate the surface metric tensor

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta = a_{\beta\alpha} \tag{2.2}$$

and are the coefficients of the first fundamental form

$$d\vec{r} \cdot d\vec{r} = a_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2.3}$$

The conjugate base vectors $\vec{a}^\alpha$ are related by

$$\vec{a}_\beta = a_{\alpha\beta} \vec{a}^\alpha \quad \text{or} \quad \vec{a}^\alpha \cdot \vec{a}_\gamma = \delta^\alpha_\gamma \text{ (Kronecker symbol)} \tag{2.4}$$

and generate the conjugate surface metric tensor

$$a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\beta \tag{2.5}$$

The surface metric tensors are used to raise or lower indices of surface tensors.
The determinant of the surface metric will be denoted

$$a = \left| a_{\alpha\beta} \right| \qquad \frac{1}{a} = \left| a^{\alpha\beta} \right| \tag{2.6}$$

The first fundamental form being positive definite a is a positive quantity. Let $\varepsilon_{\alpha\beta}$ denote a twice covariant antisymmetric surface tensor, it has only one strict component $\varepsilon_{12} = \varepsilon$ $(\varepsilon_{21} = -\varepsilon)$. Hence if $e_{\alpha\beta} = e^{\alpha\beta}$ denotes the corresponding permutation symbol $(e_{12} = 1,\ e_{21} = -1,\ e_{11} = e_{22} = 0)$ we have

$$\varepsilon_{\alpha\beta} = \varepsilon\, e_{\alpha\beta}$$

The twice contravariant components of this tensor are related by

$$\varepsilon_{\alpha\beta} = a_{\alpha\gamma}\, a_{\beta\mu}\, \varepsilon^{\gamma\mu} = (a_{\alpha\gamma}\, a_{\beta\mu}\, e^{\gamma\mu})\sigma$$

where σ is the strict component of $\varepsilon^{\gamma\mu}$ $(\sigma = \varepsilon^{12})$. The quantity between brackets is $e_{\alpha\beta} a$. Hence, comparing the two expressions of $\varepsilon_{\alpha\beta}$

$$\varepsilon = \sigma\, a \tag{2.7}$$

Further we have:

$$\varepsilon^{\alpha\beta}\, \varepsilon_{\alpha\beta} = \sigma\varepsilon\, e^{\alpha\beta}\, e_{\alpha\beta} = 2\sigma\varepsilon$$

and since this is obviously invariant, it can be equated to a fixed quantity. From the invariant $\sigma\varepsilon = \kappa$ and (2.7) we conclude that all antisymmetric second order surface tensors are of the form

$$\varepsilon_{\alpha\beta} = \sqrt{\kappa a}\, e_{\alpha\beta} \qquad \varepsilon^{\alpha\beta} = \sqrt{\frac{\kappa}{a}}\, e^{\alpha\beta}$$

where a is the determinant of the surface metric tensor. The usual ε-tensor for the surface corresponds to the choice $\kappa = 1$:

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \qquad \varepsilon_{12} = \sqrt{a} \tag{2.8}$$

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \qquad \varepsilon^{12} = 1/\sqrt{a}$$

Like the space metric tensor it enjoys the property that its twice covariant and twice contravariant representations are reciprocal : $\varepsilon^{\alpha\beta}\,\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$.
The unit vector $\vec{a}_3 = \vec{a}^3$ being normal to the surface

$$\vec{a}_3 . \vec{a}^{\alpha} = 0 \qquad \vec{a}_3 . \vec{a}_{\alpha} = 0 \qquad \vec{a}_3 . \vec{a}_3 = 1 \tag{2.9}$$

It can be defined by

$$\vec{a}_3 = \varepsilon^{\alpha\beta}\, \vec{a}_{\alpha} \times \vec{a}_{\beta} \qquad \vec{a}_{\alpha} \times \vec{a}_{\beta} = \varepsilon_{\alpha\beta}\, \vec{a}_3 \tag{2.10}$$

as is easily verified by scalar multiplication by $\vec{a}_{\gamma}$ and $\vec{a}_3$.
Turning now to the base vectors for space tensors, they are defined by

$$\vec{g}_i = D_i \vec{R} \tag{2.11}$$

In view of (1.8) they are related to the surface base vectors through the following equations

$$\vec{g}_{\alpha} = \vec{a}_{\alpha} + x^3\, D_{\alpha} \vec{a}_3 \tag{2.12}$$

$$\vec{g}_3 = \vec{a}_3$$

and generate the fundamental metric tensor

$$\vec{g}_\alpha \cdot \vec{g}_\beta = g_{\alpha\beta} = a_{\alpha\beta} - 2x^3 b_{\alpha\beta} + (x^3)^2 c_{\alpha\beta} \tag{2.13}$$

$$\vec{g}_\alpha \cdot \vec{g}_3 = g_{\alpha 3} = 0 \tag{2.14}$$

$$\vec{g}_3 \cdot \vec{g}_3 = g_{33} = 1 \tag{2.15}$$

(2.14) and (2.15) follow directly from (2.9). (2.13) contains the definitions of the following, obviously symmetric, surface tensors

$$-2b_{\alpha\beta} = \vec{a}_\alpha \cdot D_\beta \vec{a}_3 + \vec{a}_\beta \cdot D_\alpha \vec{a}_3 \tag{2.16}$$

$$c_{\alpha\beta} = D_\alpha \vec{a}_3 \cdot D_\beta \vec{a}_3 \tag{2.17}$$

Simpler formulas for the first are obtained by transformations such as

$$\vec{a}_\alpha \cdot D_\beta \vec{a}_3 = D_\beta(\vec{a}_\alpha \cdot \vec{a}_3) - \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = - \vec{a}_3 \cdot D_\beta \vec{a}_\alpha$$

justified by (2.9) and also

$$D_\beta \vec{a}_\alpha = D_\beta D_\alpha \vec{r} = D_\alpha \vec{a}_\beta \tag{2.18}$$

there comes

$$b_{\alpha\beta} = \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = - \vec{a}_\alpha \cdot D_\beta \vec{a}_3 \tag{2.19}$$

The elements of this tensor are the coefficients of the second fundamental form

$$- d\vec{r} \cdot d\vec{a}_3 = b_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

They appear also when expressing the surface derivative of the unit normal vector to the surface

$$D_\beta \vec{a}_3 = - b_{\alpha\beta}\, \vec{a}^\alpha \tag{2.20}$$

This formula is immediatly justified by (2.4), (2.9) and(2.19) when taking scalar multiplication with the surface base vectors. Substituting (2.20) into (2.17) it is found that the coefficients of the third fundamental form

$$d\vec{a}_3 \cdot d\vec{a}_3 = c_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2.21}$$

are expressible in terms of those of the second

$$c_{\alpha\beta} = b_{\gamma\alpha}\, b_{\varepsilon\beta}\, \vec{a}^\gamma \cdot \vec{a}^\varepsilon = b_{\gamma\alpha}\, b_{\varepsilon\beta}\, a^{\gamma\varepsilon} = b^\varepsilon_\alpha\, b_{\varepsilon\beta} \tag{2.22}$$

It is obvious from the geometrical signifiance of (2.20) or from its equivalent formulation in mixed components,known as the differential equation of Weingarten,

$$D_\beta \vec{a}_3 = - b^\alpha_\beta\, \vec{a}_\alpha \tag{2.23}$$

that the surface tensor generated by the second fundamental form characterizes completely the curvature of the middle surface. With respect to a change of curvilinear surface coordinates, the fundamental invariants of this tensor are

$$H = \frac{1}{2} b^\alpha_\alpha \tag{2.24}$$

the so-called mean curvature, and

$$K = \frac{1}{2} \varepsilon^{\alpha\beta}\, \varepsilon_{\lambda\mu}\, b^\lambda_\alpha\, b^\mu_\beta = b^1_1\, b^2_2 - b^2_1\, b^1_2 \tag{2.25}$$

the Gaussian curvature.

3. Covariant surface derivatives.

The surface Christoffel symbols are defined by the following decomposition of the derivatives with respect to the surface curvilinear coordinates of the surface base vectors themselves :

$$D_\beta \vec{a}_\alpha = \mathring{\Gamma}^\gamma_{\beta\alpha}\, \vec{a}_\gamma + \mathring{\Gamma}^3_{\beta\alpha}\, \vec{a}_3 \tag{3.1}$$

In view of (2.18) the Christoffel symbols are symmetrical with respect to the lower subscripts. Scalar multiplication of (3.1) by $\vec{a}_3$ and consideration of the orthogonality of $\vec{a}_3$ with respect to the $\vec{a}_\gamma$ and of (1.10) yields

$$\mathring{\Gamma}^3_{\beta\alpha} = \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = b_{\alpha\beta} \tag{3.2}$$

as shown by (2.19). Hence (3.1) will be replaced by

$$D_\beta \vec{a}_\alpha = \mathring{\Gamma}^\gamma_{\beta\alpha}\, \vec{a}_\gamma + b_{\alpha\beta}\, \vec{a}_3 \tag{3.3}$$

In this form it is known as the differential equation of Gauss. The remaining Christoffel symbols can be calculated from the derivations of the fundamental metric surface tensor. Indeed, scalar multiplication of (3.1) by $\vec{a}_\varepsilon$ produces

$$\vec{a}_\varepsilon \cdot D_\beta \vec{a}_\alpha = \overset{\circ}{\Gamma}{}_{\beta}{}^{\gamma}{}_{\alpha} \, a_{\gamma\varepsilon} = \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha}$$

However the left hand side is also

$$D_\beta(\vec{a}_\varepsilon \cdot \vec{a}_\alpha) - \vec{a}_\alpha \cdot D_\beta \vec{a}_\varepsilon = D_\beta a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon}$$

Hence by comparison,

$$D_\beta a_{\varepsilon\alpha} = \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon} + \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} \tag{3.4}$$

Similarly, by changing the order of the subscripts,

$$D_\varepsilon a_{\alpha\beta} = \overset{\circ}{\Gamma}_{\varepsilon\beta\alpha} + \overset{\circ}{\Gamma}_{\varepsilon\alpha\beta}$$

$$D_\alpha a_{\beta\varepsilon} = \overset{\circ}{\Gamma}_{\alpha\varepsilon\beta} + \overset{\circ}{\Gamma}_{\alpha\beta\varepsilon}$$

Combining those relations and using the symmetry properties of the Christoffel symbols

$$D_\beta a_{\varepsilon\alpha} - D_\varepsilon a_{\alpha\beta} + D_\alpha a_{\beta\varepsilon} = 2\,\overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} \tag{3.5}$$

In this formula, the central subscript can further be raised by multiplication by the reciprocal surface metric tensor $a^{\varepsilon\gamma}$, producing

$$2\,\overset{\circ}{\Gamma}{}_{\beta}{}^{\gamma}{}_{\alpha} = a^{\varepsilon\gamma}\,(D_\beta a_{\varepsilon\alpha} - D_\varepsilon a_{\alpha\beta} + D_\alpha a_{\beta\varepsilon}) \tag{3.6}$$

Consider now a vector defined on the middle surface

$$\vec{u} = u^\alpha(x^1, x^2)\,\vec{a}_\alpha + u^3\,(x^1, x^2)\,\vec{a}_3$$

and calculate its surface derivative

$$D_\beta \vec{u} = (D_\beta u^\alpha)\,\vec{a}_\alpha + (D_\beta u^3)\,\vec{a}_3 + u^\alpha D_\beta \vec{a}_\alpha + u^3 D_\beta \vec{a}_3$$

Substituting (2.23) and (3.3) and reorganizing the terms

$$D_\beta \vec{u} = (u^\alpha||_\beta - b^\alpha_\beta\, u3)\,\vec{a}_\alpha + (D_\beta u3 + b_{\alpha\beta} u^\alpha)\,\vec{a}_3 \tag{3.7}$$

where

$$u^\alpha||_\beta = D_\beta u^\alpha + u^\gamma \overset{\circ}{\Gamma}{}^\alpha_{\beta\gamma} \tag{3.8}$$

is, by definition, the covariant surface derivative of u^α.
In a change of curvilinear coordinates on the middle surface, $\vec{a}_3$ and u^3 are invariant, $D_\beta\vec{u}$, $b^\alpha_\beta\vec{a}_\alpha$, $D_\beta u^3$ and $b_{\alpha\beta}u^\alpha$ are obviously once covariant and the same must then be true for $u^\alpha||_\beta\,\vec{a}_\alpha$. Consequently $u^\alpha||_\beta$ is a mixed surface tensor.
Let us now look after a formula analogous to (3.1)for the surface derivatives of the dual surface base vectors. Scalar multiplication of (3.1) by $\vec{a}^\varepsilon$ yields in view of (2.4) and (2.9)

$$\overset{\circ}{\Gamma}{}^\varepsilon_{\beta\alpha} = \vec{a}^\varepsilon \cdot D_\beta\vec{a}_\alpha = D_\beta\delta^\varepsilon_\alpha - \vec{a}_\alpha \cdot D_\beta\vec{a}^\varepsilon = \vec{a}_\alpha \cdot D_\beta\vec{a}^\varepsilon \tag{3.9}$$

while $\vec{a}_3 \cdot D_\beta\vec{a}^\varepsilon = - \vec{a}^\varepsilon \cdot D_\beta\vec{a}_3$

or, considering (2.23)

$$\vec{a}_3 \cdot D_\beta\vec{a}^\varepsilon = b^\alpha_\beta\,\vec{a}_\alpha \cdot \vec{a}^\varepsilon = b^\alpha_\beta\delta^\varepsilon_\alpha = b^\varepsilon_\beta \tag{3.10}$$

Finally (3.9) and (3.10) are equivalent to the required formula

$$D_\beta\vec{a}^\varepsilon = - \overset{\circ}{\Gamma}{}^\varepsilon_{\beta\alpha}\vec{a}^\alpha + b^\varepsilon_\beta\,\vec{a}^3 \tag{3.11}$$

If we now express the surface vector $\vec{u}$ by its covariant components

$$\vec{u} = u_\varepsilon\vec{a}^\varepsilon + u_3\vec{a}^3$$

and use the results (3.11) and (2.20) ($\vec{a}^3 = \vec{a}_3$) there comes

$$D_\beta\vec{u} = (u_\alpha||_\beta - u_3 b_{\alpha\beta})\,\vec{a}^\alpha + (D_\beta u_3 + b^\alpha_\beta u_\alpha)\,\vec{a}^3 \tag{3.12}$$

where $$u_\alpha||_\beta = D_\beta u_\alpha - \overset{\circ}{\Gamma}{}^\gamma_{\beta\alpha}u_\gamma \tag{3.13}$$

is a twice covariant surface tensor : the covariant surface derivative of u_α. Equations (3.8) and (3.13) can be generalized to express the covariant surface derivative of a surface tensor of any order. Take the particular case of the twice covariant fundamental metric tensor $a_{\varepsilon\alpha}$:

$$a_{\varepsilon\alpha}||_\beta = D_\beta a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}{}_\beta{}^\gamma{}_\varepsilon a_{\gamma\alpha} - \overset{\circ}{\Gamma}{}_\beta{}^\gamma{}_\alpha a_{\varepsilon\gamma}$$

This is equivalent to

$$a_{\varepsilon\alpha}||_\beta = D_\beta a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon} - \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} = 0 \tag{3.14}$$

by virtue of (3.4). This result is known as Ricci's Lemma. The same result is easily established by direct calculation for $a^{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$. It shows that those tensors behave as constants when calculating surface covariant derivatives.
Another important result concerns the covariant surface differentiation of the $b_{\alpha\beta}$ tensor. From direct application of (3.13)

$$b_{\alpha\beta}||_\gamma = D_\gamma b_{\alpha\beta} - \overset{\circ}{\Gamma}{}_\gamma{}^\varepsilon{}_\alpha b_{\varepsilon\beta} - \overset{\circ}{\Gamma}{}_\gamma{}^\varepsilon{}_\beta b_{\alpha\varepsilon} \tag{3.15}$$

If we now differentiate (2.20) and use (3.11)

$$D_\gamma D_\beta \vec{a}_3 = - D_\gamma b_{\alpha\beta} \vec{a}^\alpha - b_{\alpha\beta} (- \overset{\circ}{\Gamma}{}^\alpha_{\gamma\varepsilon} \vec{a}^\varepsilon + b^\alpha_\gamma \vec{a}^3)$$

the result can be put in the form

$$D_\gamma D_\beta \vec{a}_3 = - b_{\alpha\beta}||_\gamma \vec{a}^\alpha - \overset{\circ}{\Gamma}{}_\gamma{}^\varepsilon{}_\beta b_{\alpha\varepsilon} \vec{a}^\alpha - c_{\gamma\beta} \vec{a}^3$$

The left-hand side and the two last terms of the right-hand side are symmetrical with respect to the subscripts β and γ. The same must then be true of the remaining term. This establishes the Mainardi-Codazzi equation

$$b_{\alpha\beta}||_\gamma = b_{\alpha\gamma}||_\beta \tag{3.16}$$

Should we have started from (2.23) instead of (2.20) we would have obtained

$$b^\varepsilon_\beta||_\gamma = b^\varepsilon_\gamma||_\beta \tag{3.17}$$

That this is an equivalent formulation of the Mainardi-Codazzi equation is obvious when (3.16) is multiplied by $a^{\alpha\varepsilon}$ and use made of Ricci's Lemma.
The Mainardi-Codazzi equation shows that the curvature tensor $b_{\alpha\beta}$ of a surface imbedded in euclidean space is not independant of the surface metric.
From this viewpoint we should consider the system of differential equations of Weingarten (2.23) and Gauss (3.3) and investigate all the integrability conditions to be satisfied for the existence of a field of surface base vectors. In this light the Mainardi-Codazzi equation appears as the integrability condition for the unit normal vector.
The integrability conditions for the other base vectors is obtained by a similar statement of symmetry of second differentials.
Differentiating (3.3) and repeating the use of (2.23) and (3.3)

$$D_\gamma D_\beta \vec{a}_\alpha = \{ D_\gamma \overset{\circ}{\Gamma}{}_\beta{}^\phi{}_\alpha + \overset{\circ}{\Gamma}{}_\beta{}^\varepsilon{}_\alpha \overset{\circ}{\Gamma}{}_\gamma{}^\phi{}_\varepsilon - b_{\alpha\beta} b_\gamma{}^\phi \} \vec{a}_\phi$$

$$+\{ b_{\gamma\varepsilon} \overset{\circ}{\Gamma}{}_\beta{}^\varepsilon{}_\alpha + D_\gamma b_{\alpha\beta} \} \vec{a}_3$$

Considering the symmetry of the left-hand side, the brackets in the right-hand side should also be symmetrical with respect to the interchange of β and α. That this is true for the bracket of $\vec{a}_3$ is immediately seen by substituting its second term from (3.15), producing

$$b_{\alpha\beta||\gamma} + \overset{\circ}{\Gamma}{}_{\beta}{}^{\varepsilon}{}_{\alpha}\, b_{\gamma\varepsilon} + \overset{\circ}{\Gamma}{}_{\gamma}{}^{\varepsilon}{}_{\alpha}\, b_{\beta\varepsilon} + \overset{\circ}{\Gamma}{}_{\gamma}{}^{\varepsilon}{}_{\beta}\, b_{\alpha\varepsilon}$$

While the group of the 3 last terms is clearly symmetrical with respect to the interchange of β and γ, the same is true of the first term by virtue of the Mainardi-Codazzi condition. The symmetry condition applied to the first bracket yields after rearranging terms.

$$b_{\alpha\beta} b^{\phi}_{\gamma} - b_{\alpha\gamma} b^{\phi}_{\beta} = R^{\phi}{}_{\cdot\alpha\gamma\beta}$$

where
$$R^{\phi}{}_{\cdot\alpha\gamma\beta} = D_{\gamma}\, \overset{\circ}{\Gamma}{}_{\beta}{}^{\phi}{}_{\alpha} - D_{\beta}\, \overset{\circ}{\Gamma}{}_{\gamma}{}^{\phi}{}_{\alpha} + \overset{\circ}{\Gamma}{}_{\beta}{}^{\varepsilon}{}_{\alpha}\, \overset{\circ}{\Gamma}{}_{\gamma}{}^{\phi}{}_{\varepsilon} - \overset{\circ}{\Gamma}{}_{\gamma}{}^{\varepsilon}{}_{\alpha}\, \overset{\circ}{\Gamma}{}_{\beta}{}^{\phi}{}_{\varepsilon} \tag{3.18}$$

depends only on the surface metric. Premultiplication by $a^{\alpha\psi}$, producing

$$b^{\psi}_{\beta}\, b^{\phi}_{\gamma} - b^{\psi}_{\gamma}\, b^{\phi}_{\beta} = R^{\phi\psi}{}_{\cdot\cdot\gamma\beta} \tag{3.19}$$

brings out clearly the antisymmetry with respect to the pair of upper and to the pair of lower indices. As a consequence there is only one scalar condition

$$K = b^{1}_{1}\, b^{2}_{2} - b^{1}_{2}\, b^{2}_{1} = R^{21}{}_{\cdot\cdot 21} \tag{3.20}$$

This is celebrated equation discovered by Gauss from which it appears that the Gaussian curvature depends on the surface metric only and is consequently accessible to measurements made on the surface only. It appears here as the second integrability condition placed on the curvature tensor $b_{\alpha\beta}$.

4. Covariant space derivatives.

The space Christoffel symbols appear as the covariant or contravariant components of derivatives of the space base vectors

$$D_j\, \vec{g}_i = \Gamma_j{}^{h}{}_{i}\, \vec{g}_h = \Gamma_{jhi}\, \vec{g}^{h} \tag{4.1}$$

Since $D_j\, \vec{g}_i = D_j\, D_i\, \vec{R} = D_i\, \vec{g}_j$ they are also symmetrical with respect to their first and last subscripts. Furthermore, from $D_j\, (\vec{g}_i \cdot \vec{g}^{k}) = D_j \delta_i{}^{k} = 0$, we obtain

$$\vec{g}_i \cdot D_j \vec{g}^{k} = - \vec{g}^{k} \cdot D_j \vec{g}_i = - \Gamma_{jhi}\, \vec{g}^{k} \cdot \vec{g}^{h} = - \Gamma_{jhi} g^{kh} = - \Gamma_j{}^{k}{}_{i}$$

or equivalently,

$$D_j \vec{g}^{k} = - \Gamma_j{}^{k}{}_{i}\, \vec{g}^{i} \tag{4.2}$$

Then

$$D_j g_{ih} = D_j (\vec{g}_i \cdot \vec{g}_h) = \vec{g}_i \cdot D_j\, \vec{g}_h + \vec{g}_h \cdot D_j \vec{g}_i = \Gamma_{jih} + \Gamma_{jhi} \tag{4.3}$$

from which follows by the same type of linear combinations as used in the surface case:

$$2\Gamma_{jih} = D_j g_{ih} - D_i g_{jh} + D_h g_{ji} \tag{4.4}$$

$$\Gamma_j{}^{k}{}_{h} = g^{ik}\, \Gamma_{jih}$$

Consider now a space vector defined by its contravariant components

$$\vec{u} = u^{i} \vec{g}_i$$

Its derivative formulated by means of (4.1) is

$$D_j\vec{u} = u^i|_j \, \vec{g}_i \tag{4.5}$$

where $$u^i|_j = D_j u^i + \Gamma_j{}^i{}_k \, u^k \tag{4.6}$$

is defined as the covariant space derivative of u^i, a mixed space tensor. The same vector defined by its covariant components $\vec{u} = u_k \vec{g}^k$ yields for the same derivative, using (4.2)

$$D_j\vec{u} = u_k|_j \, \vec{g}^k \tag{4.7}$$

with $$u_k|_j = D_j u_k - \Gamma_j{}^m{}_k \, u_m \tag{4.8}$$

The extension of formulas (4.6) and (4.8) to form covariant derivatives of tensors of higher order is immediate.

Clearly, since (4.5) and (4.7) represent the same quantity, expanded first in the system of covariant base vectors and later in the dual system, we must have

$$u_k|_j = g_{ki} \, u^i|_j$$

However the left-hand side is also

$$(g_{ki} \, u^i)|_j = g_{ki}|_j \, u^i + g_{ki} u^i|_j$$

From the comparison and the arbitrariness of the vector $\vec{u}$ it can be concluded that

$$g_{ki}|_j = 0 \tag{4.9}$$

This is Ricci's Lemma for the space case. Its direct computational proof relies on (4.3)

$$g_{ih}|_j = D_j g_{ih} - \Gamma_j{}^m{}_i \, g_{mh} - \Gamma_j{}^m{}_h \, g_{im}$$

$$= D_j g_{ih} - \Gamma_{jhi} - \Gamma_{jih} = 0$$

Let $e_{ijk} = e^{ijk}$ denote the permutation symbol, equal to unity if ijk is some even permutation of the sequence 123, to minus unity if the permutation is odd , to zero for all other cases. Any completely antisymmetric tensor is of the form

$$\varepsilon_{ijk} = \varepsilon \, e_{ijk}$$

where $\varepsilon = \varepsilon_{123}$ is the only strict component of the tensor. Similarly the contravariant form of this tensor will have a single strict component σ

$$\varepsilon^{mnp} = \sigma e^{mnp} \qquad\qquad \sigma = \varepsilon^{123}$$

The relation between the covariant and contravariant forms

$$\varepsilon_{ijk} = \varepsilon^{mnp} \, g_{mi} g_{nj} g_{pk} = \sigma(e^{mnp} g_{mi} g_{nj} g_{pk})$$

becomes equivalent to $\varepsilon = \sigma g$, when the quantity between brackets is recognized as the expansion of the determinant g of the fundamental metric tensor

$$e^{mnp} g_{mi} g_{nj} g_{pk} = e_{ijk} g$$

Furthermore the product of the strict components is easily recognized to be an invariant

$$\sigma\varepsilon = \frac{1}{6} \varepsilon^{ijk} \varepsilon_{ijk}$$

All completely antisymmetric tensors of this type differ only by the value of this invariant. For the choice $\sigma\varepsilon = 1$, corresponding to

$$\varepsilon = \sqrt{g} \qquad \sigma = 1/\sqrt{g}$$

we have the usual definition of the space ε- tensor :

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk} \qquad \varepsilon^{mnp} = \frac{1}{\sqrt{g}} e^{mnp} \tag{4.10}$$

An equivalent definition is

$$\varepsilon_{ijk} = (\vec{g}_i \times \vec{g}_j).\vec{g}_k \qquad \varepsilon^{mnp} = (\vec{g}^m \times \vec{g}^n).\vec{g}^p \tag{4.11}$$

justified by the complete antisymmetry of the mixed products and the fact that when the base vectors are cartesian, g = 1 and (4.11) reduces effectively to the permutation symbol.
In conjunction with (4.1) or (4.2) the expressions (4.11) are convenient to compute derivatives of the ε tensor.

$$\begin{aligned} D_h \varepsilon_{ijk} &= (D_h\vec{g}_i \times \vec{g}_j) . \vec{g}_k + (\vec{g}_i \times D_h\vec{g}_j).\vec{g}_k + (\vec{g}_i \times \vec{g}_j). D_h\vec{g}_k \\ &= (\vec{g}_j \times \vec{g}_k).D_h\vec{g}_i + (\vec{g}_k \times \vec{g}_i) .D_h\vec{g}_j + (\vec{g}_i \times \vec{g}_j). D_h\vec{g}_k \\ &= \varepsilon_{jkm} \Gamma^{m}_{h\ i} + \varepsilon_{kim} \Gamma^{m}_{h\ j} + \varepsilon_{ijm} \Gamma^{m}_{h\ k} \end{aligned} \tag{4.12}$$

This result is equivalent to stating that the ε-tensor also obeys Ricci's Lemma

$$\varepsilon_{ijk|h} = 0 \tag{4.13}$$

In fact we need only consider the case i = 1, j = 2, k = 3, whence, retaining only the non zero terms on the right hand side

$$D_h\varepsilon_{123} = \varepsilon_{231} \Gamma^{1}_{h\ 1} + \varepsilon_{312} \Gamma^{2}_{h\ 2} + \varepsilon_{123} \Gamma^{3}_{h\ 3} = \varepsilon_{123}\Gamma^{m}_{h\ m} \tag{4.14}$$

a result that will be useful later.

5. The structure of space metric in terms of surface metric for a shell.

Between space and surface base vectors exist the following relations deduced from (2.12) and (2.23)

$$\vec{g}_\alpha = \mu^\beta_\alpha \vec{a}_\beta \tag{5.1}$$

$$\vec{g}_3 = \vec{a}_3 \tag{5.2}$$

where although dependant on x^3,

$$\mu_\alpha^\beta = \delta_\alpha^\beta - x^3\, b_\alpha^\beta \tag{5.3}$$

is a mixed surface tensor, i.e. exhibiting the covariant and contravariant behavior associated with a change of curvilinear coordinates on the surface. The correspondant relations between metric tensors are

$$g_{\alpha\beta} = \mu_\alpha^\beta\, \mu_\beta^\lambda\, a_{\gamma\lambda} \tag{5.4}$$

equivalent to (2.13).
Taking the determinant on both sides and noting that

$$g = |g_{ij}| = |g_{\alpha\beta}| \text{ in view of (2.14) and (2.15)}$$

$$g = \mu^2 a \tag{5.5}$$

where $\mu = |\mu_\alpha^\gamma| = 1 - 2\,H\,x^3 + (x^3)^2\,K$ (5.6)

as is easily verified by direct expansion of this determinant from (5.3) and consideration of the definitions (2.24) and (2.25). We also need the tensor reciprocal to μ_α^γ and denote it, following Naghdi, by

$$(\mu^{-1})_\alpha^\beta = \frac{1}{\mu}\, e_{\alpha\gamma}\, e^{\beta\varepsilon}\, \mu_\varepsilon^\gamma = \frac{1}{\mu}\{\, \delta_\alpha^\beta + x^3 (b_\alpha^\beta - 2\,H\,\delta_\alpha^\beta)\} \tag{5.7}$$

Then from (5.1) in succession

$$(\mu^{-1})_\gamma^\alpha\, \vec{g}_\alpha = \delta_\gamma^\beta\, \vec{a}_\beta = \vec{a}_\gamma$$

$$(\mu^{-1})_\gamma^\alpha\, \vec{g}_\alpha \cdot \vec{g}^\varepsilon = (\mu^{-1})_\gamma^\alpha\, \delta_\alpha^\varepsilon = (\mu^{-1})_\gamma^\varepsilon = \vec{a}_\gamma \cdot \vec{g}^\varepsilon \quad \text{and finally}$$

$$\vec{g}^\varepsilon = (\mu^{-1})_\gamma^\varepsilon\, \vec{a}^\gamma \tag{5.8}$$

while

$$\vec{g}^3 = \vec{a}^3 \;(= \vec{a}_3 = \vec{g}_3) \tag{5.9}$$

The reciprocal space metric tensor becomes

$$g^{\varepsilon\phi} = (\mu^{-1})_\gamma^\varepsilon\, (\mu^{-1})_\lambda^\phi\, a^{\gamma\lambda} \tag{5.10}$$

$$g^{\varepsilon 3} = 0 \; . \qquad g^{33} = 1 \tag{5.11}$$

We now proceed to calculate the space Christoffel symbols :

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \vec{g}^\gamma \cdot D_\alpha \vec{g}_\beta = (\mu^{-1})_\lambda^\gamma\, \vec{a}^\lambda \cdot D_\alpha\, (\mu_\beta^\varepsilon\, \vec{a}_\varepsilon)$$

$$= (\mu^{-1})_\lambda^\gamma\, \{ D_\alpha\, \mu_\beta^\lambda + \mu_\beta^\varepsilon\, \overset{\circ}{\Gamma}{}_{\alpha\ \varepsilon}^{\ \lambda} \}$$

Use was here made in succession of (5.1), (5.8) and (3.1) and (2.9).
The expression in brackets is modified to introduce the surface covariant derivative

$$\mu_{\beta||\alpha}^\lambda = D_\alpha \mu_\beta^\lambda + \mu_\beta^\varepsilon\, \overset{\circ}{\Gamma}{}_{\alpha\ \varepsilon}^{\ \lambda} - \mu_\varepsilon^\lambda\, \overset{\circ}{\Gamma}{}_{\alpha\ \beta}^{\ \varepsilon}$$

This produces

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = (\mu^{-1})_\lambda^\gamma \{\, \mu_{\beta||\alpha}^\lambda + \mu_\varepsilon^\lambda\, \overset{\circ}{\Gamma}{}_{\alpha\ \beta}^{\ \varepsilon} \} = \overset{\circ}{\Gamma}{}_{\alpha\ \beta}^{\ \gamma} + (\mu^{-1})_\lambda^\gamma\, \mu_{\beta||\alpha}^\lambda \tag{5.12}$$

Finally, in view of (5.7) and (5.3) this can be placed in the form

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \overset{\circ}{\Gamma}{}_{\alpha\ \beta}^{\ \gamma} - \frac{x^3}{\mu}(1 - 2\,H\,x^3)\; b^{\gamma}_{\beta||\alpha} - \frac{(x^3)^2}{\mu}\; b^{\gamma}_{\lambda}\; b^{\lambda}_{\beta||\alpha} \qquad (5.13)$$

where the dependance on x^3 is more apparent. In particular, as the notation suggested

$$\overset{\circ}{\Gamma}{}_{\alpha\ \beta}^{\ \gamma} = (\Gamma_{\alpha\ \beta}^{\ \gamma})_{x^3 = 0}$$

Moreover the symmetry in the subscripts α and β is guaranteed by the Mainardi-Codazzi condition (3.17).
Next calculate

$$\Gamma_{\alpha\ \beta}^{\ 3} = \vec{g}^{3}.D_{\alpha}g_{\beta} = \vec{a}^{3}.D_{\alpha}\ (\ \mu^{\varepsilon}_{\beta}\ \vec{a}_{\varepsilon})$$

Expanding the derivative and using the orthogonality between $\vec{a}_{\varepsilon}$ and $\vec{a}^{3}$,

$$\Gamma_{\alpha\ \beta}^{\ 3} = \mu_{\beta}^{\ \varepsilon}\vec{a}^{3}.(\ \overset{\circ}{\Gamma}{}_{\alpha\ \varepsilon}^{\ \lambda}\vec{a}_{\lambda} + b_{\alpha\varepsilon}\ \vec{a}_{3}) = \mu^{\varepsilon}_{\beta}\ b_{\alpha\varepsilon} \qquad (5.14)$$

or, as explicit function of x^3,

$$\Gamma_{\alpha\ \beta}^{\ 3} = b_{\alpha\varepsilon}\ (\ \delta^{\varepsilon}_{\beta} - x^3\ b^{\varepsilon}_{\beta}) = b_{\alpha\beta} - x^3\ c_{\alpha\beta} \qquad (5.15)$$

For $\Gamma_{\beta\ 3}^{\ \alpha} = \Gamma_{3\ \beta}^{\ \alpha}$ we have

$$\Gamma_{\beta\ 3}^{\ \alpha} = \vec{g}^{\alpha}.D_{\beta}\vec{a}_{3} = (\mu^{-1})^{\alpha}_{\lambda}\vec{a}^{\lambda}.(-b^{\gamma}_{\beta}\ \vec{a}_{\gamma}) = -\ b^{\lambda}_{\beta}\ (\mu^{-1})^{\alpha}_{\lambda} \qquad (5.16)$$

or again as an explicit function of x^3

$$\Gamma_{\beta\ 3}^{\ \alpha} = -\frac{1 - 2\,H\,x^3}{\mu}\ b^{\alpha}_{\beta} - \frac{x^3}{\mu}\ b^{\lambda}_{\beta}\ b^{\alpha}_{\lambda} \qquad (5.17)$$

Finally it is easily verified that the remaining space Christoffel symbols vanish

$$\Gamma_{3\ 3}^{\ \alpha} = 0 \qquad \Gamma_{3\ \beta}^{\ 3} = 0 \qquad \Gamma_{3\ 3}^{\ 3} = 0 \qquad (5.18)$$

6. The divergence operator and the divergence theorems.

The divergence theorem

$$\int_S (\vec{n}.\vec{a})\ ds = \int_V \mathrm{div}\ \vec{a}\ dV \qquad (6.1)$$

applied to a vector field $\vec{a}$ within a volume bounded by a simple closed surface S, provides an intrinsic definition of the divergence operator

$$\mathrm{div}\ \vec{a} = \lim_{dV \to 0} \frac{\int_S (\vec{n}.\vec{a})dS}{dV} \qquad (6.2)$$

that can be used to obtain its formulation in general curvilinear coordinates. As elementary volume we take the parallelopiped built on the three infinitesimal vectors $\vec{g}_1\ dx^1$, $\vec{g}_2\ dx^2$, $\vec{g}_3\ dx^3$, issued from the point P where the divergence of the field is to be calculated.

The surface vector $(\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2$ proportional to the aera of one of the facets of the parallelopiped through P, is oriented as the inward normal to this facet. Hence the entering flux of $\vec{a}$ through this facet is

$$(\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2 \cdot (a^i\, \vec{g}_i) = a^3\, dx^1\, dx^2\, (\vec{g}_1 \times \vec{g}_2)\cdot\vec{g}_3$$
$$= dx^1\, dx^2\, a^3\, \varepsilon_{123}$$

The outgoing flux through the opposite facet is

$$dx^1\, dx^2\, a^3\, \varepsilon_{123} + dx^1\, dx^2\, dx^3\, D_3\, (a^3\, \varepsilon_{123})$$

and the net contribution to the outcoming flux is

$$dx^1\, dx^2\, dx^3\, D_3\, (a^3\, \varepsilon_{123})$$

Taking account of the analogous contributions from the two other pairs of facets and dividing by the volume of the parallelopiped

$$dV = dx^1\, dx^2\, dx^3\, (\vec{g}_1 \times \vec{g}_2)\cdot\vec{g}_3 = dx^1\, dx^2\, dx^3\, \varepsilon_{123}$$

the application of definition (6.2) yields

$$\operatorname{div} \vec{a} = \frac{1}{\varepsilon_{123}}\, D_i\, (a^i\, \varepsilon_{123})$$

Expanding the derivative and using our previous result (4.14)

$$\operatorname{div} \vec{a} = D_i\, a^i + a^i\, \Gamma_{i\ m}^{\ m} = a^i|_i \tag{6.3}$$

This allows to write the divergence theorem in the form

$$\int_S (n_i a^i)\, dS = \int_V a^i|_i\, dV \tag{6.4}$$

where n_i are the covariant components of the outward unit normal to the surface. A similar result will prove to be useful for the flux of surface vectors. Let an elementary parallelogram be drawn from the vectors $\vec{a}_1\, dx^1$ and $\vec{a}_2\, dx^2$ in a tangent plane to the middle surface of the shell, and let $\vec{h} = h^\alpha\, \vec{a}_\alpha$ be a vector field defined on this middle surface. We calculate the outgoing flux $\Sigma\, \vec{\nu}\cdot\vec{h}\, ds$, where $\vec{\nu}$ is the outward unit normal to the contour of the parallelogram. Along the side $\vec{a}_1\, dx^1$ the contribution is $(\vec{a}_1 \times \vec{a}_3)\, dx^1 \cdot h^\alpha\, \vec{a}_\alpha = dx^1\, h^2\, (\vec{a}_1 \times \vec{a}_3)\cdot\vec{a}_2 = -\, dx^1\, h^2\, \varepsilon_{12}$

where use was made of

$$\varepsilon_{12} = (\vec{a}_1 \times \vec{a}_2)\cdot\vec{a}_3 \tag{6.5}$$

which is a direct consequence of (2.10). Adding the contribution from the opposite side, the net result is $dx^1\, dx^2\, D_2\, (h^2\, \varepsilon_{12})$.

Finally, adding the analogous contribution from the other pair of sides :

$$dx^1\, dx^2\, D_\alpha\, (h^\alpha\, \varepsilon_{12}) = \Sigma\, \vec{v}.\vec{h}\, ds \tag{6.6}$$

The next result is a direct consequence of (2.23), (3.3) and (2.9) :

$$D_\alpha\, \varepsilon_{12} = D_\alpha\, \{(\vec{a}_1 \times \vec{a}_2).\vec{a}_3\} = \varepsilon_{12}\, \overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta}{}_{\beta} \tag{6.7}$$

It constitutes also the verification that $\varepsilon_{\alpha\beta}$ obeys Ricci's Lemma for the surface derivative. Substituted into (6.6) it gives

$$\begin{aligned}\Sigma\, \vec{v}.\vec{h}\, ds &= (D_\alpha\, h^\alpha + \overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta}{}_{\beta}\, h^\alpha)\, \varepsilon_{12}\, dx^1\, dx^2 \\ &= h^\alpha||_\alpha\, \varepsilon_{12}\, dx^1\, dx^2\end{aligned} \tag{6.8}$$

Then, by the usual argument of decomposition into elementary cells, we obtain the surface divergence theorem

$$\oint_c \nu_\alpha h^\alpha\, ds = \int_A h^\alpha||_\alpha\, dA \tag{5.9}$$

relating a contour integral to an integral extended over an area A of the middle surface. The surface element is indeed easily observed to be

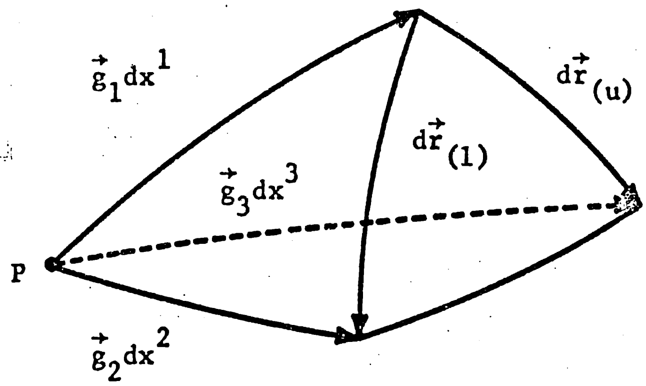
$$dA = \varepsilon_{12}\, dx^1\, dx^2 \tag{5.10}$$

7. Three-dimensional elasticity in curvilinear coordinates.

The approach towards a shell theory will consist in reducing the general three-dimensional elasticity equations to two-dimensional ones by suitable assumptions concerning the structure of the problem with respect to the x^3 coordinate, normal to the curved middle surface. Consequently the three-dimensional elasticity equations in curvilinear coordinates will play an important role and will be briefly recapitulated hereunder.
The basic stress definition will be lagrangian in nature. This implies that infinitesimal forces, that have to be defined as acting on infinitesimal aeras of the strained elastic body, are kept invariant but, for the purpose of defining stresses, are manipulated in the metric of a "reference configuration". In the reference configuration, also called initial configuration, the body is by definition unstrained. The geometrical considerations to follow will then apply essentially to the reference configuration.
An oriented surface vector is defined in general as the cross product

$$d\vec{r}_{(1)} \times d\vec{r}_{(2)} = (\vec{g}_i\, dx^i_{(1)}) \times (\vec{g}_j\, dx^j_{(2)})$$

between two infinitesimal vectors. In modulus it is proportional to the area of the parallelogram constructed on these vectors; in orientation it is normal to the plane of the parallelogram.
Consider now an elementary tetrahedron, three lateral facets of which are generated by infinitesimal vectors taken along the local base vectors and issued from a point P.



The fourth facet, called the "inclined" facet, is half the parallelogram constructed on the vectors

$$d\vec{r}_{(1)} = \vec{g}_2\, dx^2 - \vec{g}_1\, dx^1 \qquad d\vec{r}_{(2)} = \vec{g}_3\, dx^3 - \vec{g}_1\, dx^1$$

and has an oriented surface vector that, considering the antisymmetric properties of the cross product, can be written in cyclic form as

$$d\vec{r}_{(1)} \times d\vec{r}_{(2)} = (\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2 + (\vec{g}_2 \times \vec{g}_3)\, dx^2\, dx^3 + (\vec{g}_3 \times \vec{g}_1)\, dx^3\, dx^1 \tag{7.1}$$

Each term on the right represents the surface vector of one of the lateral facets; moreover, if the orientation of the surface vector of the inclined facet is outwards, those of the lateral facets are inwards. Hence (7.1) expresses that the sum of all outward surface vectors of the tetrahedron vanishes.

In (7.1) the left hand side can be replaced by $2\,\vec{n}\, dS$, where $\vec{n}$ is the unit outward normal and dS the area of the inclined facet, while the first term in the right-hand side can be replaced by

$$\vec{g}_1 \times \vec{g}_2\, dx^1\, dx^2 = \varepsilon_{123}\, dx^1\, dx^2\, \vec{g}^3 = 2\, \frac{\vec{g}^3}{\sqrt{g^{33}}}\, dS_3$$

since $\vec{g}^3 / \sqrt{g^{33}}$ is obviously the (inward) unit normal, dS_3 denoting the area, of the lateral facet generated by $\vec{g}_1\, dx^1$ and $\vec{g}_2\, dx^2$. Thus (7.1) becomes

$$\vec{n}\, dS = \frac{\vec{g}^1}{\sqrt{g^{11}}}\, dS_1 + \frac{\vec{g}^2}{\sqrt{g^{22}}}\, dS_2 + \frac{\vec{g}^3}{\sqrt{g^{33}}}\, dS_3$$

or, introducing the covariant components n_i of the unit normal $\vec{n}$

$$\vec{n} = n_i\, \vec{g}^i \quad , \tag{7.2}$$

and identifying

$$dS_i = \sqrt{g^{ii}}\, n_i \, dS \qquad \text{(without summation)} \qquad (7.3)$$

Now let $\vec{t}$ and $\vec{t}^{\,i}$ denote respectively the stress vectors acting on the inclined facet and on the lateral facets. Imagining the tetrahedron to be convected in some strained state of the body, those stresses are defined as the ratio of the force acting in the stained state on each facet to the area of the unstrained facet. Hence the translational equilibrium condition of the tetrahedron is, the resultant of mass forces being negligibly small,

$$\vec{t}\,dS = \vec{t}^{\,i}\, dS_i \qquad (7.4)$$

The substitution of (7.3) into (7.4) suggests that a lagrangian stress tensor t^{ij} be defined according to equation

$$\sqrt{g^{ii}}\, \vec{t}^{\,i} = t^{ij}\, \vec{g}_j \qquad (7.5)$$

so that the equilibrium condition is now expressed by

$$\vec{t} = n_i t^{ij}\, \vec{g}_j \qquad (7.6)$$

It follows that the t^{ij} are not physical stress components. The physical components derived from definition (7.5) are given by

$$\sqrt{\frac{g_{jj}}{g^{ii}}}\; t^{ij} \qquad \text{(no summation)} \qquad (7.7)$$

If the points of the strained body are submitted to an additional small displacement field $\delta\vec{u}\,(x^1, x^2, x^3)$, the statement of energy conservation is

$$\int_S \vec{t}\cdot\delta\vec{u}\; dS + \int_V \vec{X}\cdot\delta\vec{u}\; dV = \int_V \delta W\; dV \qquad (7.8)$$

where surface and volume integrals are taken in the initial configuration. W is the elastic strain energy and $\vec{X}$ the body force field per unit volume in this initial configuration. In curvilinear coordinates this becomes

$$\int_S (n_i t^{ij}\, \delta u_j)\; dS + \int_V X^j\, \delta u_j\; dV = \int_V \delta W\; dV \qquad (7.9)$$

When the surface integral is transformed in a volume integral by the divergence theorem (6.4), the local statement of conservation of energy (per unit volume) is

$$(t^{ij}\, \delta u_j)|_i + X^j\, \delta u_j = \delta W \qquad (7.10)$$

If the additional displacement field is of a rigid body type, there is no energy increase; this can be used to extract from (7.10) the translational and rotational equilibrium equations. In the translational case

$$D_i\,(\delta u_j\, \vec{g}^{\,j}) = \delta u_j|_i\, \vec{g}^{\,j} = 0 \qquad \text{or} \qquad \delta u_j|_i = 0 \qquad (7.11)$$

so that (7.10) with $\delta W = 0$ reduces to

$$(t^{ij}|_i + X^j)\, \delta u_j = 0$$

and, on account of the arbitrariness of the translation vector

$$t^{ij}{}_{|i} + X^j = 0 \qquad (j = 1, 2, 3) \qquad (7.12)$$

Those equilibrium equations satisfied by the lagrangian stresses are linear; they allow a first simplification of the energy equation (7.10)

$$t^{ij} \; \delta u_{j|i} = \delta W \qquad (7.13)$$

For the rotational case we can write the additional displacement field as

$$\delta\vec{u} = \vec{d\omega} \times (\vec{R} + \vec{u}) \qquad (7.14)$$

where $\vec{d\omega}$ is a constant infinitesimal rotation vector and $\vec{u}$ the already prevailing displacement field from the initial configuration. Then

$$D_m \, \delta\vec{u} = \vec{d\omega} \times (\vec{g}_m + D_m \vec{u}) \qquad \text{or}$$

$$\delta u_{i|m} \, \vec{g}^i = (\delta^p_m + u^p|_m) \, \vec{d\omega} \times \vec{g}_p$$

Scalar multiplication by $\vec{g}_j$ and transformation of

$$(\vec{d\omega} \times \vec{g}_p).\vec{g}_j = d\omega^r \, (\vec{g}_r \times \vec{g}_p).\vec{g}_j = \varepsilon_{rpj} \, d\omega^r$$

furnishes

$$\delta u_{j|m} = \varepsilon_{rpj} \, (\delta^p_m + u^p|_m) \, d\omega^r \qquad (7.15)$$

For this value of the covariant derivative of δu_j, the energy increase vanishes again in (7.13) and, since the $d\omega^r$ are arbitrary

$$\varepsilon_{rpj} \, (t^{pj} + t^{ij} \, u^p|_i) = 0 \qquad \text{all } r \text{ and } j$$

Considering the complete antisymmetry of ε_{rpj} this rotational equilibrium condition is satisfied if and only if the tensor between bracket is symmetrical in the superscripts p and j. This produces the final rotational equilibrium equations connecting the lagrangian stress tensor components

$$t^{pj} + t^{ij} \, u^p|_i = t^{ip} \, u^j|_i \;\; R + t^{jp} \qquad (7.16)$$

There are only three distinct ones, since they are identically satisfied for p = j. They reduce to a statement of symmetry of the lagrangian stress tensor only in the case where the covariant derivatives of the contravariant displacement tensor are negligibly small compared to unity (geometrical linearity). A method for automatically satisfying (7.16) consists in writing

$$t^{ij} = (\delta^j_q + u^j|_q) \, s^{qi} \quad \text{with } s^{qi} = s^{iq} \qquad (7.17)$$

The new symmetrical stress tensor s^{qi} is that of Kirchhoff and Trefftz, it transforms the energy equation (7.13) into

$$s^{qi} \, (\delta u_{q|i} + u^j|_q \, \delta u_{j|i}) = \delta W$$

or, making use of the symmetry of the new stress tensor

$$\frac{1}{2} s^{qi} \, (\delta u_{q|i} + \delta u_{i|q} + u^j|_q \, \delta u_{j|i} + u^j|_i \, \delta u_{j|q}) = \delta W \qquad (7.18)$$

Because the formulation is lagrangian, the order sequence of the δ operator and the derivatives with respect to the coordinates x^i may be interchanged. It is then recognized that the tensor between brackets is a perfect differential, provided one also recognizes, e.g. by applying Ricci's Lemma, the equivalence

$$u^j|_i \, \delta u_j|_q = u_j|_i \, \delta u^j|_q$$

In fact, introducing the symmetrical tensor

$$\gamma_{qi} = \frac{1}{2} (u_q|_i + u_i|_q + u^j|_q \, u_j|_i) \tag{7.19}$$

the energy statement (7.18) reduces to

$$\delta W = s^{qi} \, \delta\gamma_{qi} \tag{7.20}$$

It shows that the definition (7.17) is not altogether artificial since it now appears that the strain energy density is a function of the displacement gradients through six distinct combinations of them only, the components of a symmetrical tensor γ_{qi}, and that the Kirchhoff-Trefftz stresses are the corresponding partial derivatives. Obviously the γ_{qi} must represent strains, they are in fact the most commonly used lagrangian measures of strain and are usually derived from a statement of conservation of distance between neighboring points in a rigid body displacement. It is easily verified that

$$d(\vec{R} + \vec{u}).d(\vec{R} + \vec{u}) - d\vec{R}.d\vec{R} = 2\gamma_{qi} \, dx^q \, dx^i \tag{7.21}$$

While the lagrangian viewpoint offers many advantages, the hybrid nature of the stress tensor definitions it requires is well illustrated by the Kirchhoff-Trefftz tensor. They can be interpreted as the same lagrangian stress vectors but decomposed in the metric of the strained body. The base vectors of this metric are naturally defined by

$$\vec{G}_q = D_q(\vec{R} + \vec{u}) = \vec{g}_q + u^j|_q \, \vec{g}_j = (\delta_q^{\ j} + u^j|_q) \, \vec{g}_j \tag{7.22}$$

Then, starting from definition (7.6) of the lagrangian stress tensor and substituting (7.17)

$$\vec{t} = n_i \, s^{qi} \, (\delta_q^j + u^j|_q) \, \vec{g}_j = n_i \, s^{qi} \, \vec{G}_q \tag{7.23}$$

which proves the assertion. This geometrically hybrid character is reflected in the nature of the translational equilibrium equations they satisfy, the Signorini equations, which depend on the displacement field :

$$s^{ij}|_i + (s^{qi} \, u^j|_q)|_i + X^j = 0 \tag{7.24}$$

Due to the symmetry of the Kirchhoff-Trefftz stresses, (7.20) shows that the energy density is in fact a function of the six strain measures.

$$\gamma_{11}, \gamma_{22}, \gamma_{33}, \quad \text{and} \quad \begin{aligned} \eta_{12} &= \gamma_{12} + \gamma_{21} \\ \eta_{23} &= \gamma_{23} + \gamma_{32} \\ \eta_{31} &= \gamma_{31} + \gamma_{13} \end{aligned} \tag{7.25}$$

with the following general form of the stress-strain relations

$$\sigma^{ii} = \frac{\partial W}{\partial \gamma_{ii}} \qquad \sigma^{ij} = \frac{\partial W}{\partial \eta_{ij}} \quad (i \neq j) \tag{7.26}$$

If, however, it is agreed to distinuish γ_{ij} from γ_{ji} in (7.25), the general formula

$$\sigma^{ij} = \frac{\partial W}{\partial \gamma_{ij}} \tag{7.27}$$

holds true by the chain rule of differentiation.
In writing down an explicit form of the energy density we shall limit ourselves to the generalized Hooke situation (physical linearity).
The energy is then a homogeneous quadratic form in the components of the strain tensor :

$$W = \frac{1}{2} E^{ijkl} \gamma_{ij} \gamma_{kl} \tag{7.28}$$

Because the tensor of elastic moduli is symmetrical with respect to the pair of indices ij and kl and also with respect to the interchange of those pairs, its total number of independant components is not larger than 21.

In the isotropic case, the energy is explicitly

$$W = G \frac{1 - \nu}{1 - 2\nu} I_1^2 - 2GI_2 \tag{7.29}$$

with G the shear modulus and ν Poisson's ratio. The invariants I_1 and I_2 are obtained as coefficients of the powers of γ in the expansion of the determinant.

$$\begin{vmatrix} \gamma_1^1 - \gamma & \gamma_1^2 & \gamma_1^3 \\ \gamma_2^1 & \gamma_2^2 - \gamma & \gamma_2^3 \\ \gamma_3^1 & \gamma_3^2 & \gamma_3^3 - \gamma \end{vmatrix} = - \gamma^3 + I_1 \gamma^2 - I_2 \gamma + I_3$$

$$I_1 = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = \gamma_i^i$$

$$I_2 = \gamma_1^1 \gamma_2^2 + \gamma_2^2 \gamma_3^3 + \gamma_3^3 \gamma_1^1 - \gamma_1^2 \gamma_2^1 - \gamma_2^3 \gamma_3^2 - \gamma_3^1 \gamma_1^3 = \frac{1}{2} (\gamma_i^i \gamma_j^j - \gamma_i^j \gamma_j^i)$$

Going back to the doubly covariant components of strain

$$I_1 = g^{ij} \gamma_{ij} \qquad I_2 = \frac{1}{2} g^{mi} g^{nj} (\gamma_{mi} \gamma_{nj} - \gamma_{mj} \gamma_{ni}) \tag{7.30}$$

and applying formally (7.27)

$\sigma^{pq} = E^{pq\,ij} \gamma_{ij}$ with

$$E^{pqij} = G(g^{pi} g^{qj} + g^{pj} g^{qi} + \frac{2\nu}{1 - 2\nu} g^{pq} g^{ij}) \tag{7.31}$$

8. Stress assumption for shell theory.

The basic stress assumption will be that the direct Kirchhoff-Trefftz σ^{33} normal to the middle surface is zero; it generalizes the assumption common to plate theories. Then, in cartesian local axes, the third axis being oriented like $\vec{a}_3$, the isotropic stress-strain relations reduce to

$$\sigma_1^1 = \frac{E}{1-\nu^2}\left(\gamma_1^1 + \nu\gamma_2^2\right) \quad , \quad \sigma_2^2 = \frac{E}{1-\nu^2}\left(\gamma_2^2 + \nu\gamma_1^1\right) , \quad \sigma_3^3 = 0$$

$$\sigma_i^j = \frac{E}{1-\nu^2}\left(1-\nu\right)\gamma_i^j \qquad (i \neq j)$$

where E is Young's modulus. Building up the corresponding energy density by means of Clapeyron's theorem

$$W = \frac{1}{2}\sigma_i^j \gamma_j^i$$

the resulting expression is easily put in the form

$$W = \frac{E}{2(1-\nu^2)}\left\{(\gamma_\alpha^\alpha)^2 + 2(1-\nu)\gamma_\alpha^3\,\gamma_3^\alpha - (1-\nu)(\gamma_\alpha^\alpha\gamma_\beta^\beta - \gamma_\alpha^\beta\gamma_\beta^\alpha)\right\}$$

It is clearly valid also in the curvilinear system of coordinates of the shell in view of the invariance of the different terms with respect to any change in metric not involving the third axis.
Turning then to the covariant strains measures

$$W = \frac{E}{2(1-\nu^2)}\left\{(g^{\alpha\beta}\gamma_{\alpha\beta})^2 + 2(1-\nu)\,g^{\alpha\beta}\gamma_{\alpha 3}\gamma_{3\beta} - (1-\nu)g^{\alpha\lambda}g^{\beta\mu}(\gamma_{\alpha\lambda}\gamma_{\beta\mu} - \gamma_{\alpha\mu}\gamma_{\beta\lambda})\right\} \quad (8.1)$$

By taking formal derivatives it is found that

$$\left.\begin{aligned} E^{\lambda\mu\alpha\beta} &= \frac{E}{2(1-\nu^2)}\left\{2\nu g^{\lambda\mu}g^{\alpha\beta} + (1-\nu)(g^{\lambda\alpha}g^{\mu\beta} + g^{\lambda\beta}g^{\mu\alpha})\right\} \\ E^{\lambda 3\alpha 3} &= 2\,G\,g^{\lambda\alpha} \\ E^{\lambda\mu\alpha 3} &= 0 \qquad E^{\lambda\mu 33} = 0 \qquad E^{\lambda 3\,33} = 0 \qquad E^{33\,33} = 0 \end{aligned}\right] \quad (8.2)$$

9. Displacement assumptions for a shell theory.

Following the basic stress assumption $\sigma^{33} = 0$, there are, like in plate theory, different ways open for reaching the goal of reducing a shell theory to essentially two dimensions. Further assumptions can be made on stress distributions, generalizing E. Reissner's approach to the plate bending formulation. Displacement assumptions can be used instead, as in Hencky's plate bending theory, or both systems can be mixed to take advantage of two-field variational principles.
We shall here discuss a displacement formulation, first because it is more widely in use, secondly because it can be specialized to include a discussion of the Kirchhoff-Love hypothesis. The simplest assumption is that of linearity across the shell thickness of the displacement vector :

$$\vec{u} = \vec{u}_{(0)} + x^3\,\vec{u}_{(1)} \quad (9.1)$$

where the two component vectors depend only on the middle surface coordinates. Covariant components of surface displacement tensors are consequently introduced by the definitions

$$\vec{u}_{(0)} = v_\alpha \vec{a}^\alpha + w \vec{a}^3 \tag{9.2}$$

$$\vec{u}_{(1)} = w_\alpha \vec{a}^\alpha + p \vec{a}^3 \tag{9.3}$$

The v_α are middle surface displacements, w is a transversal displacement normal to the middle surface, the w_α are sectional rotations and finally p is a transverse "pinch". The pinch component is usually omitted; it is however essential to preserve the ability of (9.1) to represent a finite rigid body rotation and moreover it lends more symmetry to the subsequent derivations. We next introduce the surface tensors generated by taking surface derivatives of the component vectors

$$D_\beta \vec{u}_{(0)} = \lambda_{\gamma\beta} \vec{a}^\gamma + \phi_\beta \vec{a}^3 = \lambda^\gamma_{.\beta} \vec{a}_\gamma + \phi_\beta \vec{a}_3 \tag{9.4}$$

$$D_\beta \vec{u}_{(1)} = \rho_{\alpha\beta} \vec{a}^\gamma + \psi_\beta \vec{a}^3 = \rho^\gamma_{.\beta} \vec{a}_\gamma + \psi_\beta \vec{a}_3 \tag{9.5}$$

where, from application of formula (3.12), we find

$$\lambda_{\gamma\beta} = v_{\gamma||\beta} - b_{\gamma\beta} w \tag{9.6}$$

$$\rho_{\gamma\beta} = w_{\gamma||\beta} - b_{\gamma\beta} p \tag{9.7}$$

$$\phi_\beta = D_\beta w + b^\varepsilon_\beta v_\varepsilon \tag{9.8}$$

$$\psi_\beta = D_\beta p + b^\varepsilon_\beta w_\varepsilon \tag{9.9}$$

To prepare the computation of the Green strain tensor, we calculate the various derivatives $u_{i\,|\,q}$.

$$\left.\begin{aligned}
u_{\alpha|\beta} &= \vec{g}_\alpha \cdot D_\beta \vec{u} = \mu^\varepsilon_\alpha \vec{a}_\varepsilon \cdot \{ (\lambda_{\gamma\beta} + x^3 \rho_{\gamma\beta}) \vec{a}^\gamma + (\phi_\beta + x^3 \psi_\beta) \vec{a}^3 \} \\
&= \mu^\gamma_\alpha (\lambda_{\gamma\beta} + x^3 \rho_{\gamma\beta}) \\
u_{\alpha|3} &= \vec{g}_\alpha \cdot D_3 \vec{u} = \mu^\varepsilon_\alpha \vec{a}_\varepsilon \cdot \vec{u}_{(1)} = \mu^\varepsilon_\alpha w_\varepsilon \\
u_{3|\beta} &= \vec{g}_3 \cdot D_\beta \vec{u} = \vec{a}_3 \cdot D_\beta \vec{u} = \phi_\beta + x^3 \psi_\beta \\
u_{3|3} &= \vec{g}_3 \cdot D_3 \vec{u} = \vec{a}_3 \cdot \vec{u}_{(1)} = p
\end{aligned}\right] \tag{9.10}$$

Similar calculations, using (5.8), produce

$$\left.\begin{aligned}
u^\alpha|_\beta &= (u^{-1})^\alpha_\gamma (\lambda^\gamma_{.\beta} + x^3 \rho^\gamma_{.\beta}) \\
u^\alpha|_3 &= (u^{-1})^\alpha_\varepsilon w^\varepsilon \\
u^3|_\beta &= u_{3|\beta} \\
u^3|_3 &= u_{3|3}
\end{aligned}\right] \tag{9.11}$$

Substitution into the definitions (7.19) gives then

$$
(9.12)\quad \left[\begin{aligned}
2\gamma_{\alpha\beta} &= \lambda_{\alpha\beta} + \lambda_{\beta\alpha} + \lambda_{\varepsilon\alpha}\lambda^{\varepsilon}_{\cdot\beta} + \phi_{\alpha}\phi_{\beta} \\
&\quad + x^3(\rho_{\alpha\beta} + \rho_{\beta\alpha} - b^{\varepsilon}_{\alpha}\lambda_{\varepsilon\beta} - b^{\varepsilon}_{\beta}\lambda_{\varepsilon\alpha} + \rho_{\varepsilon\alpha}\lambda^{\varepsilon}_{\cdot\beta} + \rho_{\varepsilon\beta}\lambda^{\varepsilon}_{\cdot\alpha} + \phi_{\alpha}\psi_{\beta} + \phi_{\beta}\psi_{\alpha}) \\
&\quad + (x^3)^2\,(-b^{\varepsilon}_{\alpha}\rho_{\varepsilon\beta} - b^{\varepsilon}_{\beta}\rho_{\varepsilon\alpha} + \rho_{\varepsilon\alpha}\rho^{\varepsilon}_{\cdot\beta} + \psi_{\alpha}\psi_{\beta}) \\
2\gamma_{\alpha 3} &= w_{\alpha} + \phi_{\alpha} + w_{\varepsilon}\lambda^{\varepsilon}_{\cdot\alpha} + p\phi_{\alpha} + x^3(\psi_{\alpha} - b^{\varepsilon}_{\alpha}w_{\varepsilon} + w_{\varepsilon}\rho^{\varepsilon}_{\cdot\alpha} + p\psi_{\alpha}) \\
2\gamma_{33} &= 2p + p^2 + w^{\varepsilon}w_{\varepsilon}
\end{aligned}\right.
$$

Some additional physical interpretation of the surface tensors in equations (9.4) and (9.5) can be derived in the case of small material rotations. The incremental displacement field in some infinitesimal neighborhood of a point can then be analyzed, following Helmholtz, as a superposition of a pure strain and a small rotation :

$$d\vec{u} = \vec{\omega} \times d\vec{R} + \gamma_{iq}\, dx^q\, \vec{g}^i \qquad \text{with } \gamma_{qi} = \gamma_{iq} \tag{9.13}$$

The rotational component can be explicited, either by introducing an antisymmetrical tensor ω_{iq}

$$\vec{\omega} \times d\vec{R} = \omega_{iq}\, dx^q\, \vec{g}^i \tag{9.14}$$

$$\omega_{qi} = -\,\omega_{iq} \tag{9.15}$$

or a pseudo-tensor

$$\vec{\omega} = \omega^m\, \vec{g}_m \tag{9.16}$$

Then

$$\vec{\omega} \times d\vec{R} = \omega^m\, dx^q\, \vec{g}_m \times \vec{g}_q = \varepsilon_{mqr}\, \omega^m dx^q\, \vec{g}^r$$

Hence the relationship

$$\omega_{rq} = \varepsilon_{mqr}\, \omega^m \tag{9.17}$$

The antisymmetrical tensor is more convenient because its dependance on x^3 is simpler. Inserting (9.15) into (9.14), replacing

$$d\vec{u} = D_q\vec{u}\, dx^q$$

and comparing coefficients of dx^q, there comes

$$D_{\beta}\vec{u} = (\omega_{i\beta} + \gamma_{i\beta})\, \vec{g}^i$$

$$D_3\vec{u} = (\omega_{i3} + \gamma_{i3})\, \vec{g}^i$$

Scalar multiplication of those equations by $\vec{g}_i$ and evaluation of the left-hand sides as in the previous computation of the strain components, yields

$$\omega_{\alpha\beta} + \gamma_{\alpha\beta} = \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta})$$
$$\omega_{3\beta} + \gamma_{3\beta} = \phi_\beta + x^3 \psi_\beta$$
$$\omega_{\beta 3} + \gamma_{\beta 3} = \mu^{\varepsilon}_{\beta} w_\varepsilon$$
$$\gamma_{33} = p$$

From here, playing on the symmetry characteristics,

$$\left.\begin{aligned} 2\gamma_{\alpha\beta} &= \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta}) + \mu^{\varepsilon}_{\beta} (\lambda_{\varepsilon\alpha} + x^3 \rho_{\varepsilon\alpha}) \\ 2\gamma_{\beta 3} &= \phi_\beta + x^3 \psi_\beta + \mu^{\varepsilon}_{\beta} w_\varepsilon \\ 2\gamma_{33} &= 2p \end{aligned}\right] \quad (9.18)$$

and

$$\left.\begin{aligned} 2\omega_{\alpha\beta} &= \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta}) - \mu^{\varepsilon}_{\beta} (\lambda_{\varepsilon\alpha} + x^3 \rho_{\varepsilon\alpha}) \\ 2\omega_{3\beta} &= \phi_\beta + x^3 \psi_\beta - \mu^{\varepsilon}_{\beta} w_\varepsilon \end{aligned}\right] \quad (9.19)$$

Expressions (9.18) can be recognized as the linearized versions of (9.11), (9.12) and (9.13); they are formed from all the linear terms of the general strain components. The surface tensor of the "rotation of the normal" is, by definition

$$(\omega_{3\beta})_{x^3=0} = \frac{1}{2} (\phi_\beta - w_\beta) \quad (9.20)$$

while the "rotation about the normal" is

$$(\omega_{12})_{x^3=0} = \frac{1}{2} (\lambda_{12} - \lambda_{21}) = \frac{1}{2} (v_{1||2} - v_{2||1}) \quad (9.21)$$

The rate of change in the direction of the normal of the rotation about the normal, or "normal twist", plays an important role in plate theory where it can be shown to be responsible for boundary layer effects, while it vanishes under a Kirchhoff-Love hypothesis. Its expression in shell theory can be defined from

$$D_3 (\omega^3 \vec{g}_3) = (D_3 \omega^3) \vec{g}_3$$

as measured by $D_3\omega^3$. Since according to (9.17)

$$\omega_{21} = \varepsilon_{312} \omega^3 = \sqrt{g}\, \omega^3 \quad (9.22)$$

and in view of (5.5), (5.6) and (2.24) and (9.19)

$$2\sqrt{a}(1-2Hx^3 + (x^3)^2 K)\omega^3 = \lambda_{21} - \lambda_{12} + x^3(\rho_{21} - \rho_{12} - b^{\varepsilon}_{2}\lambda_{\varepsilon 1} + b^{\varepsilon}_{1}\lambda_{\varepsilon 2})$$
$$- (x^3)^2 (b_2 \rho_{\varepsilon 1} - b^{\varepsilon}_{1} \rho_{\varepsilon 2})$$

$$\omega^3 = \omega^3_{(0)} + x^3 \omega^3_{(1)} + (x^3)^2 \omega^3_{(2)} + \ldots$$

where $$2\sqrt{a}\,\omega^3_{(0)} = \lambda_{21} - \lambda_{12}$$

$$2\sqrt{a}\,(-2H\omega^3_{(0)} + \omega^3_{(1)}) = \rho_{21} - \rho_{12} - b^\varepsilon_2\lambda_{\varepsilon 1} + b^\varepsilon_1\lambda_{\varepsilon 2}$$

...

Hence for the normal twist on the middle surface

$$\omega^3_{(1)} = \frac{1}{2\sqrt{a}}\{ b^\varepsilon_\varepsilon (\lambda_{21} - \lambda_{12}) + \rho_{21} - \rho_{12} - b^\varepsilon_2\lambda_{\varepsilon 1} + b^\varepsilon_1\lambda_{\varepsilon 2}\}$$
$$= \frac{1}{2\sqrt{a}}\{ \rho_{21} - \rho_{12} + b^\varepsilon_1\lambda_{2\varepsilon} - b^\varepsilon_2\lambda_{1\varepsilon}\} = \frac{1}{2}\varepsilon^{\alpha\beta}(\rho_{\beta\alpha} + b^\varepsilon_\alpha\lambda_{\beta\varepsilon}) \tag{9.23}$$

The Kirchhoff-Love hypothesis requires $\gamma_{\beta 3} = 0$. It can be satisfied in the linearized case by setting

$$w_\beta = -\phi_\beta = -D_\beta w - b^\varepsilon_\beta v_\varepsilon \tag{9.24}$$

thus relating the sectional rotations to the middle surface and transverse displacement. Also we should have

$$\psi_\beta - b^\varepsilon_\beta w_\varepsilon = D_\beta p = 0 \tag{9.25}$$

a condition identically satisfied by suppressing the pinch component in the original displacement assumptions. Under those conditions

$$\rho_{21} = w_{2||1} = -D_1D_2w - (b^\varepsilon_2 v_\varepsilon)_{||1}$$

and

$$\rho_{21} - \rho_{12} = b^\varepsilon_1 v_{\varepsilon||2} - b^\varepsilon_2 v_{\varepsilon||2} + v_\varepsilon (b^\varepsilon_{1||2} - b^\varepsilon_{2||1})$$

The last term vanishes by virtue of the Mainardi-Codazzi condition, hence, using (9.6)

$$\rho_{21} - \rho_{12} = b^\varepsilon_1\lambda_{\varepsilon 2} - b^\varepsilon_2\lambda_{\varepsilon 1}$$

$$\omega^3_{(1)} = \frac{1}{2\sqrt{a}}\{ b^\varepsilon_1 (\lambda_{\varepsilon 2} + \lambda_{2\varepsilon}) - b^\varepsilon_2 (\lambda_{\varepsilon 1} + \lambda_{1\varepsilon})\}$$

$$= \frac{1}{2}\varepsilon^{\alpha\beta} b^\varepsilon_\alpha (\lambda_{\varepsilon\beta} + \lambda_{\beta\varepsilon}) \tag{9.26}$$

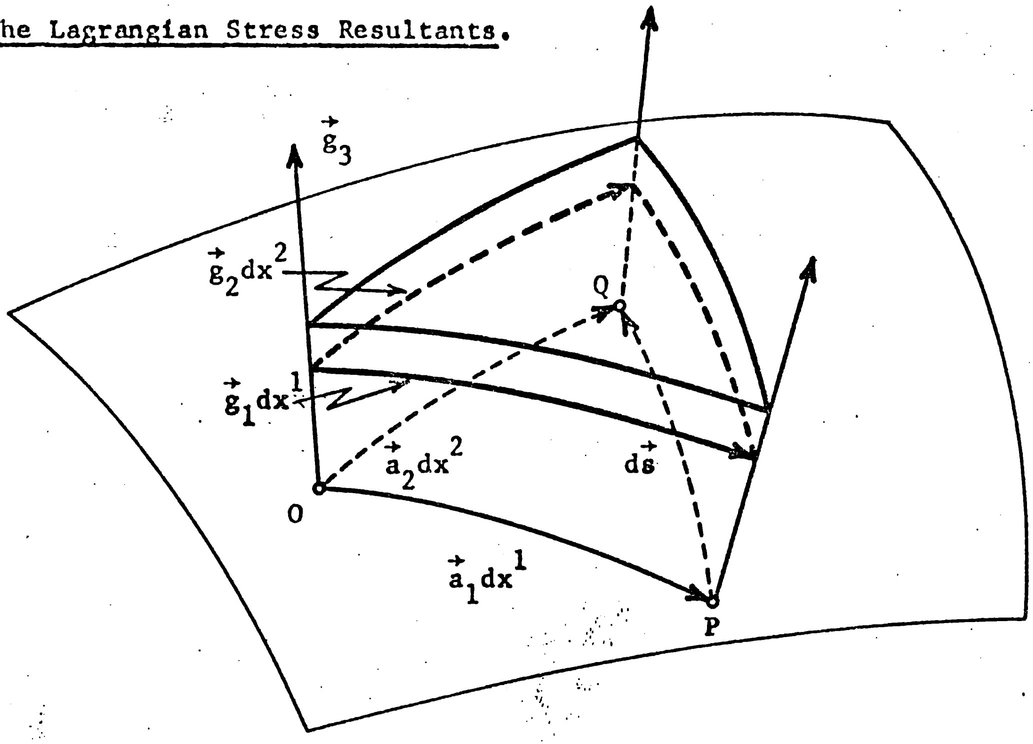
Hence, under a Kirchhoff-Love hypothesis, there remains a normal twist due to the curvature of the middle surface. Substracted from (9.23) it furnishes the "additional normal twist"

$$\Delta\omega^3_{(1)} = \frac{1}{2}\varepsilon^{\alpha\beta}(\rho_{\beta\alpha} - b^\varepsilon_\alpha\lambda_{\varepsilon\beta}) \tag{9.27}$$

Again, in view of the antisymmetry of $\varepsilon^{\alpha\beta}$ and the Mainardi-Codazzi condition, this can be put in the form

$$\Delta\omega^{3}_{(1)} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\,(w_{\beta} - b^{\varepsilon}_{\beta}\,v_{\varepsilon})_{||\alpha} \qquad (9.28)$$

10 The Lagrangian Stress Resultants.



Let O, P and Q be three points in the middle surface, $\vec{OP}$ is taken along the coordinate curve x^2 = constant, $\vec{OQ}$ along the coordinate curve x^1 = constant. The "inclined" side of the triangle, $\vec{PQ}$ is an element $d\vec{s}$ of a contour drawn on the middle surface. Erecting normals to the middle surface in those three points, and measuring equal distances x^3 and dx^3 on each, we define a triangular slab of which the translational equilibrium will be expressed. Obviously at the limit when both P and Q tend to coincide with O, the only forces to take into account will be due to the stresses acting on the lateral sides of the slab. The first lateral side has a surface vector. $\vec{g}_1 \times \vec{g}_3\, dx^1\, dx^3 = -\sqrt{g}\;\vec{g}^2\, dx^1\, dx^3$ oriented outwards. The stress vector on this is $\vec{t}^2$ and the resultant force applied to it

$$-\sqrt{g}\;\sqrt{g^{22}}\;\vec{t}^2\, dx^1\, dx^3 = -\sqrt{g}\; t^{2j}\,\vec{g}_j\, dx^1\, dx^3$$

Similarly for the second lateral side with outwardly oriented surface vector

$$-\vec{g}_2 \times \vec{g}_3\, dx^2\, dx^3 = -\sqrt{g}\;\vec{g}^1\, dx^2\, dx^3$$

on which the resultant force is

$$-\sqrt{g}\;\sqrt{g^{11}}\;\vec{t}^1\, dx^2\, dx^3 = -\sqrt{g}\; t^{1j}\,\vec{g}_j\, dx^2\, dx^3$$

Then, if $d\vec{f}$ denotes the force acting on the third (inclined) lateral side, equilibrium demands

$$d\vec{f} = \sqrt{g}\,(t^{2j}\,\vec{g}_j\,dx^1 + t^{1j}\,\vec{g}_j\,dx^2)\,dx^3 \tag{10.1}$$

Let, as in section 5, $\vec{\nu}$ denote the outward normal to a closed contour described in the positive sense on the middle surface, then

$$\vec{\nu}\,ds = d\vec{s} \times \vec{a}_3 = (-\vec{a}_1\,dx^1 + \vec{a}_2\,dx^2) \times \vec{a}_3$$

$$= \sqrt{a}\,(\vec{a}^2\,dx^1 + \vec{a}^1\,dx^2)$$

or, introducing the covariant components of $\vec{\nu}$ by

$$\vec{\nu} = \nu_\alpha\,\vec{a}^\alpha$$

and comparing

$$\nu_1\,ds = \sqrt{a}\,dx^2 \qquad\qquad \nu_2\,ds = \sqrt{a}\,dx^1 \tag{10.2}$$

This result can be substituted into (10.1) after using (5.5)

$$d\vec{f} = \mu(\nu_2 t^{2j} + \nu_1 t^{1j})\,\vec{g}_j\,ds dx^3 = \mu\nu_\alpha t^{\alpha j}\vec{g}_j ds dx^3$$

or, in view of (5.1) and (5.2),

$$d\vec{f} = \mu(\nu_\alpha t^{\alpha\beta}\mu_\beta^\gamma\,\vec{a}_\gamma + \nu_\alpha t^{\alpha 3}\,\vec{a}_3)\,ds\,dx^3 \tag{10.3}$$

In order to see what are the stress resultants to introduce, we form the scalar product of this elementary force with the incremental displacement vector corresponding to the assumptions (9.1), (9.2) and (9.3)

$$\delta\vec{u} = (\delta v_\varepsilon + x^3\delta w_\varepsilon)\,\vec{a}^\varepsilon + (\delta w + x^3\delta p)\,\vec{a}^3$$

then integrate over the thickness of the shell :

$$\int_{-h}^{h} d\vec{f}\cdot\delta\vec{u}\,dx^3 = (\nu_\alpha N^{\alpha\varepsilon}\delta v_\varepsilon + \nu_\alpha M^{\alpha\varepsilon}\delta w_\varepsilon + \nu_\alpha Q^\alpha dw + \nu_\alpha P^\alpha \delta p)\,ds \tag{10.4}$$

with the membrane resultants

$$N^{\alpha\varepsilon} = \int_{-h}^{h} \mu\mu_\beta^\varepsilon t^{\alpha\beta} dx^3 \tag{10.5}$$

the bending and twisting moments

$$M^{\alpha\varepsilon} = \int_{-h}^{h} \mu\mu_\beta^\varepsilon t^{\alpha\beta} x^3 dx^3 \tag{10.6}$$

the shear loads

$$Q^\alpha = \int_{-h}^{h} \mu t^{\alpha 3}\,dx^3 \tag{10.7}$$

and pinching loads

$$P^{\alpha} = \int_{-h}^{h} \mu t^{\alpha 3}\, x^3\, dx^3 \qquad (10.8)$$

If (10.4) is integrated over a closed contour (c) on the middle surface and transformed into an integral over the enclosed area A by means of theorem (5.9) the contribution of the stresses to the energy increase per unit initial area of the middle surface is obtained in the form

$$\delta F_1 = (N^{\alpha\varepsilon}\delta v_{\varepsilon} + M^{\alpha\varepsilon}\delta w_{\varepsilon} + Q^{\alpha}\delta w + P^{\alpha}\delta p)\,_{||\alpha} \qquad (10.9)$$

There is also a contribution from the externally applied loads, either through body forces, or through surface tractions on the faces $x^3 = \pm h$, of the form

$$\delta F_2 = N^{\varepsilon}\delta v_{\varepsilon} + M^{\varepsilon}\delta w_{\varepsilon} + Q\delta w + P\delta p \qquad (10.10)$$

From the energy equation $\delta F = \delta F_1 + \delta F_2$ the equilibium equations and the constitutive equations can be deduced by a procedure analogous to that of the general three dimensional equations in section 7.

a. Translational equilibrium equations for a piece of shell.

An additional displacement field is of translational type if

$$D_3\, \delta\vec{u} = 0, \text{ which implies } \delta w_{\beta} = 0 \text{ and } \delta p = 0 \qquad (10.11)$$

$$D_{\beta}\, \delta\vec{u} = 0, \text{ which implies } \delta\lambda_{\alpha\beta} = \delta v_{\alpha||\beta} - b_{\alpha\beta}\delta w = 0 \qquad (10.12)$$

$$\text{and } \delta\phi_{\beta} = D_{\beta}\delta w + b^{\varepsilon}_{\beta}\delta v_{\varepsilon} = 0 \qquad (10.13)$$

The energy equation can be reduced to the form

$$\delta F = N^{\alpha\varepsilon}{}_{||\alpha}\delta v_{\varepsilon} + b_{\varepsilon\alpha}N^{\alpha\varepsilon}\delta w + Q^{\alpha}{}_{||\alpha}\delta w - Q^{\alpha}b^{\varepsilon}_{\alpha}\delta v_{\varepsilon} + N^{\varepsilon}\delta v_{\varepsilon} + Q\delta w$$

and, since $\delta F = 0$, equating separately to zero the coefficients of δv_{ε} and δw

$$N^{\alpha\varepsilon}{}_{||\alpha} - Q^{\alpha}b^{\varepsilon}_{\alpha} + N^{\varepsilon} = 0 \quad (\varepsilon = 1, 2) \qquad (10.14)$$

$$Q^{\alpha}{}_{||\alpha} + b_{\varepsilon\alpha}N^{\alpha\varepsilon} + Q = 0 \qquad (10.15)$$

b. Rotational equilibrium equations.

In an expanded form of (10.9), the covariant derivatives of the membrane forces and transverse shear forces can now be substituted from the equilibrium equations (10.14) and (10.15). If, likewise, the covariant derivatives of the incremental displacements are taken from the incremental form of definitions (9.6), (9.7), (9.8) and (9.9)

$$\delta F = N^{\alpha\varepsilon}\delta\lambda_{\varepsilon\alpha} + M^{\alpha\varepsilon}\delta\rho_{\varepsilon\alpha} + (M^{\alpha\varepsilon}{}_{||\alpha} - b^{\varepsilon}_{\alpha}P^{\alpha} + M^{\varepsilon})\delta w_{\varepsilon} + Q^{\alpha}\delta\phi_{\alpha} + P^{\alpha}\delta\psi_{\alpha} + (P^{\alpha}{}_{||\alpha} + b_{\varepsilon\alpha}M^{\alpha\varepsilon} + P)\delta p \qquad (10.16)$$

It is now necessary to derive the particular values taken by the incremental displacement parameters in a small rigid body rotation superinposed on the strained state of the elastic body. Starting from (7.14) and separating the contributions with respect to the x^3 distribution

$$\delta\vec{u}_{(o)} = d\vec{w} \times (\vec{r} + \vec{u}_{(o)}) \qquad (10.17)$$

$$\delta\vec{u}_{(1)} = d\vec{w} \times (\vec{a}_3 + \vec{u}_{(1)}) \qquad (10.18)$$

In these equations we substitute (9.2) and (9.3), together with

$$\vec{r} = r^{\alpha}\vec{a}_{\alpha} + r^{3}\,\vec{a}_{3}$$

$$d\vec{w} = dw^{\lambda}\,\vec{a}_{\lambda} + dw^{3}\,\vec{a}_{3}$$

and make use of

$$\vec{a}_{\lambda} \times \vec{a}_{\alpha} = \varepsilon_{\lambda\alpha}\,\vec{a}^{3} \qquad \vec{a}_{\lambda} \times \vec{a}_{3} = \varepsilon_{\alpha\lambda}\,\vec{a}^{\alpha}$$

Then, comparing covariant components

$$\left.\begin{aligned} \delta v_{\beta} &= \varepsilon_{\beta\lambda}\,(r^{3} + w)\,d\omega^{\lambda} + \varepsilon_{\alpha\beta}\,(r^{\alpha} + v^{\alpha})\,d\omega^{3} \\ \delta w &= \varepsilon_{\lambda\alpha}\,(r^{\alpha} + v^{\alpha})\,d\omega^{\lambda} \\ \delta w_{\beta} &= \varepsilon_{\beta\lambda}\,(1 + p)\,d\omega^{\lambda} + \varepsilon_{\alpha\beta}w^{\alpha}d\omega^{3} \\ \delta p &= \varepsilon_{\lambda\alpha}\,w^{\alpha}d\omega^{\lambda} \end{aligned}\right\} \qquad (10.19)$$

Next, taking partial derivatives with respect to x^{β} on both sides of (10.17) and (10.18) and remembering (9.4) and (9.5)

$$\delta\lambda_{\gamma\beta}\vec{a}^{\gamma} + \delta\phi_{\beta}\vec{a}^{3} = \Lambda^{\alpha}_{\cdot\beta}\;d\vec{\omega} \times \vec{a}_{\alpha} + \phi_{\beta}d\vec{\omega} \times \vec{a}_{3}$$

$$\delta\rho_{\gamma\beta}\vec{a}^{\gamma} + \delta\psi_{\beta}\vec{a}^{3} = R^{\alpha}_{\cdot\beta}\;d\vec{\omega} \times \vec{a}_{\alpha} + \psi_{\beta}d\vec{\omega} \times \vec{a}_{3}$$

where, for concision, the following notations were introduced

$$\Lambda^{\alpha}{}_{\cdot\beta} = \delta^{\alpha}_{\beta} + \lambda^{\alpha}_{\cdot\beta} \qquad\qquad R^{\alpha}_{\cdot\beta} = \rho^{\alpha}_{\cdot\beta} - b^{\alpha}_{\beta} \qquad (10.20)$$

From those equations we deduce

$$\left.\begin{aligned} \delta\lambda_{\gamma\beta} &= \varepsilon_{\alpha\gamma}\Lambda^{\alpha}_{\cdot\beta}\;d\omega^{3} + \varepsilon_{\gamma\lambda}\phi_{\beta}\;d\omega^{\lambda} \\ \delta\phi_{\beta} &= \varepsilon_{\lambda\alpha}\Lambda^{\alpha}_{\cdot\beta}\;d\omega^{\lambda} \\ \delta\rho_{\gamma\beta} &= \varepsilon_{\alpha\gamma}R^{\alpha}_{\cdot\beta}\;d\omega^{3} + \varepsilon_{\gamma\lambda}\psi_{\beta}\;d\omega^{\lambda} \\ \delta\psi_{\beta} &= \varepsilon_{\lambda\alpha}R^{\alpha}_{\cdot\beta}\;d\omega^{\lambda} \end{aligned}\right\} \qquad (10.21)$$

The results (10.19) and (10.21) are to be inserted into (10.16) and since $\delta F = 0$ again, while $d\vec{w}$ is an arbitrary constant vector, the following general conditions for rotational equilibrium are found :

b.1. equilibrium about the normal to the middle surface, obtained by equating to zero the coefficient of $d\omega^{3}$

$$\varepsilon_{\gamma\beta}\left[N^{\alpha\beta}\Lambda^{\gamma}_{\cdot\alpha} + M^{\alpha\beta}R^{\gamma}_{\cdot\alpha} + S^{\beta}w^{\gamma}\right] = 0 \qquad (10.22)$$

with the definition

$$S^{\beta} = M^{\alpha\beta}{}_{||\alpha} - b^{\beta}_{\alpha}P^{\alpha} + M^{\beta} \qquad (10.23)$$

Considering the antisymmetry of $\varepsilon_{\gamma\beta}$, this is equivalent to require the symmetry of the bracket in (10.22) with respect to β and γ.

b.2. equilibrium about axes tangent to the middle surface, expressed by equating to zero the coefficient of $\varepsilon_{\gamma\lambda}\, d\omega^{\lambda}$

$$N^{\alpha\gamma}\phi_{\alpha} + M^{\alpha\gamma}\psi_{\alpha} + S^{\gamma}(1 + p) - Q^{\alpha}\Lambda^{\gamma}_{\cdot\alpha} - P^{\alpha}R^{\gamma}_{\cdot\alpha} - Tw^{\gamma} = 0 \qquad (10.24)$$

$$(\gamma = 1,\ 2)$$

with the definition

$$T = P^{\alpha}\big|\big|_{\alpha} + b_{\varepsilon\alpha}M^{\alpha\varepsilon} + P$$

As in the case of the general three-dimensional equilibrium equations satisfied by lagrangian stresses, the lagrangian resultants obey linear translational equilibrium conditiond (10.14) and (10.15), but their rotational equilibrium conditions (10.22) and (10.24) also involve deformations.

An attempt to calculate δF directly from the general result (7.13) and the relation

$$\delta W dV = \delta W\sqrt{g}\, dx^1 dx^2 dx^3 = \delta W\mu\sqrt{a}\, dx^1 dx^2 dx^3 = \delta W\mu dx^3 dS$$

or

$$\delta F = \int_{-h}^{h} \mu\delta W\, dx^3$$

gives, in view of definitions (10.5) to (10.8) and results (9.10)

$$\delta F = N^{\beta\gamma}\delta\lambda_{\gamma\beta} + M^{\beta\gamma}\delta\rho_{\gamma\beta} + Q^{\beta}\delta\phi_{\beta} + P^{\beta}\delta\psi_{\beta} + S^{\beta}\delta w_{\beta} + T\delta p \qquad (10.26)$$

which is similar to (10.16) and shows that the tensors defined by (10.23) and (10.25) are also resultants

$$S^{\beta} = \int_{-h}^{h} \mu\mu^{\beta}_{\alpha}\, t^{3\alpha}\, dx^3 \qquad\qquad T = \int_{-h}^{h} \mu t^{33}\, dx^3 \qquad (10.27)$$

They do not, however, constitute internal loads, since they involve lagrangian stresses which are defined at interfaces between shell pieces.
As a matter of fact, the rotational equilibrium condition about the displaced normal to the middle surface does not contain S^{β} nor T.
To show it we calculate the base vectors in the deformed metric from (7.22) and (9.11) obtaining

$$\vec{G}_{\gamma} = (\Lambda^{\varepsilon}_{\cdot\gamma} + x^3 R^{\varepsilon}_{\cdot\gamma})\,\vec{a}_{\varepsilon} + (\phi_{\gamma} + x^3\psi_{\gamma})\,\vec{a}_3 \qquad (10.28)$$

$$\vec{G}_3 = w^{\varepsilon}\vec{a}_{\varepsilon} + (1 + p)\,\vec{a}_3 \qquad (10.29)$$

The displaced base vectors on the middle surface are then given by

$$\vec{A}_{\gamma} = \Lambda^{\varepsilon}_{\cdot\gamma}\vec{a}_{\varepsilon} + \phi_{\gamma}\vec{a}_3 \qquad (10.30)$$

$$\vec{A}_3 = w^{\lambda}\vec{a}_{\lambda} + (1 + p)\,\vec{a}_3 = \vec{G}_3 \qquad (10.31)$$

Thus, if we denote by $R^{\gamma\beta}$ the bracket in (10.22) and by R^{γ} the left-hand side of (10.24)

$$\varepsilon_{\gamma\beta} R^{\gamma\beta} d\omega^3 + \varepsilon_{\gamma\lambda} R^{\gamma} d\omega^{\lambda} = 0$$

is the general rotational equilibrium condition, while for a rotation $d\Omega^3$ about the displaced normal $\vec{A}^3$

$$d\omega^{\lambda} = w^{\lambda} d\Omega^3 \qquad d\omega^3 = (1 + p)\, d\Omega^3 .$$

and the corresponding condition is

$$(1 + p)\, \varepsilon_{\gamma\beta} R^{\gamma\beta} + w^{\lambda} \varepsilon_{\gamma\lambda} R^{\gamma} = 0$$

or symmetry of

$$(1 + p)\, R^{\gamma\beta} + w^{\beta} R^{\gamma}$$

In this linear combination S^{β} and T do precisely appear in symmetrical groupings

$$(1 + p)\, (w^{\gamma} S^{\beta} + w^{\beta} S^{\gamma}) - T w^{\gamma} w^{\beta}$$

and may consequently be dropped leaving the condition

$$\begin{aligned} &(1 + p)\, N^{\alpha\beta} \Lambda^{\gamma}_{\cdot\alpha} + w^{\beta} \phi_{\alpha} N^{\alpha\gamma} + (1 + p)\, M^{\alpha\beta} R^{\gamma}_{\cdot\alpha} + w^{\beta} \psi_{\alpha} M^{\alpha\gamma} \\ &\qquad - Q^{\alpha} w^{\beta} \Lambda^{\gamma}_{\cdot\alpha} - P^{\alpha} w^{\beta} R^{\gamma}_{\cdot\alpha} \qquad \text{symmetrical in } (\beta, \gamma) \end{aligned} \tag{10.32}$$

The equilibrium conditions (10.24) can now be incorporated into the energy equation (10.26) by using them to eliminate S^{β}. This gives

$$\begin{aligned} \delta F = {} & N^{\beta\gamma} \left(\delta\lambda_{\gamma\beta} - \frac{1}{1 + p} \phi_{\beta} \delta w_{\gamma} \right) + M^{\beta\gamma} \left(\delta\rho_{\gamma\beta} - \frac{1}{1 + p} \psi_{\beta} \delta w_{\gamma} \right) \\ & + Q^{\beta} \left(\delta\phi_{\beta} + \frac{1}{1 + p} \Lambda^{\gamma}_{\cdot\beta} \delta w_{\gamma} \right) + P^{\beta} \left(\delta\psi_{\beta} + \frac{1}{1 + p} R^{\gamma}_{\cdot\beta} \delta w_{\gamma} \right) \\ & + T \left(\delta p + \frac{1}{1 + p} w^{\gamma} \delta w_{\gamma} \right) \end{aligned} \tag{10.33}$$

This new form of the energy equation cannot be expected to turn into a canonical form similar to (7.20), from which constitutive equations can be rationally derived, untill the last rotational equilibrium equation (10.32) can be solved and the constraint it constitutes between the lagrangian stress resultants can be liberated.

11. Kirchhoff-Trefftz stress parameters.

Since the Kirchhoff-Trefftz stress tensor contains implicitly the rotational equilibrium conditions in its property of symmetry, an immediate solution to our problem consists in transforming the definitions (10.5) to (10.8) by means of the relations (7.17). In our case, considering equations (9.11), we can substitute

$$\begin{aligned} t^{\alpha\beta} &= s^{\alpha\beta} + u^{\beta}|_{\varepsilon} s^{\varepsilon\alpha} + u^{\beta}|_{3} s^{3\alpha} \\ &= s^{\alpha\beta} + (\mu^{-1})^{\beta}_{\gamma} (\lambda^{\gamma}_{\cdot\varepsilon} + x^3 \rho^{\gamma}_{\cdot\varepsilon})\, s^{\varepsilon\alpha} + (\mu^{-1})^{\beta}_{\gamma} w^{\gamma} s^{3\alpha} \\ t^{\alpha 3} &= s^{\alpha 3} + u^{3}|_{\beta} s^{\beta\alpha} + u^{3}|_{3} s^{3\alpha} \\ &= s^{\alpha 3} + (\phi_{\beta} + x^3 \psi_{\beta}) s^{\beta\alpha} + p s^{3\alpha} \\ t^{3\alpha} &= s^{3\alpha} + u^{\alpha}|_{\beta} s^{\beta 3} + u^{\alpha}|_{3} s^{33} \\ &= s^{3\alpha} + (\mu^{-1})^{\alpha}_{\gamma} (\lambda^{\gamma}_{\cdot\beta} + x^3 \rho^{\gamma}_{\cdot\beta}) s^{\beta 3} + (\mu^{-1})^{\alpha}_{\gamma} w^{\gamma} s^{33} \\ t^{33} &= s^{33} + u^{3}|_{\beta} s^{\beta 3} + u^{3}|_{3} s^{33} \\ &= s^{33} + (\phi_{\beta} + x^3 \psi_{\beta})\, s^{\beta 3} + p\, s^{33} \end{aligned} \tag{11.1}$$

The lagrangian stress resultants are then expressed in terms of the following Kirchhoff-Trefftz stress parameters

$$n^{\alpha\beta} = \int_{-h}^{h} \mu s^{\alpha\beta} dx^3 = n^{\beta\alpha}$$

$$m^{\alpha\beta} = \int_{-h}^{h} \mu s^{\alpha\beta} x^3 dx^3 = m^{\beta\alpha}$$

$$l^{\alpha\beta} = \int_{-h}^{h} \mu s^{\alpha\beta} (x^3)^2 \, dx^3 = l^{\beta\alpha} \qquad (11.2)$$

$$q^{\beta} = \int_{-h}^{h} \mu s^{\beta 3} \, dx^3$$

$$p^{\beta} = \int_{-h}^{h} s^{\beta 3} \, x^3 \, dx^3$$

$$n = \int_{-h}^{h} \mu s^{33} \, dx^3$$

There comes

$$N^{\alpha\varepsilon} = \Lambda^{\varepsilon}_{\cdot\gamma} \, n^{\gamma\alpha} + R^{\varepsilon}_{\cdot\gamma} \, m^{\gamma\alpha} + w^{\varepsilon} q^{\alpha}$$

$$M^{\alpha\varepsilon} = \Lambda^{\varepsilon}_{\cdot\gamma} \, m^{\gamma\alpha} + R^{\varepsilon}_{\cdot\gamma} \, l^{\gamma\alpha} + w^{\varepsilon} p^{\alpha} \qquad (11.3)$$

$$Q^{\alpha} = (1 + p) \, q^{\alpha} + \phi_{\gamma} \, n^{\gamma\alpha} + \psi_{\gamma} \, m^{\gamma\alpha}$$

$$P^{\alpha} = (1 + p) \, p^{\alpha} + \phi_{\gamma} \, m^{\gamma\alpha} + \psi_{\gamma} \, l^{\gamma\alpha}$$

and also

$$S^{\beta} = \Lambda^{\beta}_{\cdot\alpha} q^{\alpha} + R^{\beta}_{\cdot\alpha} p^{\alpha} + w^{\beta} n \qquad (11.4)$$

$$T = (1 + p) \, n + \phi_{\alpha} q^{\alpha} + \psi_{\alpha} p^{\alpha}$$

It is easily verified that all the rotational equilibrium conditions (10.22) and (10.24) or (10.32) are thereby satisfied. Furthermore the energy equation (10.26) takes the canonical form

$$\delta F = n^{\gamma\alpha} \delta \overset{\circ}{\gamma}_{\gamma\alpha} + m^{\gamma\alpha} \delta \overset{1}{\gamma}_{\gamma\alpha} + l^{\gamma\alpha} \delta \overset{2}{\gamma}_{\gamma\alpha}$$

$$+ 2q^{\alpha} \delta \overset{\circ}{\gamma}_{\alpha 3} + 2p^{\alpha} \delta \overset{1}{\gamma}_{\alpha 3} + n \delta \gamma_{33} \qquad (11.5)$$

where the strains are the coefficients of the powers in x^3 of the Green strain tensor (9.12)

As a matter of fact, this result is equivalent to a straight forward application of a Rayleigh-Ritz process based on the displacement assumptions (9.1) to (9.3) to the expression

$$\delta F = \int_{-h}^{h} \mu \delta W \, dx^3$$

where δW is given by (7.20). Applied to the form (8.1) of the energy density it introduces the following rigidity coefficients for the shell :

$$D^{\alpha\beta}_{(m)}{}^{\gamma\delta} = \frac{E}{1-\nu^2} \int_{-h}^{h} \mu g^{\alpha\beta} g^{\gamma\delta} (x^3)^m \, dx^3 \qquad (m = 0,1,\dots 4) \qquad (11.6)$$

$$C^{\alpha\beta}_{(m)} = \frac{E}{1-\nu^2} \int_{-h}^{h} \mu g^{\alpha\beta} (x^3)^m \, dx^3 \qquad (m = 0,1,2) \qquad (11.7)$$

Taking due account of the symmetries, the total number of rigidity coefficients of type (11.6) is 6 × 5 = 30, of type (11.7) 3 × 3 = 9.
The energy per mit area is then expressible as

$$\begin{aligned} 2\,F = {} & D^{\alpha\lambda}_{(0)}{}^{\beta\mu} \{ \nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{\circ}{\gamma}_{\beta\mu} + (1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{\circ}{\gamma}_{\beta\lambda} \} \\ & + 2\, D^{\alpha\lambda}_{(1)}{}^{\beta\mu} \{ \nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{1}{\gamma}_{\beta\mu} + (1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{1}{\gamma}_{\beta\lambda} \} \\ & + D^{\alpha\lambda}_{(2)}{}^{\beta\mu} \{ 2\nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + \nu \overset{1}{\gamma}_{\alpha\lambda} \overset{1}{\gamma}_{\beta\mu} + 2(1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} + (1-\nu) \overset{1}{\gamma}_{\alpha\mu} \overset{1}{\gamma}_{\beta\lambda} \} \\ & + 2\, D^{\alpha\lambda}_{(3)}{}^{\beta\mu} \{ \nu \overset{1}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + (1-\nu) \overset{1}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} \} \\ & + D^{\alpha\lambda}_{(4)}{}^{\beta\mu} \{ \nu \overset{2}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + (1-\nu) \overset{2}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} \} \\ & + 2(1-\nu) \{ C^{\alpha\beta}_{(0)} \overset{\circ}{\gamma}_{\alpha 3} \overset{\circ}{\gamma}_{\beta 3} + 2\, C^{\alpha\beta}_{(1)} \overset{\circ}{\gamma}_{\alpha 3} \overset{1}{\gamma}_{\beta 3} + C^{\alpha\beta}_{(2)} \overset{1}{\gamma}_{\alpha 3} \overset{1}{\gamma}_{\beta 3} \} \end{aligned} \qquad (11.8)$$

The constitutive equations of the shell follow by taking the partial derivatives of (11.8) as indicated in the total differential (11.5).
In particular, since (11.8) is independant of γ_{33}, we find n = 0 in accordance with the assumption $\sigma^{33} = 0$.
The integrals in (11.6) and (11.7) are not easily carried out in closed form because of the complicated dependance of the reciprocal metric tensor (5.10) on the coordinate x^3. They can be evaluated by numerical quadrature methods or expanded in powers of a (small) parameter.

It follows from (11.5) that the Kirchhoff-Trefftz stress parameters are internal conjugate variables to the arguments of the strain energy per unit surface :

$$n^{\gamma\alpha} = \frac{\partial F}{\partial \overset{\circ}{\gamma}_{\gamma\alpha}} \qquad m^{\gamma\alpha} = \frac{\partial F}{\partial \overset{1}{\gamma}_{\gamma\alpha}} \qquad l^{\gamma\alpha} = \frac{\partial F}{\partial \overset{2}{\gamma}_{\gamma\alpha}} \qquad (11.9)$$

$$2q^{\alpha} = \frac{\partial F}{\partial \overset{\circ}{\gamma}_{\alpha 3}} \qquad 2p^{\alpha} = \frac{\partial F}{\partial \overset{1}{\gamma}_{\alpha 3}} \qquad n = \frac{\partial F}{\partial \gamma_{33}}$$

If those arguments as given by (9.12) are expressed through definitions (9.6) to (9.9) in terms of v_α, w_α, w, p and their derivatives, it is easily found by the chain rule of differentiation and by the results (11.3) and (11.4) that

$$\frac{\partial F}{\partial v_{\alpha||\beta}} = N^{\beta\alpha} \qquad \frac{\partial F}{\partial w_{\alpha||\beta}} = M^{\beta\alpha}$$

$$\frac{\partial F}{\partial w_{||\alpha}} = Q^{\alpha} \qquad \frac{\partial F}{\partial p_{||\alpha}} = P^{\alpha} \qquad (11.10)$$

$$\frac{\partial F}{\partial p} = T - b_{\alpha\beta} M^{\alpha\beta} \qquad \frac{\partial F}{\partial w_{\alpha}} = b^{\alpha}_{\varepsilon} P^{\varepsilon} + S^{\alpha}$$

$$\frac{\partial F}{\partial w} = -b_{\alpha\beta} N^{\alpha\beta} \qquad \frac{\partial F}{\partial v_{\alpha}} = b^{\alpha}_{\varepsilon} Q^{\varepsilon} \qquad (11.11)$$

Equations (11.10) can then be considered as the constitutive equations of the shell in terms of Lagrangian resultants. Since they involve (11.3) and (11.4) they automatically satisfy the rotational equilibrium equations. They also allow to apply in a simple fashion the variational principle

$$\delta \left\{ \int_S F dS - \int_S (N^{\alpha} v_{\alpha} + M^{\alpha} w_{\alpha} + Qw + Pp)\, dS + F_3 \right\} = 0 \qquad (11.12)$$

where F_3 denotes the potential energy of external loads applied at the boundary. From (11.10) and (11.11) there comes

$$\int_S \{ N^{\beta\alpha} \delta v_{\alpha||\beta} + M^{\beta\alpha} \delta w_{\alpha||\beta} + Q^{\alpha} \delta w_{||\alpha} + P^{\alpha} \delta p_{||\alpha}$$

$$+ (b^{\alpha}_{\varepsilon} Q^{\varepsilon} - N^{\alpha})\, \delta v_{\alpha} + (S^{\alpha} + b^{\alpha}_{\varepsilon} P^{\varepsilon} - M^{\alpha})\, \delta w_{\alpha}$$

$$- (Q + b_{\alpha\beta} N^{\alpha\beta})\, \delta w + (T - b_{\alpha\beta} M^{\alpha\beta} - P) \delta p \}\, dS + \delta F_3 = 0$$

After integration by parts, the following Euler-Lagrange equilibrium equations are obtained :

for δv_{α} $\qquad - N^{\beta\alpha}{}_{||\beta} + b^{\alpha}_{\varepsilon} Q^{\varepsilon} - N^{\alpha} = 0$ (11.13)

δw_{α} $\qquad - M^{\beta\alpha}{}_{||\beta} + S^{\alpha} + b^{\alpha}_{\varepsilon} P^{\varepsilon} - M^{\alpha} = 0$ (11.14)

δw $\qquad - Q^{\alpha}{}_{||\alpha} - Q - b_{\alpha\beta} N^{\alpha\beta} = 0$ (11.15)

$$\delta p \qquad - P^{\alpha}{}_{||\alpha} + T - b_{\alpha\beta} M^{\alpha\beta} - P = 0 \qquad (11.16)$$

Associated with boundary conditions resulting from the variational equation

$$\delta F_3 + \phi \nu_\beta | N^{\beta\alpha} \delta v_\alpha + M^{\beta\alpha} \delta w_\alpha + Q^{\alpha} \delta w + P^{\alpha} \delta p)\, ds = 0 \qquad (11.17)$$

In equations (11.13) and (11.15) we recognize the translational equilibrium equations already obtained in (10.14) and (10.15), while (11.14) and (11.16) are respectively equivalent to the definitions (10.23) and (10.25) of S^{α} and T.
The field equations of the two-dimensional non linear shell problem consist of the 6 linear equilibrium equations (11.13) to (11.16) together with the 15 resultants-displacement derivatives equations (11.10). Those are to be worked out by using in succession (11.8), (9.12) and (9.6) to (9.9). The total number of unknowns is also 21 : the 6 displacements (v_α, w_α, w, p) and the 15 components of resultants in $N^{\alpha\beta}$, $M^{\alpha\beta}$, Q^{α}, P^{α}, S^{α}, T. Because of the identities

$$\frac{\partial F}{\partial v_\alpha} = b^{\alpha}_{\varepsilon} \frac{\partial F}{\partial \phi_\varepsilon} = b^{\alpha}_{\varepsilon} \frac{\partial F}{\partial w_{||\varepsilon}} \quad \text{and} \quad \frac{\partial F}{\partial w} = - b_{\alpha\beta} \frac{\partial F}{\partial \lambda_{\alpha\beta}} = - b_{\alpha\beta} \frac{\partial F}{\partial v_{\alpha||\beta}}$$

stemming from (9.9) and (9.6), equations (11.11) must not be counted in the constitutive equations.