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DUAL ANALYSIS BY FINITE ELEMENTS: LINEAR AND NON LINEAR APPLICATIONS

B. FRAEIJS DE VEUBEKE
G. SANDER
P. BECKERS

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FOREWORD

This report was prepared by the Aeronautics and Space Laboratory, University of Liege, Belgium, under Contract F61052-69-C-0004, Project No. 1467, "Structural Analysis Methods," Task No. 146705, "Automatic Computer Methods of Analysis for Flight Vehicle Structures." The work was administered under the direction of the Air Force Flight Dynamics Laboratory by Mr. James R. Johnson, Project Engineer, and through the European Office of Aerospace Research (OAR), United States Air Force. Lt. Colonel Richard T. Boverie, and later Major Richard Peterson, were the Project Officers for the European Office of Aerospace Research.

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This report has been reviewed and is approved.

FRANCIS J. JANIK, JR.
Chief, Solid Mechanics Branch
Structures Division

ABSTRACT

This document presents the results obtained during the aforementioned period concerning the linear and non linear applications of the dual analysis technique. New formulation for finite element derivation are proposed including equilibrium, conforming and hybrid elements. A new method of solution of the structural problem is based on the stress function formulation. The slab analogies are used to discover additional finite element families.

A useful complementary energy principle is derived for large deformations. The stability criterion is obtained from the second variation of the total energy.

A non linear shell theory is presented which avoids the difficulties of defining the constitutive equations.

A computer program for linear applications is documented.

TABLE OF CONTENTS

4. A non linear theory of shells

1. Coordinate system
2. Base vectors for surface and space tensors
3. Covariant surface derivatives
4. Covariant space derivatives
5. The structure of space metric in terms of surface metric for a shell
6. The divergence operator and the divergence theorems
7. Three-dimensional elasticity in curvilinear coordinates
8. Stress assumption for shell theory
9. Displacement assumptions for a shell theory
10. The Lagrangian stress resultants
11. Kirchhoff-Treffz stress parameters

Appendix : The ASEF Computer Program - Version 20

A1 General description

A2 Organisation of the program

A3 Input data preparation

A4 Description of the finite elements library

A5 Examples

1. INTRODUCTION

The present report is written in the spirit of presenting the results obtained after 5 years of development of finite element models and methods of solution adapted to a dual analysis of aerospace structures.

It is subdivided in four main parts : the first covers the linear elastic applications of the finite element method and deals primarily with the development of finite element models and the relevant methods of solutions. The second part presents the basic formulations of the non linear problems, including stability. The third is devoted to a formulation of non linear shell theory. The final section describes the general computer program that has been developped to solve linear applications.

As regards the first part, considerable improvements were achieved in the formulation of the general theories, allowing the derivation not only of conforming and equilibrium models but also of hybrid models based on two field variational principles. The cases where the number of parameters is smaller than the number of interface degrees of freedom was ruled out of previous formulations for conforming elements or led to spurious kinematic deformation modes for equilibrium elements. The present formulation shows that in these cases it is always possible to derive an element but with some associated constraints. Elimination of such constraints within a group of elements appears to be a general procedure for these situations. It justifies a posteriori and generalizes the derivation of certain previously develop elements like our conforming quadrilateral plate bending element. The possibility of introducing internal displacement functions (also called bubble modes) is systematically considered. It allows a better stress representation without increasing the final number of degrees of freedom.

The static-kinematic analogies of Southwell were used to generate families of equilibrium elements dual of each existing family of conforming elements. Sub equilibrium and hyper equilibrium elements have been defined as the analogues of non-conforming or over-conforming elements.

The formulation of equilibrium models in terms of stress functions opens new avenues. In fact by discretizing the stress functions field instead of stress field, the formulation becomes strictly identical to that of the displacement element. But instead of yielding a stiffness matrix, it yields a flexibility matrix and the natural unknowns are local values of the stress functions instead of local values of displacements.

In the particular case where the static-kinematic analogies apply, the derivation of the flexibility matrix is trivial and consists simply in using the program generating the stiffness matrix of the analogous displacement element with a different Hooke's matrix. As the natural unknowns are the local values of the stress functions and as these can be interpreted as self stressing states of limited diffusion, one is inclined to use a force program for the solution and take advantage of the fact that the selection of the redundancies is no longer necessary.
However a closer examination reveals that the easiest solution is obtained for a scheme similar to the so-called combined method. This idea has been successfully applied to the solution of practical problems and its application presented so simply that it has been incorporated in the general computer program without any difficulty.

Important improvements were obtained by rationalizing the library of models and by automating the algebraic operations, treatment of polynomials and analytic formal integration. In this way, finite element families rather than individual finite element can be programmed. The stage is reached where displacement, equilibrium and hybrid models can be generated entirely automatically for polynomial approximations of any order, as specified by the user.

The systematic use of polynomial elements of variable degree, containing eventually bubble modes of higher polynomial degree, simplifies considerably the idealization of the structure in regions of high stress gradients. The mesh size may remain the same as in other regions, only the degrees of the various elements are assigned different values. Preliminary studies of difficult stress interpretation problems including singularities have shown that this technique allows a better, and at the same time cheaper, stress representation.
The part devoted to large displacement formulations goes further than Report AFFDL-TR-70-34. In this previous report an attempt was made to develop a variational principle of eulerian type, wherein translational and rotational equilibrium conditions would have been incorporated from start. It would them have been an extension of the complementary energy principle of linear elasticity and a useful tool for extending dual analysis to geometrical non-linearities. The idea behind the eulerian formulation was that true (eulerian) stresses do satisfy linear translational equilibrium equations, while satisfying already rotational equilibrium by their symmetry. It is however now recognized that, while such a principle is

feasible, it must be restricted to isotropic material behavior and is consequently too narrow. Hence all considerations about eulerian strain measures and variations have now been dropped.

In the lagrangian formulation an important step was achieved by avoiding a too early satisfaction of the rotational equilibrium. In the earlier derivations of large displacement variational principles, rotational equilibrium was implicitly assumed from start by working with the symmetrical stress tensor of Kirchhoff and Trefftz. A new approach makes explicit use of the polar decomposition of the neighborhood transformation and so introduces explicitly the finite rotation operator, whose variation results in Euler-Lagrange equations expressing rotational equilibrium. A new symmetrical stress tensor is naturally involved in this polar decomposition; it is formed from partial derivatives of the strain energy density with respect to the engineering strain measures of the polar decomposition. In this way a complete filiation of variational principles can be formulated as in the linear elasticity case. The equivalent of the complementary energy principle does now contain as Euler-Lagrange variational equations not only the compatibility equations but also the rotational equilibrium conditions. It shows that rotational equilibrium is not mandatory for the upper bound character of the approximate solutions and can be relaxed. Furthermore, relaxation of the rotational equilibrium, opens new avenues for the derivation of three-dimensional equilibrium finite elements in the linear case. All the implications of this new principle must still be evaluated and applied in numerical analysis.

The part devoted to stability of the elastic equilibrium is an amplified theoretical analysis version of Report AFFDL-TR-70-35, wherein the numerical applications as contained in there and in Reports AFFDL-TR-70-36 and 70-37 are not reproduced.

The starting point is again the second variation criterion of the total energy, that is applicable to both bifurcation and snap through phenomena. It can however be formulated both in the initial unstressed and in the equilibrium configuration as reference configurations. Comparison between both formulations have been further investigated and the results correlated with the quite different intuitive approach initiated by Timoshenko. In this way a rather satisfactory justification of the usual plate stability criterion can be obtained.

In the case of buckling of straight rods under combined flexure and torsion a similar discussion was performed and, in addition, a complete justifica-

tion given for the flexure-torsion center position in relation to warping. A considerable effort was made towards a scientific approach to non-linear deep shell analysis in order to provide the correct background for later development of finite element models. The report contains a synthesis of the available theory with probably some original features.
Particular attention was devoted to the significance of resultant loads and moments in terms of lagrangian or Kirchhoff-Trefftz stress tensors.
The final part of the report is devoted to a description of the ASEF Computer Program. The main function of this program is to provide a tool for investigations of new finite elements and new methods of solution. The second function is to provide the capability of solving large, practical problems and evaluate on them the impact of the advantages brought by new theories or new elements. The existing library of finite elements contains various equilibrium conforming and hybrid elements of membrane and flat plate. The degree of most of these elements is variable. The program works basically according to the displacement method, but additional linear constraints, in any number and of any kind, can be processed efficiently. This capability allows to solve equilibrium problems by the stress functions method. A direct substructures method of solution can be used as well as an iterative method based on conjugate gradients. An important effort has been devoted to reduce the difficulty of input data preparation to a minimum. If due consideration is given to the large number of finite element models proposed, this goal seems to be reached.

CHAPTER II

LINEAR APPLICATIONS

2.1. Development of finite elements with internal degrees of freedom. The concept of generalized internal degree of freedom.

The question of whether it is better to idealize a structure by many simple finite elements or by a smaller number of more complex elements has been often discussed (Ref. 2.1, 2.2). The purpose of this section is to introduce a new concept in the idealization of structures into finite elements and in the development of the finite elements themselves. This concept of internal degrees of freedom, coupled to that of finite element of variable degree, leads to a reconsideration of how to idealize a structure and to achieve it in a simpler as well as more efficient way.

From both theoretical and experimental points of view it appears advantageous to use the more sophisticated elements. The accuracy for a given number of degrees of freedom is improved and the difficulties in interpreting the stresses are reduced (Ref. 2.1, 2.2, 2.3). The practicians however often state that in a large number of structural problems the physical subdivision of the structure leads to a natural idealization into finite elements so fine that even the simplest elements are sufficient for obtaining a good approximation to the displacements. In such problems, the advantages offered by sophisticated elements are almost evanescent since many more degrees of freedom result only in minor improvement of the results. The use of sophisticated elements seems therefore to be restricted to sophisticated situations or to problems where large regions are free of physical discontinuities (like in field problems as opposed to structural problems).

In certain structural problems however, parts of the structure exist where many simple elements are used, while it is clear that few more complex elements would do much better. Such a situation is typical of regions of high stress gradient or of particular interest to the analyst. Using refined elements in the whole structure is however not a practical procedure because it is usually too expensive.

As a result it seems desirable to look for the possibility of mixing elements of different degree of complexity, while still preserving the compatibility (or equilibrium) of the solution (figure 2.1). The development of elements based on polynomials of variable degree, described in the following sections, allows for this possibility. It avoids the cost of useless degrees of freedom in those parts of the structure where

stress gradients are known to be small.

From recent work, it appears that the goal of improving the behavior of finite element models is to be looked for in new directions. After the initial, very simple, elements the evolution tendancy was to increase the degree of the polynomial approximations in the displacement or stress fields (Ref. 2.1, 2.2). It was rapidly recognized that in pursuing this idea the number of degrees of freedom increased rapidly because the additional connections involve only pairs of interfaces instead of pertaining to vertices where they would be shared by more elements. The same problem is at the origin of the lack of popularity of the equilibrium models.

The trend to avoid simple interface connections appears clearly in the derivation of conforming plate bending models, where the need for a normal slope connection at mid-edge was present. Some authors restrained the degree of the polynomials along the edge so that the mid edge variables are no longer necessary (Ref. 2.4, 2.5). Others have used, at connection vertices, the curvature tensor although this is not required by the compatibility conditions and is known by its additional constraints to have a detrimental effect on the results (Ref. 2.6, 2.7). In both cases the results plotted versus the number of degrees of freedom nevertheless show an advantage : the deterioration of the results due to the additional constraints is ofset by the reduction in the number of degrees of freedom.

A closer look at the deficiencies of simple elements, reveals that it is the stress representation that is especially poor under certain conditions. The approximation computed for the displacements is much better and, consequently, that of the strain energy. This holds for equilibrium as well as for displacement models.

From a mathematical point of view this is clearly due to the fact that the deformations are derivatives of the displacements and that the guarantee of convergence in energy does not imply a similar rate of convergence for both displacements and stresses, nor even a convergence of stresses at all when singularities are present.

A possible improvement appears perhaps more clearly from the following physical argument. Let us take the example of a cantilever deep beam such as indicated in figure 2.2, and let us focus the attention on the upper element in the built-in section. Even the simplest approximation of linear displacement is already reasonnably good for the contour of the

element. However the stress field presents a singularity at the built-in boundary since the shear stress is zero on the free edge and has a finite important value along the other edge. A good solution requires that the stress or deformation field contains such a singularity or at least is able to represent it approximately. As the approximation of these fields is achieved by polynomials, a decent picture of the stress field will be obtained in the vicinity of the corner only if very high order polynomials are used. These considerations suggest that, if the number of generalized displacements along the interfaces is to be kept small, more internal degrees of freedom should be given to the deformation or to the stress field.

Such a possibility was introduced first by the hybrid models of Pian (Ref. 2.8) which can be derived by application of Reissner's principle. In those elements, the interface displacements are approximated by simple polynomials, the coefficients of which are determined by a (small) number of generalized displacements. Inside the element the stress field is independantly approximated up to any order. The more parameters the stress field contains, the more accurate is the solution. Note that in this context, the "exact" solution is that of the elasticity problem with the constraints imposed on the displacements along the interfaces. Obviously an optimum exists between the introduction of additional internal parameters for the stress field, which does not increase the final number of degrees of freedom of the structure, and the introduction of degrees of freedom along the interfaces or the increase of the number of interfaces. The drawback of such models is that even though convergence in energy is guaranteed, it is not monotonic and no guideline exists in the interpretation of the position of the results with respect to the exact solution.

The same idea can be applied to displacement or equilibrium models with the advantage that their purely conforming or purely equilibrium characteristics are preserved. It is possible to add to the displacement (or stress) field defined naturally by the analytic continuation of the polynomials choosen along the edges, other displacement (or stress) modes of higher order but such that they vanish along all the interfaces of the element. For such modes, sometimes called "bubble" functions (Ref. 2.9), one can choose polynomials up to any high order, or even non polynomial functions. As they are purely internal to the element, their intensities can be computed in terms of the boundary modes by a preliminary minimization of the strain energy at the element level of generation.

One could compare that procedure to that of developping super-elements obtained by an assemblage of a certain number of simple elements where the internal degrees of freedom are eliminated at the element generation. It should be noted that such an elimination of internal degrees of freedom, even if it is achieved numerically, is a cheap procedure compared to the processing of the same number of degrees of freedom at the structural level. The concept of internal or "bubble" modes generalizes this scheme in this sense that these modes are not restricted to those that can be represented by an assemblage of simple elements.

Coupling the idea of elements of variable degree along the interfaces with that of internal degrees of freedom, a structure can now be idealized by deciding first what contours of physical importance to retain as interfaces in the analysis. One then decides what degree of the polynomial approximation to choose for the displacements or stresses along each interface separately. This will depend on the particular interest attached to the region or of the expected strength of the stress gradient. One finally decides how many internal degrees of freedom should be used inside each element. An important practical advantage of this new idealization scheme is that if one wishes to refine a given idealization, it is no longer necessary to prepare a new set of input data (a bulky task in practice). The only modifications to introduce consist in stating in what elements one wishes to increase the degree of the approximation, either along the interfaces or internally.

The following sections give the theoretical support for the derivation of such elements, within the framework of displacement, equilibrium and hybrid models respectively.

2.2. Formulation for conforming displacement models.

Conforming finite element models are derived by application of the minimum total energy principle :

$$\delta\ (U + P) = 0 \tag{2.1}$$

where, using matrix notations

$$U = \int_{vol} W(\varepsilon)\,dvol = \frac{1}{2}\int_{vol} \varepsilon'\ H\ \varepsilon\ dvol \tag{2.2}$$

is the strain energy expressed in terms of the strain tensor components ε_{ij} and in which H stands for a symmetric matrix of elastic coefficients; the prime denotes transposition.

$$P = - \int_{vol} \bar{X}' \, u \, dvol - \int_{\Gamma_\sigma} \bar{t}' \, u \, d\Gamma_\sigma \qquad (2.3)$$

is the potential energy of the body forces X_i and of the surface tractions $\bar{t}_i$. The displacement vector is denoted by u and Γ_σ is that part of the boundary where surface tractions are prescribed. The bar denotes prescribed values. The case where the body forces are zero is only considered in the following for simplicity of presentation.

This principle may be applied if the displacement field satisfies a priori the compatibility requirements :

$\varepsilon_{ij} = D_j \, u_i$ or, in matrix form $\varepsilon = \partial \, u$ in the volume

$u_i = \bar{u}_i$ on Γ_u, which is the part of the boundary

where displacements are prescribed.
The derivation of a finite element follows the discretization of the displacement field in the form of a linear combination of assumed displacement modes $P_{ik} \, (x_m)$ of the space variables x_m

$$u_i = \sum_k \alpha_{ik} \, P_{ik} \, (x_m) \qquad k = 1, \ldots n$$

or using matrix notations :

$$u = P \, a \qquad (2.4)$$

The n_a coefficients α_{ik}, grouped in the column matrix a, are the intensities of these modes and can be taken as generalized (unknown) displacements. Such an assumption automatically satisfies the compatibility conditions in the volume.
The column of the strain tensor components is written

$$\varepsilon = \partial P \, a \qquad (2.5)$$

and the strain energy becomes

$$U = \frac{1}{2} \int_{vol} a' \, \partial P' \, H \, \partial P \, a \, dvol = \frac{1}{2} a' \, I \, a \qquad (2.6)$$

When using the minimum total energy principle, the stresses are defined by

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

With the discretization adopted, the stress field is described by a set of generalized stress parameters denoted β_{ik}, or in matrix form b

$$b = \frac{\partial U}{\partial a} = I \, a \qquad (2.7)$$

The knowledge of b alone would correspond to a "weak" knowledge of the stress field, as these parameters are weighted averages of the stresses

$$b = \left[\int_{vol} \sigma' \, \partial P \, dvol \right] a \qquad (2.8)$$

where σ denotes the column of the stress tensor components.

The matrix I, sometimes called the integral matrix of the element is symetric and singular. This last property is due to the fact that the P_{ik} modes contain the rigid body modes of the element which do not contribute to the strain energy.

A subset a^* of the parameters a can be defined which does not contain the parameters affecting the rigid body modes

$$a^* = D \, a \qquad (2.9)$$

where D is a matrix of constants. The strain tensor and strain energy can be expressed in terms of a^* only

$$\begin{aligned} \varepsilon &= \partial P^* \, a^* \\ U &= \frac{1}{2} a^{*\prime} \, I^* \, a^* \end{aligned} \qquad (2.10)$$

A reduced set of generalized stress parameters can also be defined

$$b^* = \frac{\partial U}{\partial a^*} = I^* a^* \tag{2.11}$$

The matrix I^* is still symmetric but non singular. It involves in (2.11) a generalized form of the stress-strain relations. A better denomination for I is therefore an extended form of generalized stress-strain relations. Between the two expressions, the following relations hold

$$\begin{aligned} I &= D' I^* D \\ b &= D' b^* \end{aligned} \tag{2.12}$$

Turning next to the satisfaction of the compatibility requirements along the interfaces of elements, the assumed displacement field generates along these interfaces a certain number n_c of interface displacement modes. The element is "conforming" if these interface modes are exactly transmitted across the interfaces to guarantee the single valuedness of the complete displacement field. This will be the case if :

- the same interface modes are defined in adjacent elements,
- the intensities of these modes are identical,
- the interface modes are independant and define uniquely the displacement field along each interface.

The n_c interface mode intensities are listed in a column matrix q_c and their expression in terms of the parameters α_{ik} is

$$q_c = C\, a \tag{2.13}$$

As the displacement field has to be unique at a vertex common to different elements, q_c always contains the local values of the displacement components at the vertices. In addition other local values, derivatives or integrals of displacements can be used along the interfaces. The matrix C is called the kinematic matrix of the element. It is not necessarily a square matrix and hence not necessarily invertible.

In the absence of body forces, the virtual work T consist only in the work done by the prescribed surface tractibus

$$T = \int_{\Gamma_\sigma} u' \, \bar{t} \, d\Gamma_\sigma = a' \left[\int_{\Gamma_\sigma} P' \, \bar{t} \, d\Gamma_\sigma \right] = a'b \qquad (2.14)$$

which yields the expression of the generalized stress parameters b in terms of the applied loads. An alternative form of the virtual work is, using (2.13)

$$T = q'_c g_c = a'C'g_c \qquad (2.15)$$

where g_c is another set of generalized loads conjugate to the interface modes. Comparison of (2.14) and (2.15) yields

$$b = C'g_c \qquad (2.16)$$

The transpose of the kinematic matrix turns out to be a static matrix relating the stress parameters to the applied loads.

The problem of the derivation of the finite element can now be presented as that of finding the minimum of the functional

$$\{ \frac{1}{2} a' \, I \, a - q'_c g_c \} \min \qquad (2.17)$$

subject to the constraints (2.13). Three cases have now to be distinguished :

a) $n_a = n_c$

The number of parameters of the displacement field equals the number of interface connection modes. The matrix C is a square, non singular matrix. This allows to solve the constraints

$$a = C^{-1} q_c \qquad (2.18)$$

and hence

$$\delta \{ \frac{1}{2} q'_c \, C^{-1'} \, I \, C^{-1} \, q_c - q'_c g_c \} = 0$$

or

$$\delta \{ \frac{1}{2} q'_c \, K_c \, q_c - q'_c g_c \} = 0$$

which yields the familiar stiffness relation

$$g_c = K_c q_c \tag{2.19}$$

between the interface modes and the corresponding generalized forces. The latter can now be interpreted, using (2.14) and (2.18)

$$g_c = C^{-1'} \int_{\Gamma_\sigma} P' \, \bar{t} \, d\Gamma_\sigma \tag{2.20}$$

They are weighted averages of the prescribed surface tractions, giving therefore only a "weak" knowledge of the stress state on Γ_σ.

b) $n_a > n_c$

In this case the homogeneous system

$$C\, a = 0 \tag{2.21}$$

has $(n_a - n_c)$ non trivial solutions. Let s_i denote such solutions and group them in a matrix S

$$S = (s_1, s_2 \ldots, s_{n_a - n_c})$$

The non zero solutions of the homogeneous system (2.21) can be written

$$a = S\, w$$

where w is any arbitrary vector. Such modes of the parameters α_i exist with zero displacement prescribed along the interfaces. The constraints (2.13) can now be solved in the form

$$a = T\, q_c + S\, a_x \tag{2.22}$$

where $T\, q_c$ is a particular solution and $S\, a_x$ the complementary solution. Substitution into (2.17) yields :

$$\delta \{ \frac{1}{2} q'_c K_{qq} q_c + \frac{1}{2} a'_x K_{xx} a_x + \frac{1}{2} q'_c K_{qx} a_x - q'_c g_c \} = 0 \quad (2.23)$$

when $K_{qq} = T' I T$

$K_{xx} = S' I S$

$K_{qx} = K_{xq}' = T' I S$

Variation of q_c and a_x finally yields

$$g_c = (K_{qq} - K_{qx} K_{xx}^{-1} K_{xq}) q_c = K_c q_c \quad (2.24)$$

In general the form (2.22) could be obtained numerically, by a Gauss-Jordan elimination technique for instance. However, this operation can always be avoided by one of the following methods :

Add to (2.13) a set of $(n_a - n_c)$ internal values of the displacements q_i

$$q_i = C_i a \quad (2.25)$$

and form with (2.13) the non singular relation

$$q = \begin{vmatrix} q_c \\ q_i \end{vmatrix} = \begin{vmatrix} C \\ C_i \end{vmatrix} a = C^* a$$

$$a = C^{*-1} q \quad (2.26)$$

Partitioning of (2.26) yields the definition of the particular and complementary solutions

$$a = \left| C^{*-1} \right|_c q_c + \left| C^{*-1} \right|_i q_i = T q_c + S q_i \quad (2.27)$$

Another simpler possibility arises when the displacement field can be split in two parts

$$u = P a = P_c a_c + P_b a_b \quad (2.28)$$

where the interface modes q_c are uniquely defined by the modes P_c, so that

$$q_c = C_c\, a_c$$

and
$$a_c = C_c^{-1}\, q_c \tag{2.29}$$

and when P_b are "bubble modes" giving zero contribution to the interface modes. The relation (2.17) can be written

$$\delta \{ \frac{1}{2} a'_b\, I_{bb}\, a_b + \frac{1}{2} a'_c\, I_{cc}\, a_c + a'_b\, I_{bc}\, a_c - a'_c\, C_c^{-1'}\, g_c \} = 0 \tag{2.30}$$

from which

$$g_c = C_c^{-1'}\, (I_{cc} - I_{cb}\, I_{bb}^{-1}\, I_{bc})\, C_c^{-1}\, q_c = K_c\, q_c \tag{2.31}$$

When the number of bubble modes becomes important, this last form represent a significant saving of numerical work, compared to the preceding forms (2.24) or (2.27).

c) $n_a < n_c$

The constraints (2.13) cannot be inverted and this case was ruled out of previous formulations. It is however possible to derive a finite element in this case.

Considering the static relation (2.16), the number of generalized stress parameters n_b is smaller than the number n_g of generalized interface loads g_c. This means that $(n_g - n_b)$ non zero solutions exist to the problem

$$C'\, g_c = 0 \tag{2.32}$$

These solutions correspond to combinations of loads applied along the interfaces which do not generate stresses in the element.
By analogy with the terminology adopted for equilibrium models these modes might be called "spurious static modes".

To derive the finite element properties, let us split the constraints in two parts

$$q_c^* = C^* a$$

$$q_R = C_R a \tag{2.33}$$

where q_c^* is a subset of q composed of n_a components arbitrarily selected. q_R forms the complementary subset of q_c^*
The matrix C^* is non singular and (2.33) yields

$$a = C^{*-1} q_c \tag{2.34}$$

Substitution in the expression of the minimum principle (2.17) gives

$$\delta \{ \frac{1}{2} q_c^{*\prime} C^{*-1\prime} I C^{*-1} q_c^* - q_c^{*\prime} g_c^* - q_R^{\prime} g_R \} = 0 \tag{2.35}$$

with the remaining contraints

$$q_R = C_R C^{*-1} q_c^* \tag{2.36}$$

Introducing these remaining constraints in (2.35) affected by a column of lagrangean multiplier λ

$$\delta \{ \frac{1}{2} q_c^{*\prime} K_c^* q_c^* - q_c^{*\prime} g_c^* - q_R^{\prime} g_R + \lambda (q_R - C_R C^{*-1} q_c^*) \} = 0 \tag{2.37}$$

it turns out that

$$K^* q_c^* = g_c^* + C^{*-1\prime} C_R \lambda \tag{2.38}$$

$$g_R = \lambda$$

$$q_R = C_R C^{*-1} q_c$$

This procedure yields a stiffness matrix K_c^* which can be used if the satisfaction of the remaining constraints is achieved at the structural level. It appears sometimes in practice that the remaining constraints

can be eliminated in a group of finite elements. This allows a derivation of super-elements free of such constraints. The procedure was used implicitly in the derivation of elements by superposition of fields in a quadrangle subdivided in four triangles. (Ref. 2.10, 2.11)

2.3. Formulation for stress-diffusing equilibrium models.

Stress diffusing equilibrium finite element models are derived by application of the minimum complementary energy principle

$$\delta (\Psi + Q) = 0$$

where

$$\Psi = \int_{vol} \phi(\sigma_{ij}) \, dvol = \frac{1}{2} \int_{vol} s' \, H^{-1} \, s \, dvol \qquad (2.39)$$

is the complementary strain energy (stress energy). The column matrix s groups the stress tensor components σ_{ij} and H^{-1} is a matrix of elastic coefficients relating strains to stresses. It is the inverse of the matrix defined by equation (2.2).

$$Q = - \int_{\Gamma u} t' \, \bar{u} \, d\Gamma u \qquad (2.40)$$

is the complementary potential energy of the surface tractions t over the prescribed displacements $\bar{u}$ evaluated on that part of the contour Γu where displacements are prescribed.

This principle may be applied if the stress field satisfies a priori the equilibrium requirements

$$\begin{aligned} & D_j \, \sigma_{ij} + X_i = 0 && \text{in the volume} \\ & t_i = \ell_j \, \sigma_{ji} = \bar{t}_i && \text{on } \Gamma_\sigma \text{ which is the part of} \end{aligned} \qquad (2.41)$$

the boundary when surface tractions are prescribed. The ℓ_j are direction cosines of the outward normal to Γ_σ and X_i denotes the body forces. Again the case without body forces only will be considered in the following for simplicity. A presentation which includes them can be found in reference (2.12).

The part of the stress field satisfying the homogeneous equilibrium equations can be discretized by two different procedures :

a) The stress tensor can be represented by a linear combination of assumed modes $M_k\ (x_m)$ where intensities are controlled by a set of unknown parameters β_k

$$\sigma_{ij} = \sum_k M_{k(ij)}\ (x_m)\ \beta_k$$

or in matrix form

$$s = M\ b \tag{2.42}$$

These modes must be such that the homogeneous part of the equilibrium equations (2.41) is a priori satisfied.

b) The other procedure is rather new. It might appear, at first sight, unnecessarily sophisticated but it reveal extremely fruitfull for the development of the finite elements as well as it provides a new method of solution of the structural problem.

The stress tensor is derived from a set of stress functions

$$\sigma_{ij} = \epsilon_{imr}\ \epsilon_{jns}\ \phi_{rs,mn} \qquad \text{(ij not summed)} \tag{2.43}$$

where, in the three dimensional case ϕ_{rs} is a symmetric tensor of stress functions while ϵ_{ijk} is the alternator symbol. These stress functions can be approximated as a linear combination of modes

$$P_{k(rs)}\ (x_m) \text{ of intensity } \alpha_k$$

$$\phi_{rs} = \sum_k P_{k(rs)}\ (x_m)\ \alpha_k \tag{2.44}$$

or on matrix form

$$r = P\ a \tag{2.45}$$

Following this procedure the homogeneous equilibrium equations are automatically satisfied.

The column of stress components derived according to (2.43) is denoted :

$$s = \partial^2 P\, a \tag{2.46}$$

The elements of the matrices $\partial^2 P$ and M are in general functions of the space variables x_m. The number of parameters α_i is always larger than that of the β_i by the number of integration constants of the system (2.43) which are the "rigid body modes" of the stress functions. In some cases the parameters β_k can simply be a subset of the α_k but in general the relation between the two sets is denoted

$$b = D\, a \tag{2.47}$$

when D is a rectangular matrix.

The complementary strain energy ψ can be alternatively written

$$\psi = \frac{1}{2} \int_{vol} b'\, M'\, H^{-1}\, M\, b \; dvol = \frac{1}{2}\, b'\, J^*\, b \tag{2.48}$$

$$\psi = \frac{1}{2} \int_{vol} a'\, \partial^2 P'\, H^{-1}\, \partial^2 P\, a \; dvol = \frac{1}{2}\, a'\, J\, a \tag{2.49}$$

where J and J^* are two symmetric flexibility matrices.

When using the complementary energy principle, the strains are defined by

$$\frac{\partial \phi(\sigma_{ij})}{\partial \sigma_{ij}} = \varepsilon_{ij}$$

With the first discretization adopted, the strain field is described by a set of generalized strain parameters π_{bi} or in matrix form π_b, conjugate to the generalized stresses b

$$\pi_b = \frac{\partial \Psi}{\partial b} = J^*\, b \tag{2.50}$$

This corresponds to a "weak" knowledge of the strain field as these parameters are weighted averages of the strains

$$\pi_b = \left[\int_{vol} \varepsilon' M \, dvol \right] b \qquad (2.51)$$

In view of (2.50) the matrix J^* can be interpreted as yielding a generalized form of the strain-stress relations. The parameters β_i are independant and each of them represents a non zero contribution to the strain energy. The matrix J^* is therefore non singular.

With the second dicretization adopted above, another set of generalized strains, conjugate of the α_i parameters, can be defined

$$\pi_a = \frac{\partial \psi}{\partial a} = J \, a \qquad (2.52)$$

or
$$\pi_a = \left[\int_{vol} \varepsilon' \, \partial^2 P \, dvol \right] a$$

The matrix J generates another form of generalized strain-stress relations, but it is singular since the parameters α_i contains the "rigid body modes" of the stress functions which do not contribute to the strain energy. Using (2.47) the relation between the two flexibility matrices is

$$J = D' \, J^* \, D \qquad (2.53)$$

which shows that J^* is a non singular kernel of J.

Let us turn next to the equilibrium requirements along the boundary. They concern only the interfaces as we have ruled out the body forces. The surface tractions can be expressed in terms of the stress field by

$$t = L \, s \qquad (2.54)$$

when L is a matrix of the direction cosines of the outward normal to the interface. Using the first description for the stress field (2.42)

$$t = L \, M \, b \qquad (2.55)$$

The surface traction modes described by t have to be uniquely determined by a set of n_c forces, denoted g_c .

These can be simply local values of the surface tractions (2.55) or derivatives, integrals or any combinations of them provided that along each edge the definition is unique. The expression of these generalized forces in terms of the stress field parameters is written

$$g_c = C\ b \tag{2.56}$$

where C is called the static matrix of the element. The matrix C is in general rectangular but even if it is square, the relation is singular since the generalized forces are a priori not independant since they satisfy the global equilibrium equations of the element. Note that in contrast to the formulation of the displacement elements, the stress tensor has no unique value at a vertex. Therefore in pure equilibrium models, they are no variables associated with a vertex and the forces g_c are essentially interface variables.

The virtual work consists only in the work done by the prescribed displacements

$$T = \int_{\Gamma u} u'\ t\ d\Gamma_u = \int_{\Gamma u} u'\ LMb\ d\Gamma_u = p_b' b \tag{2.57}$$

or alternatively

$$T = q'_c\ g_c = q'_c\ C\ b \tag{2.58}$$

where q_c are the generalized displacements conjugate to g_c.
Comparison of the two expressions yields

$$p_b = C'\ q_c \tag{2.59}$$

The transpose static matrix turns out to be a kinematic matrix relating the generalized (average) strains to the generalized interface displacements.

The problem of the derivation of the finite element reduces now to that of finding the minimum of the functional

$$\{ \frac{1}{2} b' J^* b - g'_c q_c \} \min \qquad (2.60)$$

subject to the constraints (2.56). It yields

$$J^* b = C' q_c$$

and, as J^* is non singular

$$g_c = C b = C J^{*-1} C' q_c = K_c q_c \qquad (2.61)$$

which is the familiar form of the stiffness matrix for an equilibrium element. In contrast to the formulation of displacement models, it is not necassary in order to derive a stiffness matrix to distinguish between to three cases mentionned there. The distinction is however still implicitly present and necessary if one wishes to use the relation (2.61) in practice.

Consider first the static matrix (2.56), the generalized forces g_c are not necessarily sufficient to determine all the parameters β_i.
In other words the homogeneous system

$$C b = 0 \qquad (2.62)$$

can have non-trivial solutions representing internal stress modes which exist with zero prescribed surface tractions on the interfaces. These n_i modes are "bubble" stress modes of the element.
One could think about defining additional internal generalized forces to determine these bubble stress modes in a similar way to the internal displacements defined by (2.26). This is useless here since the solution of the constraints (2.56) is not required and the bubble stress modes do not contribute to the complementary potential energy. In fact the reduction of these modes, whatever they are, is automatically included in the expression (2.61).

Consider next the kinematic matrix (2.59). Non trivial solutions of

$$C' q_c = 0 \qquad (2.63)$$

or, more physically, displacement modes of the interfaces which do not produce generalized strains, certainly exist. Among them are the rigid body modes of the element. Let z_k denote such solutions in number n_k and group them into a matrix Z

$$Z = (z_1, z_r \ldots, z_k) \tag{2.64}$$

The solutions of (2.63) can be expressed by

$$q_c = Z\, w$$

where w is any arbitrary vector. These displacement modes generate zero strain and hence their virtual work is equal to zero

$$T = g'_c\, q_c = g'_c\, Z\, w = 0 \tag{2.65}$$

As w is arbitrary, it implies

$$Z'\, g_c = 0 \tag{2.66}$$

It means that among the n_c forces g_c, n_k are non independant and related to the others by these relations. As the solutions of (2.63) are also solutions of (2.61), the stiffness matrix of the element is n_k times singular. If the n_k kinematic modes consist only in the n_r rigid body modes, the situation is normal as the rigid body modes are eliminated by fixation of the structure. The relation (2.66) simply reduces to the global or rigid body equilibrium equations of the element. If n_k contains other kinematic modes, $n_k - n_r$ relations have to be kept in (2.66) as remaining constraints to be satisfied by the solution of (2.61).
They correspond to what has been called previously spurious kinematic deformation modes. Just as in the corresponding case for displacement elements, the stiffness matrix derived in (2.61) can not be used alone, but with these remaining constraints. Here again it appears often that such constraints can be eliminated within a group of such elements, leading to a super element free of such constraints. The procedure has been applied implicitly in the derivation of a certain number of equilibrium membrane elements. The advantage of the present formulation is to show that the procedure is general and that the problem of deriving stress diffusing

equilibrium elements is strictly the same as that of deriving conforming displacement elements. Finally, from the preceding discussion, it appears that the condition to fulfil to avoid remaining constraints (or spurious kinematic deformation modes) is

$$n_b - n_i = n_c - n_r$$

The strict similarity between the derivation of the two types of element becomes even more transparent if the second formulation, in terms of stress functions, is used. In that case the requirements of surface traction continuity has to be translated into requirements of stress function continuity. They certainly involve the continuity of stress function local values but in addition, they eventually involve the continuity of some of its derivatives (specific examples are given in the following). Those requirements are to be satisfied by the single valuedness of m_c local values of the stress functions and its derivatives, denoted f_c. These are equivalent to the n_c forces g_c. They can be expressed in terms of the intensities of the assumed stress function modes α_i by

$$f_c = T\, a \qquad (2.67)$$

The matrix T is another static matrix of the element.
Note that the m_c interface connection modes necessarily define the "rigid body" modes of the stress functions and that their number m_c is equal to the number n_c. In contrast with the first formulation, the generalized stress function modes f_c are not necessarily interface variables only, as the continuity requirements are often such that a single value of the stress function has to be achieved at a vertex. This situation is therefore similar to that encountered in the derivation of displacement elements.

The virtual work of the surface tractions can be alternatively written, using (2.54), (2.46) and (2.50)

$$T = \int_{\Gamma u} u'\, t\, d\Gamma u = \int_{\Gamma u} u'\, L\, \partial^2 P\, a\, d\Gamma u = p'_a\, a \qquad (2.68)$$

which yields the meaning of the generalized strains p_a

$$p_a = \left[\int_{\Gamma_u} \partial^2 p' \, L' \, u \, d\Gamma_u \right] \tag{2.69}$$

With (2.52) it corresponds to another "weak" knowledge of the strain field. Another expression of the virtual work is, using (2.67)

$$T = p'_a \, a = f'_c \, d_c = a' \, T' \, d_c \tag{2.70}$$

where d_c are generalized interface displacements, conjugate to the generalized stress function modes f_c. It shows that

$$p_a = T' \, d_c \tag{2.71}$$

which again indicates that the transpose static matrix is a kinematic matrix relating strains to displacements.

The problem of the derivation of the finite element properties can now be presented as that of finding the minimum of

$$\{ \frac{1}{2} a' \, J \, a - f'_c \, d_c \} \text{ min} \tag{2.72}$$

with the constraints (2.67). Note that J is singular and therefore the procedure given in (2.61) does not apply. Comparison with (2.17) shows that the situation is in fact completely similar to that of the derivation of the displacement elements. Three cases have again to be considered :

a) $\underline{n_a = n_c}$

In this case, it turns out that

$$a = T^{-1} \, f_c \tag{2.73}$$

and

$$d_c = T^{-1} \, J \, T^{-1} \, f_c = F_c \, f_c \tag{2.74}$$

where F_c is the flexibility matrix of the element.

b) $\underline{n_a > n_c}$

The homogeneous system

$$T \, a = 0 \tag{2.75}$$

has $(n_a - n_c)$ non trivial solutions which represent bubble stress function modes. The two procedures for deriving the flexibility matrix are formally identical to those used for deriving the stiffness matrix in the corresponding situation for displacement elements.

c) $n_a < n_c$

The homogeneous system

$$T' d_c = 0 \tag{2.76}$$

has $(n_c - n_a)$ non trivial solutions which represents spurious kinematic deformation modes. In this case a flexibility matrix can still be derived if the system (2.67) is split in two parts :

$$\begin{aligned} f_c^* &= T^* a \\ f_{cR} &= T_R^* a \end{aligned} \tag{2.77}$$

where f_c^* is a subset of f_c composed of n_a components arbitrarily selected. f_{cR} forms the complementary subset of f_c^*. The matrix T^* is now non singular, so that

$$a = T^{*-1} f_c^*$$

and

$$f_{cR} = T_R^* T^{*-1} f_c^* \tag{2.78}$$

which are the remaining constraints. By analogy with (2.38) it is possible to derive a flexibility matrix

$$F_c = T^{*-1'} J T^{*-1} \tag{2.79}$$

but which cannot be used without imposing the constraints (2.78). Again, such constraints can often be eliminated in practice within a group of such elements, yielding a super-element free of constraints.

Finally the two formulations for the equilibrium elements are related to each other by the following relations

$$g_c = C\, b = C\, D\, T^{-1}\, f_c = C^*\, f_c \tag{2.80}$$

Note that although the matrix of correspondance C^* is square, it is always singular as the forces g_c are non independant.
By virtual work

$$T = g'_c\, q_c = f'_c\, C^{*\prime}\, q_c = f'_c\, d_c \tag{2.81}$$

from which

$$d_c = C^{*\prime}\, q_c \tag{2.82}$$

which completes the analogy.

2.4. Formulations for finite elements based on two field principles

The possibility of deriving finite elements from simultaneous assumptions on the displacement and on the stress fields is well known. Of particular concern in this chapter are those formulations where the assumptions are introduced on one field inside the element and on the other field on its boundaries. One type of such elements has been proposed by PIAN (Ref. 2.8). It derives from the Reissner principle. A dual formulation is also presented in the following which derives from another two field principle proposed by FRAEIJS de VEUBEKE.

2.4.1. Finite elements based on the Reissner principle

The Reissner principle can be written

$$\{ \int_{vol} \left[\sigma_{ij}\, D_j\, u_i - \phi(\sigma_{ij}) \right] \mathrm{dvol} + \int_{\Gamma_u} \ell_j\, \sigma_{ji}\, (\bar{u}_i - u_i) d\Gamma_u - \int_{\Gamma_\sigma} \overline{\ell_j\, \sigma_{ji}}\, u_i\, d\Gamma_\sigma \}\ \text{stat.} \tag{2.83}$$

where σ_{ij} and u_i are the independant stress and displacement fields.

$\phi(\sigma_{ij}) = \frac{1}{2}\, s' H^{-1} s$ is the complementary stress energy density

ℓ_j are the direction cosines of outward normal to the boundary Γ

Γ_u, Γ_σ are the parts of the boundary where displacements or stresses are prescribed (the bar denotes a prescribed value)

the body forces are removed for simplicity of the presentation.

Let us assume inside the element a stress field which satisfies the equilibrium equations

$$D_j \sigma_{ji} = 0 \qquad \text{in the volume} \qquad (2.84)$$

but which does not necessarily satisfy the equilibrium of the surface tractions t_i on the interfaces

$$t_i = \ell_j \sigma_{ji} = \overline{\ell_j \sigma_{ji}} = \bar{t}_i \qquad \text{on } \Gamma_\sigma \qquad (2.85)$$

Integrating by parts the first terms of the volume integral in (2.83) yields, using (2.84)

$$\{ \int_{vol} \Phi(\sigma_{ij})\, dvol - \int_{\Gamma_u} \ell_j \sigma_{ji} \bar{u}_i \, d\Gamma_u + \int_{\Gamma_\sigma} \ell_j (\bar{\sigma}_{ji} - \sigma_{ji}) u_i \, d\Gamma_\sigma \} \qquad \text{stationary} \qquad (2.86)$$

In this modified principle, the stress field inside the element is discretized in the form

$$s = M b \qquad (2.87)$$

where the stress modes $M_{ij}(x_m)$ are selected in such a way that they satisfy a priori the homogeneous equilibrium equations. The unknown intensities β_i of these modes are the corresponding generalized stresses.

The volume integral in (2.86) becomes

$$\Psi = \int_{vol} \Phi(\sigma_{ij})\, dvol = \frac{1}{2} \int_{vol} s' H^{-1} s \, dvol = \frac{1}{2} b' J b \qquad (2.88)$$

where $$J = \int_{vol} M' H^{-1} M \, dvol$$

is a non singular generalized strain-stress matrix of the element.

On the part Γ_u of the contour, the surface tractions are those generated by that stress field

$$t = L\, s = L\, M\, b \tag{2.89}$$

where L is a matrix of direction cosines of the outward normal.
Just as for equilibrium models, the surface tractions can be defined uniquely by a set of n_{cu} generalized forces g_{cu} on that part of the boundary. These forces can be local values of the surface tractions or any convenient combinations. Their expression in terms of the stress field parameters is denoted

$$g_{cu} = C\, b \tag{2.90}$$

where C is the static matrix relative to these forces.

The first line integral in (2.86) can now be written

$$-Q_u = \int_{\Gamma_u} \ell_j\, \sigma_{ji}\, \bar{u}_i\, d\Gamma_u = \int_{\Gamma_u} b'\, M'\, L'\, u\, d\Gamma_u = b'\, p_{bu} \tag{2.91}$$

where p_b are a set of generalized deformations conjugate to the generalized stresses b. Their expression yields a weak knowledge of the deformations on Γ_u

$$p_{bu} = \int_{\Gamma_u} M'\, L'\, u\, d\Gamma_u \tag{2.92}$$

An alternative form of Q_u is

$$-Q_u = g'_{cu}\, q_{cu} = b'\, C'\, q_{cu} \tag{2.93}$$

if q_{cu} denotes the generalized displacements conjugate of the forces g_{cu}. Comparison of (2.91) and (2.93) shows that

$$p_{bu} = C'\, q_{cu} \tag{2.94}$$

Here again the transposed static matrix is a kinematic matrix.

On the complementary part of the boundary, Γ_σ , the displacements can be independantly discretized in the form

$$u = P\, q_{cs} \tag{2.95}$$

where $P_{ij}\ (\Gamma_\sigma)$ are displacement modes of the boundary Γ_σ of intensity q_s. The displacement are therefore uniquely determined in that line. The second line integral in (2.86) is now expressed by

$$Q_\sigma = Q_{\bar{\sigma}} + Q_{\sigma u}$$

with
$$Q_{\bar{\sigma}} = \int_{\Gamma_\sigma} \ell_j\, \bar{\sigma}_{ji}\, u_i\, d\Gamma_\sigma \tag{2.96}$$
$$= \int_{\Gamma_\sigma} t'\, P\, q_{cs}\, d\Gamma_\sigma = g'_{cs}\, q_{cs}$$

where
$$g_{cs} = \int_{\Gamma_\sigma} P'\, t\, d\Gamma_\sigma$$

are generalized forces conjugate of the displacement q_{cs}. They give a weak knowledge of the surface tractions on Γ_σ. Then :

$$Q_{\sigma u} = -\int_{\Gamma_\sigma} \ell_j\, \sigma_{ji}\, u_i\, d\Gamma_\sigma = -\int_{\Gamma_\sigma} b'\, M'\, L'\, P\, q_{cs}\, d\Gamma_\sigma \tag{2.97}$$
$$= -\, b'\, R\, q_{cs} \qquad = -\, b'\, p_{bs}$$

defines another set of average generalized strains conjugate of the b parameters

$$p_{bs} = R\, q_{cs} = \left[\int_{\Gamma_\sigma} M'\, L'\, P\, d\Gamma_\sigma \right] q_{cs} \tag{2.98}$$

The matrix R can therefore be considered as another kinematic matrix relative to the displacements q_{cs}. Reintroducing the results (2.88), (2.93), (2.96) and (2.97) into the expression of the modified principle (2.86) yields

$$\left\{ \frac{1}{2}\, b'\, J\, b - b'\, c'\, q_{cu} + g'_{cs}\, q_{cs} - b'\, R\, q_{cs} \right\} \text{ stat} \tag{2.99}$$

where the independant variables are b and q_{cs}. Their variation gives

$$J\, b = C'\, q_{cu} + R\, q_{cs} \tag{2.100}$$

$$g_{cs} = R'\, b$$

As J is non singular

$$b = J^{-1}\, C'\, q_{cu} + J^{-1}\, R\, q_{cs} \tag{2.101}$$

and

$$g_{cs} = R'\, J^{-1}\, C'\, q_{cu} + R'\, J^{-1}\, R\, q_{cs} \tag{2.102}$$

Using (2.90) also yields

$$g_{cu} = C\, J^{-1}\, C'\, q_{cu} + C\, J^{-1}\, R\, q_{cs} \tag{2.103}$$

Grouping (2.102) and (2.103) yields the final stiffness matrix of the element

$$g = \begin{vmatrix} g_{cu} \\ g_{cs} \end{vmatrix} = \begin{vmatrix} C\, J^{-1}\, C' & C\, J^{-1}\, R \\ R'\, J^{-1}\, C' & R'\, J^{-1}\, R \end{vmatrix} = \begin{vmatrix} q_{cu} \\ q_{cs} \end{vmatrix} = K\, q \tag{2.104}$$

or

$$K = \begin{vmatrix} C & R' \end{vmatrix} \begin{vmatrix} J^{-1} \end{vmatrix} \begin{vmatrix} C' \\ R \end{vmatrix}$$

It is worth recalling here that the variables related by the stiffness matrix are of two different natures : g_{cu} and q_{cs} are precisely surface traction modes and displacement modes of the two parts of boundary Γ_u and Γ_σ while g_{cs} and q_{cu} are average conjugate quantities.

A particular form of such models is the equilibrium model when all the interfaces belong to Γ_u. Another particular form is the hybrid model of PIAN where all the interfaces belong to Γ_σ. In these special cases, the stiffness matrix reduces respectively to the upper or lower diagonal terms in (2.104).

The existence of bubble modes or kinematic modes can also be put in evidence in these models following the standard procedure of examination of the static-kinematic matrix. The global kinematic matrix of the element can be taken as

$$C^{*} = \begin{vmatrix} C' \\ R \end{vmatrix} \tag{2.105}$$

If the homogeneous system

$$C^{*} q = 0 \tag{2.106}$$

has non-trivial solutions

$$q = Z w \tag{2.107}$$

when Z is a base matrix of these modes and w an arbitrary vector, then the virtual work vanishes for such solutions

$$T = g' q = g' Z w = 0 \tag{2.108}$$

and therefore, as w is arbitrary, the forces g are non independant but satisfy the relations

$$Z'g = 0 \tag{2.109}$$

Among these n_k relations are certainly the n_r rigid body equilibrium equations of the element. The other relations (if any) are additionnal constraints that have to be kept together with the stiffness matrix at the structural level to avoid exciting the corresponding kinematic deformation modes (2.107)

If on the other hand the homogeneous static system

$$C^{*'} b = 0 \tag{2.110}$$

has n_i non trivial solutions, these are bubble stress modes

$$b = Y w \tag{2.111}$$

where Y is a base matrix of such modes and w still an arbitrary vector. Just as for the equilibrium models, the bubble stress modes are automatically reduced by the procedure of derivation of the stiffness matrix.

A finite element is free of additional constraints if the relation

$$n_b - n_i = n_g - n_r \tag{2.112}$$

is satisfied when n_b denotes the number of the stress field parameters.

If there are additional constraints to check the kinematic deformation modes, the procedure of elimination of these constraints in a group of elements can still be applied.

2.4.2. Finite elements based on the Fraeijs de Veubeke principle

Another two field principle, proposed by Fraeijs de Veubeke (Ref. 2.6) is

$$\{ \int_v \left[W(\varepsilon_{ij}) - \sigma_{ij}\, \varepsilon_{ij} \right] \mathrm{dvol} + \int_{\Gamma_u} \ell_j\, \sigma_{ji}\, \bar{u}_i\, d\Gamma_u \} \quad \text{stat} \tag{2.113}$$

where σ_{ij} and ε_{ij} are independant stress and deformation fields

$W(\varepsilon_{ij}) = \frac{1}{2}\varepsilon' H\, \varepsilon$ is the strain energy density

ℓ_j are direction cosines of the outward normal to the boundary Γ

Γ_u, Γ_σ are the parts of the boundary where displacements or stresses are prescribed (the bar denotes a prescribed value)

The body forces are again not considered for simplicity.
Inside the element we assume that the deformation field satisfies the compatibility (or integrability) conditions

$$\varepsilon_{ij} = D_j\, u_i \qquad \text{in the volume} \tag{2.114}$$

but it does not necessarily satisfy the compatibility conditions on the boundary

$$u_i = \bar{u}_i \qquad \text{on } \Gamma_u \tag{2.115}$$

Integrating (2.113) by parts, after substitution of (2.114) and remembering that the stress field satisfy the equilibrium conditions we obtain

$$\{ \int_v W(\varepsilon_{ij})\, dvol - \int_{\Gamma_\sigma} \ell_j \bar{\sigma}_{ji} u_i \, d\Gamma_\sigma + \int_{\Gamma_u} \ell_j \sigma_{ij} (\bar{u}_i - u_i) d\Gamma_u \} \text{ stat} \tag{2.116}$$

In the modified principle, the displacement field inside the element is discretized in the form

$$u = P\, a \qquad \text{in the volume} \tag{2.117}$$

where the displacement modes $P_{ik}(x_m)$ contain also the rigid modes. Their unknown intensities α_k are the corresponding generalized coordinates. The volume integral in (2.116) becomes

$$U = \int_v W(\varepsilon_{ij})\, dvol = \frac{1}{2} a' I\, a \tag{2.118}$$

where $$I = \int_v \partial P'\, H\, \partial P \, dvol$$

is a singular generalized stress-strain relation matrix.

On the part Γ_σ of the contour, the displacements are those generated by the assumed field (2.117). Their modes along the corresponding interfaces can be described uniquely by a set of n_{cs} interface modes

$$q_{cs} = C\, a \tag{2.119}$$

where C is a kinematic matrix relating these modes to the general ones. The first line integral in (2.116) can now be written

$$-\Pi_s = \int_{\Gamma_\sigma} \ell_j \bar{\sigma}_{ji} u_i \, d\Gamma_\sigma = \int_{\Gamma_\sigma} t'\, P\, a \, d\Gamma_\sigma = b'_{as}\, a \tag{2.120}$$

where b_{as} are a set of generalized stresses conjugate to the generalized displacements a.

$$b_{as} = \int_{\Gamma_\sigma} P'\, t \, d\Gamma_\sigma \tag{2.121}$$

They yield a weak knowledge of the surface tractions on Γ_σ.
An alternative form for Π_s is

$$- \Pi_s = g'_{cs}\, q_{cs} = g'_{cs}\, Ca \tag{2.122}$$

where g_{cs} denotes interface generalized forces conjugate to the interface displacements q_{cs}. Comparison of (2.122) with (2.120) shows again that the transposed kinematic matrix is a static matrix

$$b_{cs} = C'\, g_{cs} \tag{2.123}$$

On the complementary part of the boundary, Γ_u, the surface tractions can be independantly discretized by

$$t = S\, g_{cu} \tag{2.124}$$

where the generalized surface tractions g_{cu} are selected in such a way that they define uniquely the surface tractions on Γ_u.
Introduction of the assumptions (2.117) and (2.124) in the second line integral of (2.116) yields

$$\Pi_u = \int_{\Gamma_u} \ell_j\, \sigma_{ji}\, (\bar{u}_i - u_i)\, d\Gamma_u = \int_{\Gamma_u} \left[g'_{cu}\, S'\, \bar{u} - g'_{cu}\, S\, P\, a \right] d\Gamma_u$$
$$= g'_{cu}\, q_{cu} - g'_{cu}\, R\, a = g'_{cu}\, q_{cu} - b'_{au}\, a \tag{2.125}$$

where
$$q_{cu} = \int S'\bar{u}\, d\Gamma_u \tag{2.126}$$

are the average displacements conjugate of the forces g_{cu}, and where

$$R = \int_{\Gamma_u} S\, P\, d\Gamma_u \tag{2.127}$$

is a matrix whose transpose is the static matrix relating the generalized stresses b_{au} to the generalized loads g_{cu}

$$b_{au} = R'\, g_{cu} \tag{2.128}$$

Reintroducing the results (2.118), (2.122) and (2.125) into the expression of the modified principle (2.116) yields

$$\{ \frac{1}{2} a' \, I \, a - g'_{cs} \, C \, a + g'_{cu} \, q_{cu} - g'_{cu} \, R \, a \} \text{ stat.} \tag{2.129}$$

where the independant variables are a and g_{cu}. Their variation gives

$$\begin{aligned} I \, a &= C' \, g_{cs} + R' \, g_{cu} \\ q_{cu} &= R \, a \end{aligned} \tag{2.130}$$

Recalling (2.119) we can define a global kinematic matrix R^* for the element by

$$q = \left| \begin{matrix} q_{cs} \\ q_{cu} \end{matrix} \right| = \left| \begin{matrix} C \\ R \end{matrix} \right| a = R^* \, a \tag{2.131}$$

If the total number n_c of interface displacements q is equal to the number n_a of displacement field parameters a, that matrix can be inverted and (2.130) can be transformed into

$$g = \left| \begin{matrix} g_{cs} \\ g_{cu} \end{matrix} \right| = R^{*-1'} \, I \, R^{*-1} \, q = K \, q \tag{2.132}$$

which defines the stiffness matrix of the element.

A particular form of such models is the conforming displacement model where all the interfaces belong to Γ_σ and R^* reduces to C. Another particular form is the second type of hybrid model proposed by PIAN where all the interfaces belong to Γ_u and R^* reduces to R.

In the case where n_c is not equal to n_a, R^* cannot be inverted. However the various treatments exposed in the case of the conforming displacement models are still applicable.
Again if the homogeneous system

$$R^* \, a = 0 \tag{2.133}$$

has n_i solutions (bubble displacement modes), the condition for the derivation of a stiffness matrix free of additional constraints is that

$$n_a - n_i = n_c \tag{2.134}$$

If the static homogeneous system

$$R^{*'} g = 0 \tag{2.135}$$

has n_k solutions, n_k additional constraints have to be kept with the stiffness matrix. Their elimination within a group of element is usually possible.

2.5. Methods of solution of the global problem using equilibrium elements

Most of the applications of the finite element method have been based on displacement elements for which the displacement or direct stiffness method (Ref. 2.2) is the most natural and efficient algorithm of solution of the global problem. The force method (Ref. 2.13) has received relatively few applications due to the difficulty of obtaining a cheep algorithm of selection of the redundancies when other elements but bars and shear panels are used. The combined method (Ref. 2.14) has not received application due to the large system of equations it involves.

For equilibrium elements, practically all the applications have also been solved by the classical displacement method (Ref. 2.1). However the nature of assumptions in these elements suggests that generalized stresses might be a better choice as unknows. This section shows that the stress function formulation offers a possibility. The method proposed can be viewed as a particular and efficient form of the combined method. It presents important advantages for certain types of boundary conditions.

The formulation of the global problem using the two formulations of equilibrium elements can be presented as follows:
The complementary energy of the assembled structure Ψ_s is the sum of the contribution of all the elements k :

$$\Psi_s = \frac{1}{2} \sum_k b'_k J^*_k b_k \tag{2.136}$$

if the first formulation (2.48) is used, or

$$\Psi_s = \frac{1}{2} \sum_k a'_k J_k a_k = \frac{1}{2} \sum_k f'_k F_k f_k \qquad (2.137)$$

in the stress function formulation (2.49), (2.74).
Assembling the elements is achieved by stating that along each interface the sum of the generalized forces equals the external loading which is expressed by

$$g = \sum_k L_k g_k \qquad (2.138)$$

where L_k are the localizing matrices of the elements. Using (2.56) and (2.80) this relation can be transformed into

$$g = \sum_k L_k C_k b_k \qquad (2.139)$$

or

$$g = \sum_k L_k C^*_k f_k \qquad (2.140)$$

The solution of the structural problem can finally be presented as that of finding the minimum of the total complementary energy under the constraints represented by the connection equations.
Depending of the variables used, this is written

$$\{ \frac{1}{2} \sum_k b'_k J^*_k b_k - g' q \} \text{ minimum} \qquad (2.141)$$

with the constraints (2.139) or alternatively

$$\{ \frac{1}{2} \sum_k f'_k F_k f_k - g' q \} \text{ minimum} \qquad (2.142)$$

with the constraints (2.140).

Taking the first formulation (2.141), the connection constraints can be satisfied by substitution of g into the potential energy

$$\{ \frac{1}{2} \sum_k b'_k J^*_k b_k - (\sum_k b'_k C'_k L'_k) q \} \text{ min.}$$

which yields

$$\sum_k J^*_k b_k = \sum_k C'_k L'_k q$$

As J_k are non singular matrices, this turns out to be,

$$g = \sum_k L_k C_k b_k = \sum_k L_k C_k J_k^{*-1} C'_k L'_k q$$

$$g = \sum_k L'_k K_k L_k q = K q \qquad (2.143)$$

where K is the global stiffness matrix obtained simply by localization (or addressing) of the element stiffness matrices. This system is solved for the displacements q. Hence the stress field is obtained in each element by substituting the displacements in

$$b_k = J_k^{*-1} C'_k L'_k q \qquad (2.144)$$

This procedure is the well known displacement or stiffness method as applicable to equilibrium models (Ref. 2.6, 2.1).

The second procedure consists in solving first the constraint equations (2.139) by finding a particular or isostatic solution and a complete set of non zero solutions of the homogeneous system. This is written

$$b_k = P_k g + Q_k x \qquad (2.145)$$

where **x** are the intensities of these non zero solutions which have the physical meaning of generalized hyperstatic unknows or self stressing states. The minimum principles becomes

$$\{ \frac{1}{2} g' F_{gg} g + \frac{1}{2} x' F_{xx} x + g' F_{gx} x - g' q \} \text{ min.} \qquad (2.146)$$

where $F_{gg} = \sum_k P'_k J_k^* P_k \qquad F_{xx} = \sum_k Q'_k J_k^* Q_k$

$$F_{gx} = \sum_k P'_k J_k^* Q_k = F'_{xg}$$

It yields by variation of the self stressing state intensities

$$F_{xx} x + F'_{gx} = 0$$

$$x = - F_{xx}^{-1} F_{xg} \qquad (2.147)$$

and by variation of the loads

$$q = F_{gg}\, g + F_{gx}\, x \tag{2.148}$$

This procedure is well known the force method (Ref. 2.13) where two types of numerical problem have to be solved. The first is to find the self stressing states in (2.145) and the second is to construct and solve the linear equations (2.147) and (2.148).

The third procedure is to transform the problem (2.141) into an unconstrained minimization problem by the Lagrange multipliers technique. If ℓ denotes the column of the multipliers applied to each constraint, the problem becomes

$$\{ \frac{1}{2} \sum_k b'_k\, J^*_k\, b_k - g'q + \ell'(g - \sum_k L_k\, C_k\, b_k) \} \text{ min.} \tag{2.149}$$

which yields

$$\sum_k J^*_k\, b_k - \sum_k C'_k\, L_k\, \ell = 0$$

$$q = -\ell$$

$$g - \sum_k L_k\, C_k\, b_k = 0$$

This allows for finding both the stress and displacement fields in one operation by solving the linear system

$$\begin{vmatrix} J^*_s & C'_s \\ C_s & 0 \end{vmatrix} \begin{vmatrix} b \\ q \end{vmatrix} = \begin{vmatrix} 0 \\ g \end{vmatrix} \tag{2.150}$$

where J^*_s and C_s denote the global or assembled flexibility and static connection matrices, while b is a column matrix grouping all the b_k parameters for all the elements.
The last procedure has been named the combined method (Ref. 2.14).

The relative advantages of the three methods are now fairly well understood. In simple structures like trusses or thin walled structures modelled with simple finite elements,

the force and the displacement methods involve a comparable amount of computation. Each of them has its own type of pathological problems depending of whether the static or the kinematic connection matrices are ill-conditionned. When the degree of hyperstaticity of the structure or the refinement of the elements grows up the displacement method takes the advantage because the number of hyperstatic unknowns comes closer and closer to that of the displacements. As mentionned above, the combined method is not used in practice due to the large number of equations it involves.

Consider now the case where the problem is formulated in terms of local values of the stress functions. The first observation is that, as formulated in (2.142), the number of unknowns f is significantly larger than in (2.141) for, in each element, f_k contains the number of "rigid body" modes of the stress functions. However all these modes can be arbitrarily prescribed as they do not generate stresses. This reduces the number of unknowns to exactly the same number as in the preceding formulation. Obviously the same three methods could be applied in which case no advantage is gained by using these variables.

However if a closer examination is given to the connection rules between elements, an important simplification can be introduced in a large class of problems where the number of externally loaded interfaces is small. Indeed in the absence of external loading along the interfaces, the connections can be directly expressed by identification of the corresponding stress function modes f_k without recourse to the generalized forces g_k.
In this case it is convenient to define for each element a localizing matrix L_k^* which achieves these identifications. Note that the L_k^* matrices differ from the L_k matrices as the topology of the connections between stress function modes and surface traction modes is obviously not the same. If f denotes the column of all the unique local values of the stress functions, the connection equations are written

$$f_k = L_k^* f \tag{2.151}$$

In this process the "rigid body" modes of the stress functions in each element are automatically given the same value as these of the global, piecewise defined and continuous stress function field.

The remaining global rigid body modes will be fixed arbitrarily afterwards.

Across these interfaces where an external loading is to be applied, the stress functions are not singlevalued (Ref. 2.15, 2.16) and the loading is precisely measured by the discontinuity. Such loads are introduced by disconnecting the stress function modes along the loaded interfaces. This is achieved in practice by giving a different numbering to the corresponding stress function modes of the elements adjacent to the interface. Then the previous definition (2.80) of the loads pertaining to that interface are used as connection equations

$$\bar{g} = \sum_k \bar{L}_k \, C_k^* \, f_k \tag{2.152}$$

Note that $\bar{g}$ now contains only the non zero prescribed external loading unstead of all the external generalized forces. The localizing matrices $\bar{L}_k$ are the corresponding reductions of the matrices L_k.
The minimization problem (2.142) becomes when reformulated by using (2.151) and (2.152)

$$\left(\frac{1}{2} f' \, L_k^{*\prime} \, F_k \, L_k^* f - \bar{g}' \, \bar{q} \right) \text{ minimum} \tag{2.153}$$

with the constraints

$$\bar{g} = \bar{L}_k \, g_k^* \, L_k^* f \tag{2.154}$$

or

$$\left(\frac{1}{2} f' \, F_s \, f - \bar{g}' \, \bar{q} \right) \text{ minimum} \tag{2.155}$$

with

$$\bar{g} = C_s^* f \tag{2.156}$$

where F_s and C_s^* denote respectively the global flexibility matrix and the global constraint matrix obtained by localizing the corresponding element matrices. $\bar{q}$ denotes the reduced column of the displacements conjugate to $\bar{g}$. The number of independant parameters f in (2.155) can be much smaller than in (2.142) and indeed the smaller the number of loaded interfaces, the larger the reduction in the size of the problem.
The three methods of solution described above can be applied to the minimization problem (2.155) but their relative advantages from the numerical point of view appear to be rather different in the case where the number

of external loads is small. The first method is to satisfy the constraints (2.156) by substituting $\bar{g}$ in (2.155) which yields

$$\frac{1}{2} f' F_s f - f' C_s^{*'} \bar{q} \tag{2.157}$$

which is minimum for

$$F_s f = C_s^{*'} \bar{q} \tag{2.158}$$

If the global rigid body modes of the stress functions have been arbitrarily fixed, F_s is a non singular matrix so that (2.158) becomes

$$f = F_s^{-1} C_s^{*'} \bar{q} \tag{2.159}$$

from which $$\bar{g} = C_s^{*} f = C_s^{*} F_s^{-1} C_s^{*'} \bar{q} = \bar{K} \bar{q} \tag{2.160}$$

The final linear system is thus reduced in size to the number of external loads. However the inversion of the global flexibility matrix is in general required to obtain $\bar{K}$. But, as F_s couples all the elements it has to be inverted in whole as opposed to invert each element flexibility matrix in turn like in (2.143). The result is a large fully populated matrix which make this procedure, which is analogous to the so called displacement method, numerically inefficient.

The second method proceeds like the force method by first solving the constraint equations (2.156). The number of these constraint equations is much smaller than in the previous application (2.142) and the number of unknows f is at the same time smaller than the number of unknows b due to the many connections already expressed. The search for the self stressing states requires consequently much less numerical work. This work can be even more drastically reduced by the following observations.
In that part of the structure that has no external loading each of the stress function modes, which have been taken as unknows along the interfaces, can be interpreted as a self stressing state. This interpretation of the stress function modes was introduced in reference 2.15. It also provides the interpretation of the displacements d conjugated to the stress function modes f and defined in (2.70) as generalized cuts relaxing the hyperstaticity of the structure. If all the stress function modes of the

structure are given zero value except for one of them, it becomes evident that the diffusion of this mode is limited to the elements adjacent to the interface where the stress mode is defined or to the elements having in common a vertex, if the stress mode is defined by a local value at this vertex. Therefore the choice of the stress function modes as self stressing states or generalized redundancies has the advantage of reducing to a minimum the coupling between the redundancies in the structure.

This brings about an important simplification in the application of the force method (Ref. 2.15, 2.16). In the constraint equations (2.156) the column f of the stress function modes is partitioned in

$$f = \left| \begin{matrix} F_g \\ F_0 \end{matrix} \right| \tag{2.161}$$

where f_g groups all the stress function modes pertaining to the loaded interfaces while f_0 groups all the others. The constraint equations obviously do not depend upon the f_0 modes and (2.156) can be reduced to

$$\bar{g} = C^{*}_{sg} f_g \tag{2.162}$$

which is solved for f_g yielding

$$f_g = P \bar{g} + Q x \tag{2.163}$$

where $P \bar{g}$ is a particular solution and $Q x$ the general solution, x being a column of self stressing state intensities additionnal to the states f_0. In many practical cases the system (2.162) is so simple that it could be solved by hand. Expressing the complementary energy with the help of (2.161) and (2.163) yields

$$\Psi = \frac{1}{2} f' F_s f = \frac{1}{2} \bar{g}' P' F_{gg} P \bar{g} + \frac{1}{2} f'_0 F_{00} f_0 + \frac{1}{2} x' Q' F_{gg} Q x + x' Q' F_{gg} P \bar{g} + f'_0 F_{0g} P \bar{g} + f'_0 F_{0g} Q x \tag{2.164}$$

where F_{gg}, F_{00}, F_{0g} denote the submatrices of F_s corresponding to the partitioning defined in (2.161).
Differentiation with respect to all the self stressing states x and f_0 furnishes their value in terms of the applied loads.

$$\begin{vmatrix} Q' F_{gg} Q & Q' F_{g0} \\ F_{0g} Q & F_{00} \end{vmatrix} \begin{vmatrix} x \\ f_0 \end{vmatrix} = - \begin{vmatrix} Q' F_{gg} P \\ F_{0g} P \end{vmatrix} \bar{g} \quad . \tag{2.165}$$

Only a small part of the global flexibility matrix F_s has to be modified to yield this system of equations which is usually better conditionned than the one obtained using the generalized forces g as unknows (2.147) because its largest part F_{00} is narrow banded instead of being fully populated.

Finally however the most efficient method of solution is to use the so called combined method by introducing the constraint equations into the minimum principle using Lagrange multipliers. This presents the problem (2.155) in the form

$$\{ \frac{1}{2} f' F_s f - \bar{g}'\bar{q} + \bar{\ell}' (\bar{g} - C_s^* f) \} \text{ minimum} \tag{2.166}$$

The solution is

$$\begin{vmatrix} F_s & C_s^{*\prime} \\ C_s^* & 0 \end{vmatrix} \begin{vmatrix} f \\ \bar{q} \end{vmatrix} = \begin{vmatrix} 0 \\ \bar{g} \end{vmatrix} \tag{2.167}$$

Compared to (2.150) this system is much smaller because most of the constraint equations have already been used to reduce the size of F_s. Indeed in most practical applications the size of the system (2.167) is of the same order as the number of redundancies f of the structure. Obtaining this system is also very simple and consists merely in the localization of the element flexibility matrices and a few connection matrices.

However two types of localizing matrices are required to assemble the connection matrices. One corresponds to the topology of the connection

between the stress function modes, the other to the topology of the connection between the generalized forces. From the next section it appears that for plate bending or membrane stretching problems, these localizing matrices are particularly simple to generate in a computer program utilizing conforming and equilibrium elements and working by the displacement method.

2.6. Finite element families for membrane stretching and plate bending

Families of conforming displacement finite elements have been recognized by many authors (Ref. 2.17) for membrane stretching. For plate bending, a small number of purely conforming elements have been proposed (Ref. 2.10, 2.5, 2.4) but they are unrelated to eachother and their derivation appeared rather complicated. Families of hyper-conforming elements have also been recognized (Ref. 2.17, 2.20, 2.21).

For stress diffusing equilibrium elements, a certain number of isolated elements have been proposed (Ref. 2.1, 2.6, 2.7). The difficulty of finding a stress field free of kinematic degree of freedom is reflected by the small number of such elements. That limited library has been an obstacle to the practical use of equilibrium elements.

The stress function approach provides an alternative and often easier way of discovering equilibrium elements as well as a sometimes more economic method of solving the global problem as was outlined in the preceding section. For membrane stretching and plate bending, an additional guide is provided by the reformulation of the slab analogies in terms of finite elements (Ref. 2.16). More recently these analogies have allowed to define families of triangular and quadrangular equilibrium membrane elements and at the same time of conforming displacement plate bending elements (Ref. 2.9). The families of plate bending equilibrium elements appear also as the analog of those of conforming displacement membrane elements. In addition, sub-equilibrium and hyper-equilibrium membrane elements are recognized as the analog of non conforming or hyper-conforming plate bending elements. All these new families are described in the following, after summarizing the slab analogies.

2.6.1. The slab analogies

First analogy

In plane elasticity the derivation of an equilibrium finite element by the stress function approach is based on an Airy stress function ϕ such that, in the absence of body forces

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \tau_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} \tag{2.168}$$

The stress diffusitivity consists in obtaining surface tractions σ_n and τ_{nt} (n and t stand for coordinates normal and tangent to the boundary) which are continuous across the interfaces and this is achieved if

$$\phi \text{ and } \frac{\partial \phi}{\partial n} \tag{2.169}$$

are continuous across the interfaces (Ref. 2.15, 2.16). The problem of finding a polynomial for ϕ is therefore identical to that encountered in the Kirchhoff plate bending theory for finding the deflection function of a conforming displacement element. Indeed the analogy between the derivation of the flexibility matrix of the equilibrium element for plate stretching and the derivation of the stiffness matrix of the displacement element for plate bending can be established at all steps. The generalized displacements, deflections and slopes, have to be taken as the local stress function values and the moment-curvature relations have to be replaced by the strain-stress relations for plane elasticity. It follows that each conforming element for plate bending yields as analog an equilibrium element for plate stretching and vice versa. The same computer program can be used to generate both of them if the change in constitutive equations is built in.

Second analogy

A second analogy exists between the equilibrium solution of the plate bending problem and the conforming displacement solution of the plate stretching problem. The derivation of an equilibrium element for plate bending by the stress functions method requires (Ref. 2.15, 2.16) two stress functions U and V such that, in the absence of distributed transverse loading

$$M_x = \frac{\partial V}{\partial y} \qquad M_y = \frac{\partial U}{\partial x} \qquad M_{xy} = -\frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \tag{2.170}$$

The model will be stress diffusing if, across an interface of outward normal n, the normal bending moment and the Kirchhoff equivalent shear force

$$M_n \qquad \text{and} \qquad K_n = Q_n + \frac{\partial M_{nt}}{\partial t} \tag{2.171}$$

are continuously transmitted and if, at a common vertex i, the jumps of twisting moment

$$Z_i = M_{nt_{i+\varepsilon}} - M_{nt_{i-\varepsilon}}$$

in the different elements sum up to zero. These stress diffusitivity conditions expressed in terms of the stress functions are met by requiring continuous values for U and V across the interface.

The problem of finding the stress functions U and V and of expressing their continuity across the interfaces is therefore identical to that encountered in the derivation of conforming displacement elements for membrane stretching. This second analogy states that to each conforming displacement model of membrane corresponds a stress diffusing equilibrium model for plate bending and vice versa. If the generalized displacements of the conforming model are taken as the local values of the stress functions in the equilibrium model and if the stress-strain relations are replaced by the curvature moment relations, the stiffness matrix of the conforming element becomes the flexibility matrix of the equilibrium element. Therefore the same computer program can easily generates both of them. The extension of the analogies to thin shell elements has also been presented (Ref. 2.22, 2.23). The finite element library of existing programs is easily duplicated through systematic use of these analogies.

2.6.2. Equilibrium finite element families for plate bending

The first three members of the family of conforming membrane elements of triangular shape are represented on figure 2.3. They are based on displacement functions U and V which are complete polynomials of the first, second and third degree respectively. Using the second analogy recalled above these elements are recognized as the first three members of a family of equilibrium plate bending elements. If the stress function approach is followed the generalized variables used are formally identical to the

membrane problem but have the meaning of stress functions local values. If the generalized surface traction approach is followed the variables can be defined as indicated on the right hand side of figure 1 and have the meaning of local values of normal bending moments, normal Kirchhoff shear forces and corner loads. The second element of the family was the first discovered using the generalized forces approach (Ref. 2.1, 2.18) and rediscovered using the stress function approach more recently (Ref. 2.24).

The topological advantage of the stress function approach is evident by the fact that two variables belong to a vertex instead of one. In the first two members of the family, all the stress function parameters are determined by the connection modes denoted f_c in the theory. In the third element, as well as in all the higher degree elements, internal stress modes or bubble modes are present. In the stress function approach, they are indicated by the local values of the two stress functions and, in the generalized forces approach, local values of the bending moments M_x and M_y. These internal variables have been interpreted as generalized redundancies internal to the element.

In figure 2.4 the first three members of the family of equilibrium plate bending elements of parallelogram shape are presented. They are based on bilinear and bicubic polynomials respectively. The two types of variables are shown and are self explanatory. Internal stress modes appear already in the second member of the family. However due to high symmetry of the stress function field they can be easily suppressed, if so desired, by dropping certain symmetric crossed x, y terms in the polynomials. It is obvious that the reduces the quality of the element and that a better procedure is to keep these modes as bubble modes but eliminate them at the element generation stage by a preliminary minimization with respect to them.

2.6.3. Equilibrium finite element family for plate stretching

The application of the first analogy, between conforming plate bending elements and equilibrium membrane elements, presents some interesting peculiarities. The simplest equilibrium membrane element is the constant stress triangular element (Ref. 2.6). It is represented on figure 2.5 (b). The stress field depends upon three parameters.

$$\sigma_x = \beta_1 \qquad \sigma_y = \beta_2 \qquad \tau_{xy} = \beta_3 \tag{2.172}$$

The interface surface tractions are uniquely defined by their two constant components along the three edges. The number of interface generalized forces is six. As they are three global equilibrium equations and no internal stress mode, the balance of the number stress field parameters

$$n_b - n_i = n_c - n_r \tag{2.173}$$

is correct and the model is free of additional constraint or spurious kinematic mode.

If one tries however to built the conforming plate bending analog, that is to derive the same element by the stress function method, the procedure fails. The stress function has to be a complete second degree polynomial which depends upon six parameters. The continuity of the function ϕ and its normal derivation ϕ_n requires the definition at each vertex of the local values of $\phi$and the two slopes ϕ_x, ϕ_y as indicated figure 2.5 (a). Nine interface variables are requered, and a model free of additional constraints cannot be derived. This simply restates the well known fact that for plate bending no simple polynomial fits in the triangular geometry if strict conformity is required. The failure of the procedure is however not so surprising. One recalls first (Ref. 2.25) that the connection of equilibrium membrane elements can be visualized by pin-jointing their skeletons represented in dashed line in figure 3 (a). This due to the fact that the generalized displacements conjugated to the interface forces F_x, F_y are the simple averages of the displacements u and v along that interface. If the element is fixed by one interface, a rotation of the skeleton is still possible. This can be interpreted as an external kinematic mode, or equivalently, as an additional relation to be satisfied by the loading. Three such modes exist in the element. The analogous situation in the conforming plate bending element is that along the three edges, the two values of ϕ and the two values of ϕ_n cannot be chosen independantly to define the parabolic shape of ϕ.

It was noticed at the beginning of the use of equilibrium elements (Ref. 2.1, 2.25) that a structure cannot be modeled by an assemblage of such triangles because each cell of four triangles behaves as a four bar

linkage, introducing therefore an inter-element kinematic mode. To avoid these modes a procedure consist in using quadrilaterals subdivided by their diagonals as building blocks. In this case, if the internal modes are not loaded, it can be shown that the inter-element kinematic mode is not excited. Therefore a bar can be added to lock this mode without modifying the behavior of the assemblage. A more convenient procedure consists in building directly a stress field for the four regions of the quadrilateral such that no kinematic mode exists. This is achieved (Ref. 2.11) by chosing in the triangle I of figure 3 (d) a constant stress field.

$$\sigma_\zeta = \beta_1 \qquad \sigma_\eta = \beta_2 \qquad \tau_{\zeta\eta} = \beta_3$$

Then the stress field in the triangle II is chosen a priori in equilibrium with I along $\zeta = 0$. Therefore only

$$\sigma_\eta = \beta_4$$

can be chosen independantly as σ_ζ and $\tau_{\zeta\eta}$ have to remain identical for equilibrium at the interface. In triangle IV by symmetry

$$\sigma_\zeta = \beta_5$$

can be chosen independantly, while in triangle III the three parameters are defined by the equilibrium conditions with the two adjacent triangle. The stress field defined in the quadrilateral contains 5 parameters and using (2.173) one can check that it is free from spurious kinematic modes. This quadrilateral element has been used successfully for structural analysis (Ref. 2.1, 2.11).

However if one tries to build the same element by the stress function method, the procedure still fails due to the same reason as for the triangle. Indeed the quadrilateral still has four external kinematic modes. The reason for which it can be used in practice is that either the boundary conditions (or the stiffeners added to the structure) are such that they usually restrain automatically the kinematic modes, just as in structures idealized with bars and shear panels.

The use of the first member of the family of quadrilateral membrane equilibrium elements is therefore always precarious.

The second element of the family is in fact the first one which does not present difficulties in practical use. It is also the analog of the first member of the family of conforming plate bending element. The stress field is assumed to be linear and to satisfy the homogeneous equilibrium equations

$$\sigma_x = \beta_1 + \beta_2 x + \beta_3 y$$
$$\sigma_y = \beta_4 + \beta_5 x - \beta_6 y \qquad (2.174)$$
$$\tau_{xy} = \beta_7 - \beta_6 x - \beta_2 y$$

It contains 7 independant parameters. The surface tractions along an interface are linear and require therefore two generalized forces to define each of their two components. Using the relation (2.173) one observes that a triangular element such as indicated figure 2.6 (b) is subject to two additional constraints or contains two spurious kinematic modes. The interpretation of the forces transmission by pin-jointed skeleton triangles still allows to visualize these modes : if one skeleton in figure (2.6 b) is fixed, the two others are still free to rotate around the central pin-joint.

By the stress functions method, a complete cubic is the corresponding assumption. It contains 10 parameters. The 12 local values of the Airy stress function and its derivatives which are necessary to insure stress diffusitivity are indicated on figure 2.6 (a). These variables are the analogs of the deflections and slopes used in the conforming solution of the plate bending problem. The conclusion analog to the existence of the two kinematic modes is that 2 of these 12 variables are not independant. A special element can be derived such as indicated on figure 2.6 (c) where the normal derivatives ϕ_n are not used along the edges 1-2 and 2-3. By itself such an element is not convenient because in the assembling process an additionnal constraint equation has to be written each time an edge without ϕ_n is met. This suggests however the general procedure which consists in preassembling a set of such elements so that all the edges where a variable is non independant fall inside the assemblage.

The procedure is illustrated on figure 2.7 (a). In that case the 3

constraints written to insure the continuity of the normal derivative of the Airy stress function along the 3 internal interfaces allows the elimination of the 3 variables defined at the internal mode 0. The 12 remaining degrees of freedom are therefore independant. The application of the same procedure in the method of generalized forces is illustrated on figure 2.7 (b). The way the 6 rigid body modes lock each either is self explanatory. One notes that the choice of the point 0 is completely arbitrary.

The same super-element can be obtained by direct construction in the 3 regions of a stress field or a stress function field which satisfies the required continuity. If the complete cubic stress function is assumed in each of the 3 triangles and if the 10 parameters so defined in the region I are considered as independant, the continuity requirements along the interface I-II introduce 4 constraints leaving only 3 independant parameters in region II. Doing the same along the interface I-III reduces also to 3 the number of parameters in region III. Finally the continuity along the interface II-III introduces again 4 constraints reducing the total number of the stress function field parameters to 12.

The same procedure fails if applied to the building of four triangles into a quadrilateral. The reason is that there are 4 interface continuity conditions to express and only the 3 internal degrees of freedom attached to the internal mode can be eliminated. However if the 4 triangles are defined by the diagonals of the quadrilateral as indicated figure 2.7 (c) and (d), the 4 interface conditions reduce to 3. This allows the building of quadrilateral super-elements free of additional constraints. Indeed the derivation of the element represented in figure 2.7 (c) has been published as that of a conforming plate bending element (Ref. 2.1, 2.10). The derivation by the method of generalized forces (figure 2.7, d) can be found in references 2.26, 2.27. The triangular conforming plate bending element corresponding to the equilibrium membrane element of figure 2.7 (a) is similar to that presented in reference 2.5, but it retains the mid-edge normal slopes as variables.

The third couple of elements of the same family is derived from the assumption of a second degree stress field

$$\sigma_x = \beta_1 + \beta_2\, x + \beta_3\, y + \beta_4\, x^2 + \beta_5\, xy + \beta_6\, y^2$$

$$\sigma_y = \beta_7 + \beta_8\, x + \beta_9\, y + \beta_{10} x^2 + \beta_{11} xy + \beta_4\, y^2 \qquad (2.175)$$

$$\tau_{xy} = \beta_{12} - \beta_9\, x - \beta_2\, y - \frac{1}{2}\beta_{11} x^2 - 2\beta_4\, xy - \frac{1}{2}\beta_5 y^2$$

or of a complete quartic for the Airy stress function. The interface variables required in a triangular element are indicated on figure 2.8 (a) and (b). Just as in preceding elements, there are additional contraints to satisfy. Indeed the interpretation by skeleton triangles indicates clearly that in all triangular elements of higher degree the same kinematic will be present which consist in a free rotation of the 3 skeletons close to the vertices.

Again however a set of 3 such elements is free of spurious kinematic modes. It is illustrated in figure 2.8 (c) and (d). An additional peculiarity of this super-element is the introduction of internal stress modes which can be interpreted as hyperstatic internal unknows.

In the same element derived by the stress function method and represented in figure 2.8 (a), 3 interface degrees of freedom are not independant. They can be taken as the 3 variables defined at a vertex. In the assembling process of the 3 such elements into a triangular super-element such as represented in figure 2.8 (d), 6 relations are therefore not automatically satisfied, which leads to eliminate 6 other internal degrees of freedom. One can choose to eliminate the 6 internal normal derivatives ϕ_n. But then the 3 mid edge local values of ϕ still remain which define 3 internal modes (bubble modes) of the Airy stress function. They are eliminated at the element generation.

The analog reasoning in the method of generalized forces is that the assemblage of skeleton triangles represented figure 7 (c), can be constructed by adding to the central isostatic cell of 3 triangles denoted A, the couples of triangles numbered 1 to 6. The assemblage remain isostatic. But the 3 elements denoted H are then obviously hyperstatic members.

The assemblage of 4 triangles (a) or (b) into a quadrilateral is illustrated on figure 2.9 (a) or (b). Again the element is free of kinematic mode only if the diagonals are taken as interfaces. The derivation

of this element is given in reference 2.28.

The existence of the 3 couples of elements described above indicates clearly that the procedure can be applied up to any degree. In fact the method of assembling elements where the number of interface variables does not match the number of parameters, is not restricted to the particular cases presented here.

This family provides therefore conforming plate bending elements or stress diffusing equilibrium membrane elements up to any desired degree.

Let us turn now to the topological advantage of using the stress functions method in solving membrane problems by equilibrium elements. This advantage is more important than for plate bending problems due to the fact that 3 variables are attached to a vertex if one uses the stress functions, while all the variables belong to the edges if one uses generalized forces.

2.6.4. Sub-equilibrium and hyper-equilibrium elements

The difficulties encountered in developping conforming displacement finite elements for plate bending coupled to the desire of simplifying the derivation and avoiding the connections expressed along the edges, led to the development of non conforming elements. Some of them were successfull in providing accurate and convergent results (Ref. 2.4, 2.5). They are characterized by a lack of continuity of the normal slope along the edges. The bounding properties of conforming elements are evidently lost. However, if the advantage of such models is established in practice, the same ideas can be applied to equilibrium models.
In figure 2.10 the sub-equilibrium (or non stress-diffusing) membrane element analog of the CLOUGH-MELOSH non conforming plate bending element (Ref. 2.19) is represented. The analog of the lack of conformity of the shear component of the surface traction. When the model is derived by using generalized forces, only one tangential force, equal to the average shear stress, is defined per interface instead of two required for the stress diffusivity.

On the other hand, the need of more refined elements coupled to the

desire of keeping the topological advantage of the connection expressed at the vertices and of avoiding the apparent complexity of super-elements, led to the development of hyper-conforming plate bending elements. One element of that category has been proposed by BOGNER (Ref. 2.21). It can be derived either from hermitian interpolation formulas or from a bicubic deflection field. It is illustrated in figure 2.11. The sixteen coefficients are defined by the local values of the function, its two slopes and the twist at each vertex. The transmission of the twist not only unsures the required compatibility, but introduces a non necessary constraint making the model hyper-conforming. If the corresponding equilibrium membrane element is derived using the generalized forces method, the shear stress component has to be used at each vertex as variable. This represents a non necessary constraint on the stress field. The model is therefore called hyper-equilibrium. It is obvious that a family of such elements is obtained by raising the degree of the Airy stress function to biquartic, biquintic, etc...

Another hyper-conforming plate bending triangular element has been proposed which is based on a complete quintic deflection field (Ref. 2.20, 2.29). At each vertex the deflection, the two slopes and the three components of the curvature tensor are taken as generalized displacements, while the normal slopes are also used along each interface. The element is illustrated in figure 2.12. Its equilibrium membrane analog uses as generalized forces the whole stress tensor at each vertex in addition to the four interface generalized surface tractions. Although very good results have been reported using this element, no comparison has been made with conforming elements of the same level of refinement nor in really tests of stress interpretation. The fact that the whole stress tensor is identical in each element meeting at a vertex is not necessarily an advantage since all the discrepancies, which have anyhow to exist in regions of high stress gradient, have to appear between interface stresses. These discrepancies can be larger than when avoiding the additional constraints.

Clearly since the topology of the connection is the main advantage in these elements, they are interesting as equilibrium membrane elements, only if the stress function method is used.

2.6.5. Hybrid membrane elements

In the theoretical presentation of the models based on Reissner's principle in section 2.4.1., it is stated that the choice of the assumptions for the internal stress field and for the interface displacement field are independent. This is however limited by the requirement of satisfying the relation (2.112) between the number of stress field parameters and of interface generalized displacements. When this relation is not satisfied, either bubble stress modes or kinematic deformation modes are present. In the latter case the use of the model is precarious as the additional constraints corresponding to the kinematic modes cannot usually be eliminated within a group of elements.

In the case of the families of membrane elements of triangular and parallelogram shape, the figure 2.13 indicates the combination of degrees for the stress and displacement fields which can be used without kinematic mode. The number of "bubble" modes n_i is also given. The equilibrium stress field of degree ranging from 0 to 4 is in both cases assumed to derive from an Airy stress function represented by a complete polynomial of degree 2 to 6. If a computer program is developped for such families of elements, the cases with kinematic modes have obviously to be restricted.

The figure 2.13 also indicates that in all cases but one, internal stress modes (or "bubble" modes) exist. It should be remembered when appreciating the results obtained with these elements that the numerical work involved in the derivation of the stiffness matrices is strongly dependant of the number of such modes. For a given degree of the displacements along the interfaces, the comparison with other models should only be achieved if the number of internal modes is equal.

2.7. NUMERICAL APPLICATIONS

Convergence of higher degree elements

The help that the dual analysis can bring in the solution of structural problems has been presented in previous publications (Ref. 2.1, 2.7, 2.9, 2.11). The simple box beam problem studied in reference 2.3 has been reanalyzed using the new higher degree elements. The structure is defined in figure 2.14. By symmetry only one quarter need be considered. The boundary conditions represent bending and torsion. The spar elements have been avoided as in practical structures they are usually not applicable. Therefore the vertical web and horizontal skin are idealized by the same general quadrilateral membrane elements of increasing degree.

The strain energy values are indicated on figure 2.14 and plotted in figures 2.15 and 2.16 versus the number of unknowns which is roughly proportional to the computational effort.

It appears from these comparisons that constant stress or constant strain elements yield so crude results and converge so slowly that their use should be avoided as much as possible. This is however not always possible in highly complicated structures where the number of unknows becomes rapidly prohibitive.

The elements of higher degree yield much better results and it appears that the gap in strain energy can be reduced at will. The improvement shown by the quadratic stress elements is small when plotted in terms of number of unknowns. This suggests that for problems of this nature the linear stress approximations are the best compromise. It also appears that conforming and equilibrium elements of corresponding degrees converge approximately at the same rate with a slight advantage for the equilibrium elements in certain cases.

Solution by the stress function method

The reduction of the number of degrees of freedom involved in equilibrium analyses that can be achieved by using the stress function method is illustrated in figure 2.17. The number of unknowns necessary to analyse a square plate in bending and stretching is presented. The mesh size is 10 x 10 and elements with linear stress or linear strain field are chosen. For membrane stretching the conforming element is the TYPE 4 quadrilateral of second degree if one refers to the description of the finite elements library used by the ASEF-20 computer program and given in the appendix. The equilibrium element is the TYPE 16 quadrilateral in the generalized

forces method and the TYPE 15 in the stress function method. For plate bending the elements used are respectively the TYPE 15, TYPE 13 and TYPE 4.

In the equilibrium solution of the membrane stretching problem, one notes that the reduction in unknowns using the stress function method is particularly important, reaching 34 %. Indeed if compared with the conforming solution the use of equilibrium models becomes more economical by 15 % while it is always more expensive in the classical approach (here by 13 %). The topology of the connections in the plate bending problem leads to a less spectacular reduction of unknowns.

The practical use of the stress function method requires to express the boundary conditions in terms of constraints on the stress functions. This is illustrated in figures 2.18, 2.19 and 2.20 for the bending of a cantilever U beam and in figures 2.21, 2.22 or 2.23 for the bending of a centrally loaded, point supported square plate. Taking first the cantilever U beam, figure 2.18 represents the boundary conditions which are used in a classical equilibrium analysis formulated in terms of generalized forces and solved by the displacement method. These boundary conditions expressed in terms of normal and tangential surface tractions n, t are translated into boundary conditions expressed in terms of curvatures of the stress function F(x,y) as illustrated by figure 2.19. On the same figure the arbitrary fixation of the rigid body modes of the stress function is also indicated. Note that F_x stands for $\frac{\partial F}{\partial X}$ etc. From these boundary conditions it is easy to derive the set of constraints represented on figure 2.20. The input data corresponding to an equilibrium analysis of this cantilever U beam by the two methods are given in the set of problems illustrating the use of the ASEF-20 computer program in appendix.

Turning next to the plate bending problem, the boundary conditions in terms of surface tractions are indicated in figure 2.21. They are translated in terms of stress functions in figure 2.22 where the fixation of the rigid body modes is also indicated. One notes that the introduction of a concentrated load inside the plate requires to impose a dislocation of the stress functions along a line arbitrarily selected from the load to an external boundary. The angle of the opening due to the dislocation is proportional to the load. The deformed shape of the stress function is indicated by the dashed line.

On figure 2.23 the detailed expressions of the constraints corresponding to the concentrated load are represented. The upper part of the figure indicates the general conditions to satisfy by the stress functions on

both sides of the dislocation. The lower part of the figure illustrates the application in the case when the dislocation is bordered by triangular elements with a linear stress function field. It appears that the final form of the constraints expressed point by point is extremely simple and that all constraints can be introduced with a very limited number of elementary constraints applying between 2 nodes or 2 interfaces.

Stress representation in plate bending problems using conforming and equilibrium elements

The usefulness of the dual analysis in cases of difficult stress interpretation is illustrated by the following examples of dual analysis of plates in bending. Figure 2.24 presents the convergence of the central deflection and of the strain energy for a square clamped plate under uniform distributed loading. The conforming quadrilateral TYPE 15 is opposed to the equilibrium triangle of TYPE 13. Both have linear bending moments and consistent loading is used.

The oscillatory convergence of the central deflection using the conforming model confirms the theoretical conclusion that the comparison of local displacements in dual analysis or simply in convergence study may not be substituted to comparison of strain energy. The convergence of the latter is very satisfactory and the agreement between the two solutions is almost perfect for 200 degrees of freedom. If the bending moments and shear forces are compared for the same fine mesh, the agreement is also extremely good as illustrated by figure 2.25. Note that in this example the convergence of the equilibrium model is slower than that of the conforming element.

Figure 2.26 illustrates the results for a simply supported plate skewed at 30° under uniform distributed loading. This problem is known for the strong singularity present at the obtuse angle. Although it is clear that the exact representation of the singularity cannot be expected by a finite element solution using simple polynomials, it is important to know if the accuracy of the results away from the singular point remains satisfactory. The convergence of the central deflection reveals here a much better accuracy for the equilibrium element. For the finer mesh sizes, the strain energy of the conforming solution still differs by 24 % of the equilibrium one. No exact solution is available but the analytical approximate solution reported here and due to MORLEY is probably very accurate. It differs of the equilibrium solution by only 3 %.

More imporatnt is the comparison of the principal bending moments at

the center as illustrated in figure 2.27. The superiority of the equilibrium solution is important and in fact the convergence of the compatible solution is still problematic even for 1000 degrees of freedom. The distributions of bending moments along a half diagonal and obtained by the two dual finer mesh sizes are compared in figure 2.28 with the analytical solution. The superiority of the equilibrium solution appears more important as the singular point is approached. A more detailed treatment of the problem can be found in reference 2.7.

It should however not be concluded that the stress representation given by equilibrium models is always better. In regions of high stress gradient, depending of the specific nature of the boundary conditions the interpretation of the stress output can be as difficult as with conforming elements.

The physical importance of the singularity mentionned above has been studied by a refined plate bending theory, due to HENCKY, which takes into account the shear deformations. It is similar to the REISSNER theory but modified to allow the derivation of a purely conforming solution. In this theory, 2 different idealisations of the simple support can be proposed as the rotations ϕ_s of the cross sections are independant of the slope $\frac{\partial w}{\partial s}$ of mean plane. Therefore along a simply supported edge one can impose

$$w = 0 \qquad \phi_s = 0$$

or

$$w = 0 \qquad \phi_s \neq 0$$

The same skew plate has been analyzed using the conforming plate bending element of TYPE 8 in the ASEF-20 computer program and the two types of simple support represented. The non dimensional shear deformability is :

$$\rho = \frac{E_1 \, t^2}{E_2 \, a_2} = 5.10^{-2}$$

where E_1 is the Young's modulus in bending and E_2 the transverse shear modulus. The bending moments along a half diagonal are represented on figure 2.29 where it is clear that the singularity at the obtuse angle is associated with the condition of support imposing $\phi_s = 0$ and disappears if it is relaxed. Figures 2.30 and 2.31 present the distribution of shear forces and transverse rotation along an edge and a half diagonal. The important differences between the 2 support conditions indicate that in problems where the detailed stress distribution is important, close to the obtuse angle, it is essential to take into account the stiffness of the support against a rotation ϕ_s. (Ref. 2.7)

Effect of internal degrees of freedom

It has been stated in the theory that the behavior of finite elements can be improved by the introduction of bubble modes or internal degrees of freedom (D.O.F.). The influence of these bubble modes has been compared using the PIAN hybrid membrane parallelogram and the conforming membrane parallelogram. The first example is illustrated in figure 2.32. A single element is used to idealize a cantilever beam of aspect ratio equal to 2 and submitted to bending by a constant distributed shear force. Consistant loading is used. The degree of the displacements along the boundary is kept constant and equals 2 so that the final number of degrees of freedom of the problem is always 16. The strain energy is plotted versus the number of internal D.O.F. which is also a measure of the computational effort to derive the stiffness matrix of the element.

The degree of the stress field in the hybrid element has been increased up to 8 and corresponds to a complete polynomial for the Airy stress function of up to the 10th degree. The internal displacement modes of the conforming element are given by 2 functions, identical for u and v, of the form :

$$(1 - \frac{x^2}{a^2})\ (1 - \frac{y^2}{b^2})\ P_2\ (x, y)$$

with

$$P_2\ (x,y) = (\alpha_1 + \alpha_2\, x + \alpha_3\, x^2 + \ldots + \alpha_{n+1}\, x^n)$$
$$(\beta_1 + \beta_2\, y + \beta_3\, y^2 + \ldots + \beta_{n+1}\, y^n)$$

The degree mentionned on figure 2.32 is n. Note that n = 0 corresponds to P_2 = constant which implies one internal mode. The degree n - 1 corresponds to dropping this mode.

The results of a very accurate dual analysis of this problem involving more than 2000 D.O.F. are given as reference solutions.

It appears clearly from this comparison that the solutions obtained by increasing the number of internal D.O.F. in the two models converge very rapidly toward an "exact" solution which is that of the bending of the cantilever beam with the constraints of quadratic boundary displacements. The 2 models become practically identical for more than 30 internal D.O.F.

It is also clear that the behavior of both models is strongly influenced by the first few internal D.O.F. but that over, say, 20 D.O.F. the benefit becomes marginal.

Another important point is that, referring to figure 2.13, it is impossible to use the hybrid model with a first degree stress field due to the presence of 6 kinematic deformation modes. Indeed the table 2.13 mentions that even for stresses of the second degree, one kinematic mode is to be expected. If a result can be given here, this is due to the type of boundary conditions which locks this mode. But in general the first hybrid model that can be used without restriction has a 3rd degree stress field. Such a model should be compared with the conforming element derived with n = 2 as they involve the same number of internal D.O.F. and therefore the same computational effort. Otherwise the comparison is not fair.

Obviously 2 complementary convergence curves could be obtained for the same problem by using an equilibrium element and the other type of hybrid element where the surface tractions on the boundary would be kept linear and the number of internal D.O.F. increased. The convergence would be toward an overestimated strain energy corresponding to the "exact" solution of the problem with the constraints of linear surface tractions.

Turning new to the stress representation, the results for both models with only one element are just as good or as bad depending of the stress component one considers.
The longitudinal stress due to bending is correctly represented even with very few internal D.O.F. But the shear stress remains very poor as it is influenced by the presence of the singularity at the built in. In all cases however the average of shear on the whole element is correct, but the local values computed are practically useless.

The advantage of the introduction of internal D.O.F. appears more clearly in a finer analysis. Figures 2.33 and 2.34 present the stresses obtained for the same problem by a dual analysis using a 10 x 10 grid of equilibrium or conforming elements with linear stresses or linear strains but without internal D.O.F. They involve respectively 880 and 682 D.O.F. The problem being symmetric, on the upper half of the beam the curves of constant σ_x are drawn while the lower half presents the curves of constant shear stress τ_{xy}. These curves have been obtained by averaging the stresses computed at the vertices of the elements and at the centroids. A computer graphic program has been used which interpolates linearly between these points.

Comparing the two solutions, it appears that the representation of σ_x coincide almost perfectly everywhere as well as that of τ_{xy} in the central section. Close to the 2 ends, the details of the τ_{xy} curves become very

different. As a reference solution for this problem, a very fine analysis has been achieved using a grid of 20 x 40 2d degree conforming elements and involving 5042 D.O.F. The results are presented in figure 2.35 using the same procedure. The form of the τ_{xy} field is now very well represented in the neighbourhood of the singular point.

The same problem has been solved using a grid of only 4 x 4 3rd degree conforming elements with an internal displacement field defined as above with n = 0 and n = 4. It corresponds to 200 interface D.O.F. plus 32 and 400 internal D.O.F. respectively. The stresses are presented in figures 2.36 and 2.37. For n = 0 the shear stress is still relatively poor close to the built in, while for n = 4 it corresponds almost perfectly with the reference solution. As it involves only 200 D.O.F. instead of 5042 the advantage appears very important in terms of core storage requirements and computer time. The extra cost of eliminating the 400 internal D.O.F. can be roughly evaluated by comparing the total computer time for the 3 solutions : for the 4 x 4 grid, with n = 0 : 18 sec. CPU; with n = 4 : 96 sec. CPU; for the 20 x 40 grid : 1020 sec. CPU.

The introduction of internal D.O.F. could evidently be coupled with the use of element of variable degree. These 2 techniques appear promising for the improvement of the stress results in region of high stress gradient with any type of finite element model. This certainly deserves further study.

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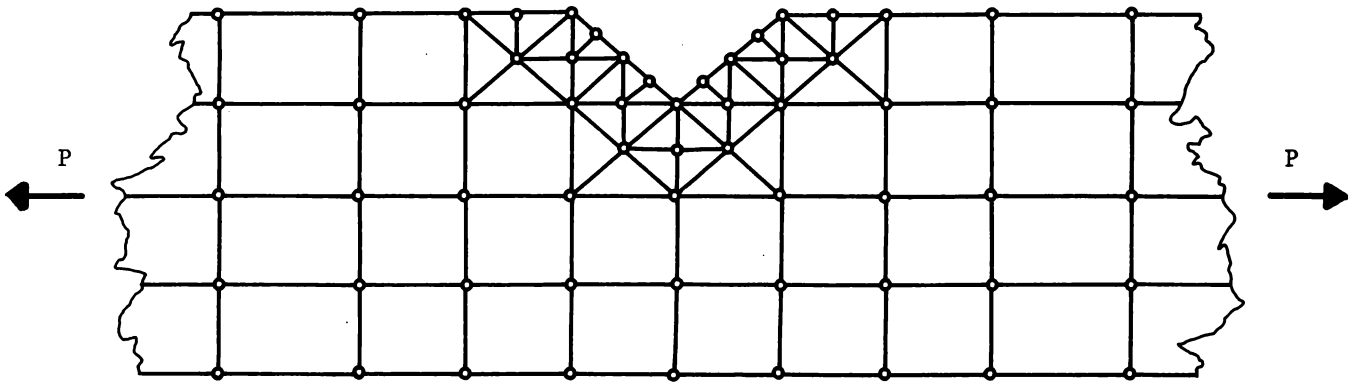
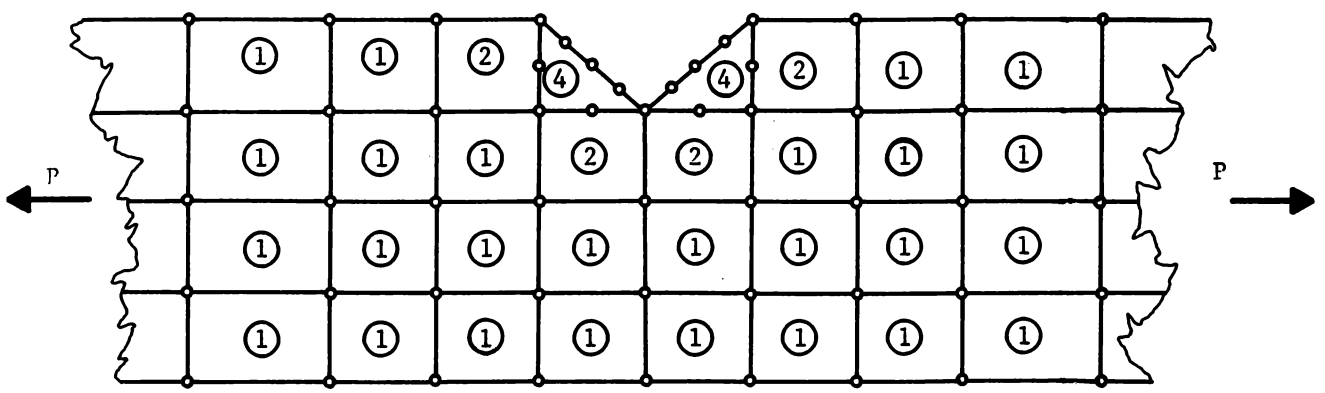
IDEALIZATION INTO FINITE ELEMENTS AROUND A CUT OUT



Idealization using only linear displacement elements

o : nodal point with 2 degrees of freedom

Total : 61 geometric nodal points
122 degrees of freedom
62 elements



Idealization using elements of variable degree, giving the same accuracy

(2) : internal degree of an element (it is reduced automatically along interfaces with simpler elements)

o : nodal point with 2 degrees of freedom

Total : 44 geometric nodal points
114 degrees of freedom
32 elements

FIGURE 2.1

IDEALIZATION OF A CANTILEVER BEAM

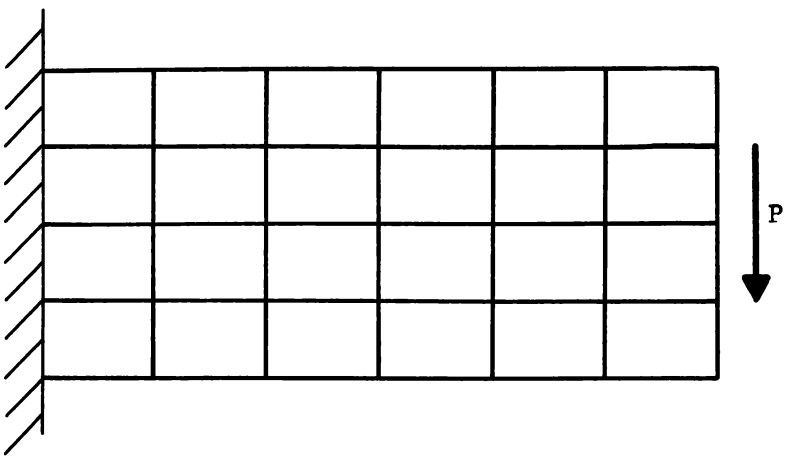
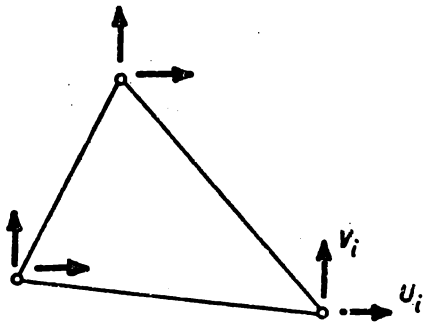
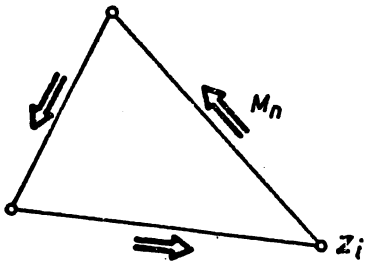


FIGURE 2.2

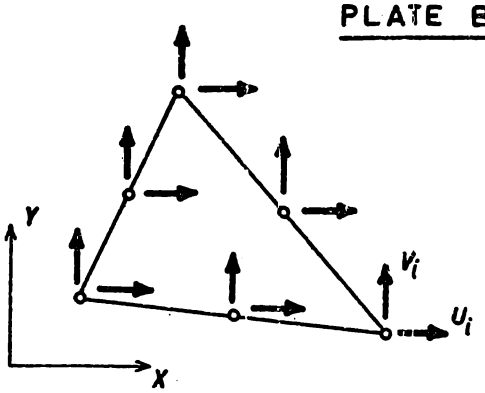


LINEAR STRESS FUNCTIONS U, V

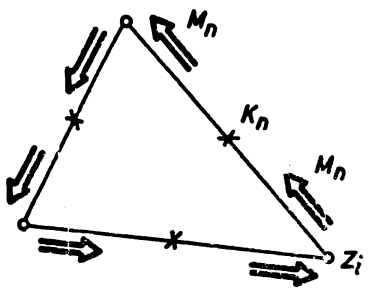


CONSTANT BENDING MOMENTS

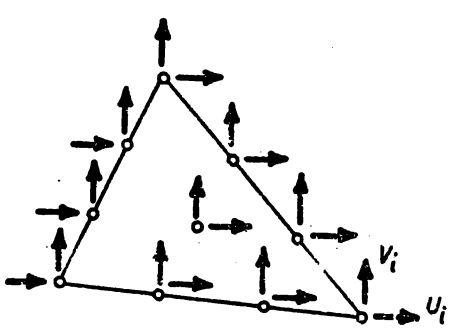
FAMILY OF EQUILIBRIUM TRIANGULAR PLATE BENDING ELEMENTS



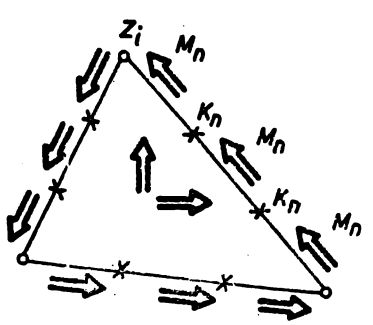
QUADRATIC STRESS FUNCTIONS U, V



LINEAR BENDING MOMENTS

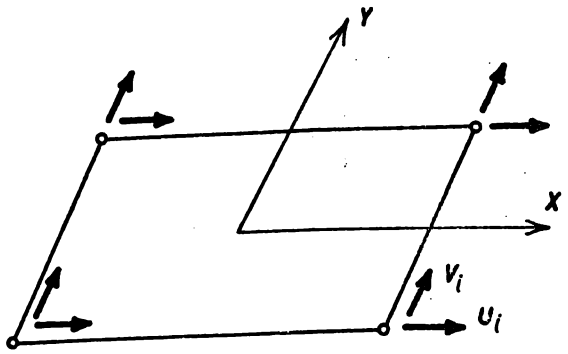


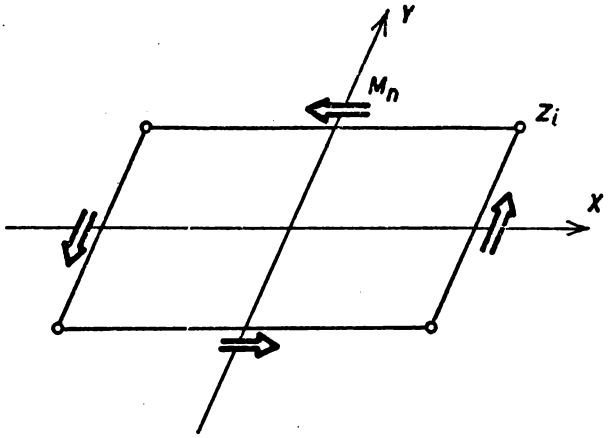
CUBIC STRESS FUNCTIONS U, V



QUADRATIC BENDING MOMENTS

FIGURE 2.3

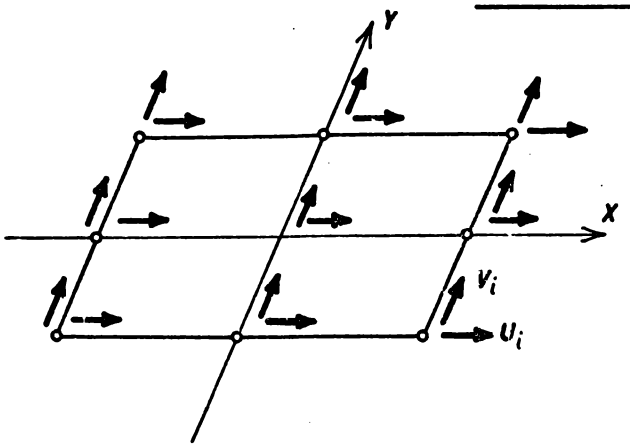


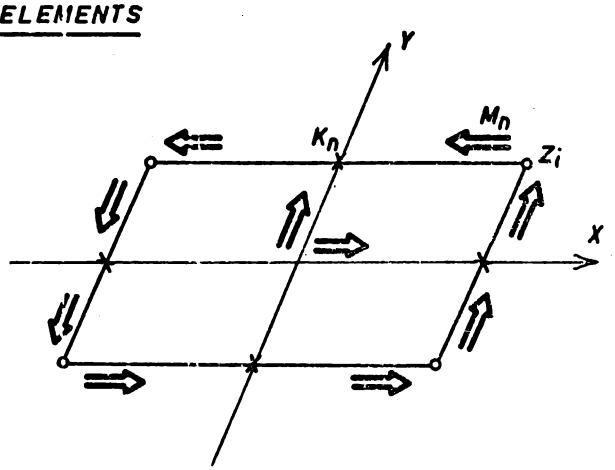


BILINEAR STRESS FUNCTION

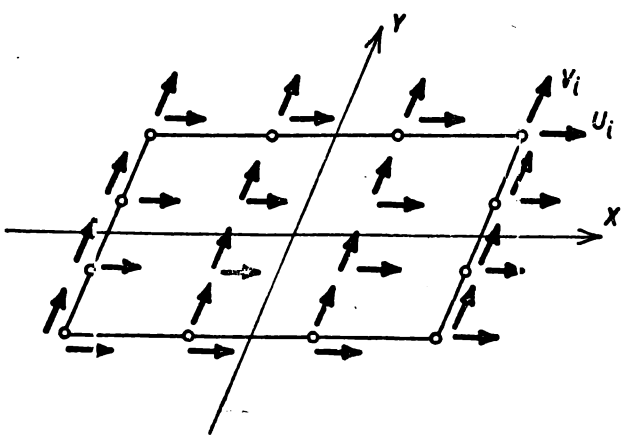
FAMILY OF EQUILIBRIUM PARALLELOGRAM

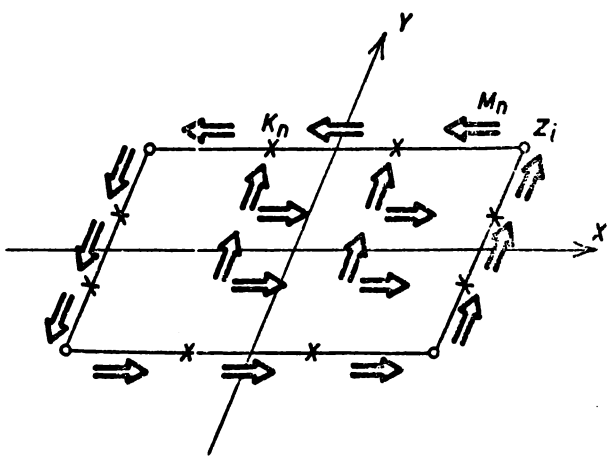
PLATE BENDING ELEMENTS





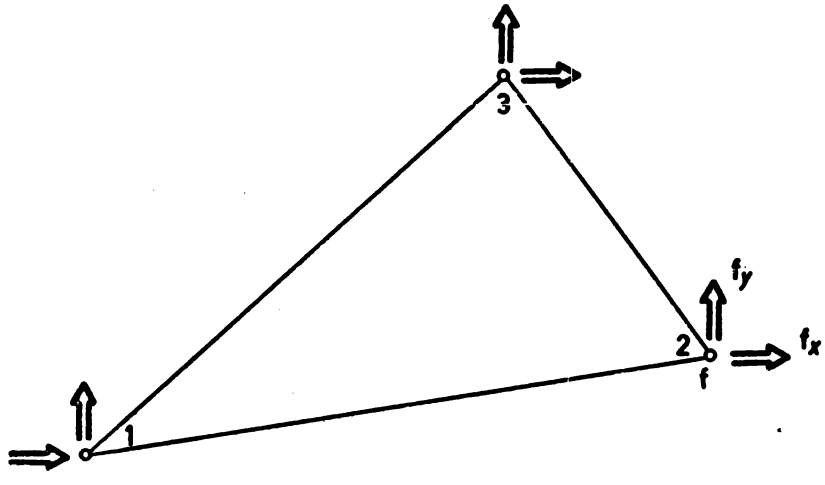
BIQUADRATIC STRESS FUNCTION



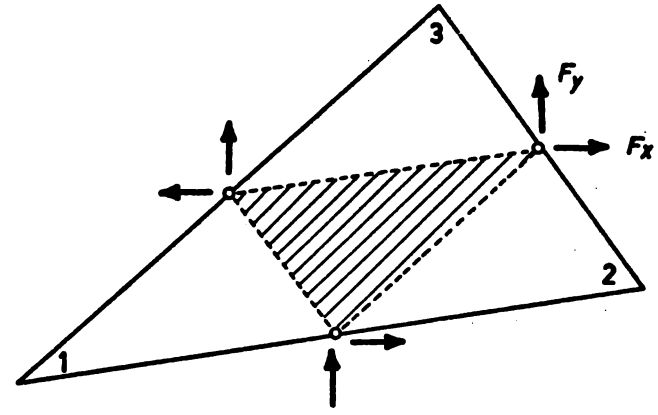


BICUBIC STRESS FUNCTION

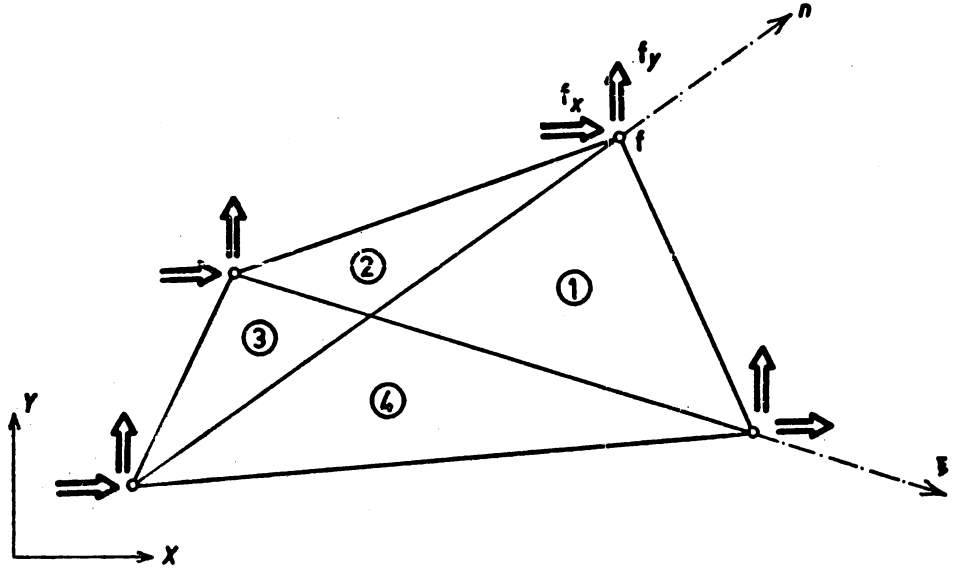
FIGURE 2.4



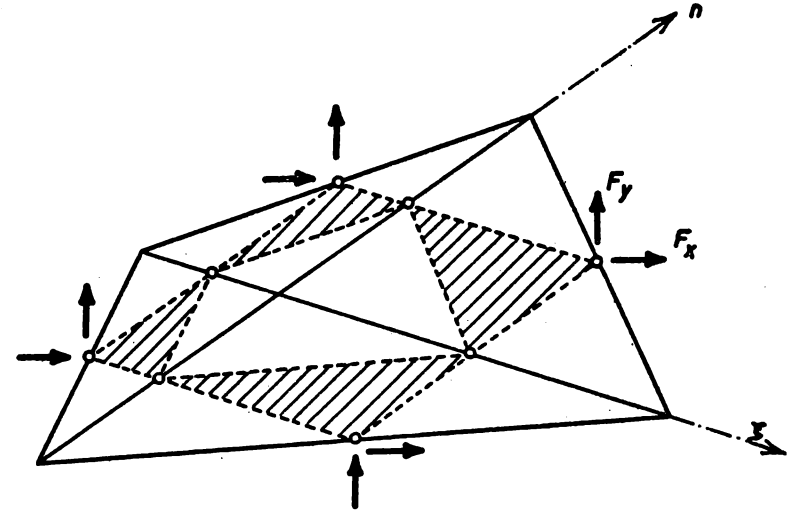
a) QUADRATIC AIRY STRESS FUNCTION



b) CONSTANT STRESS FIELD

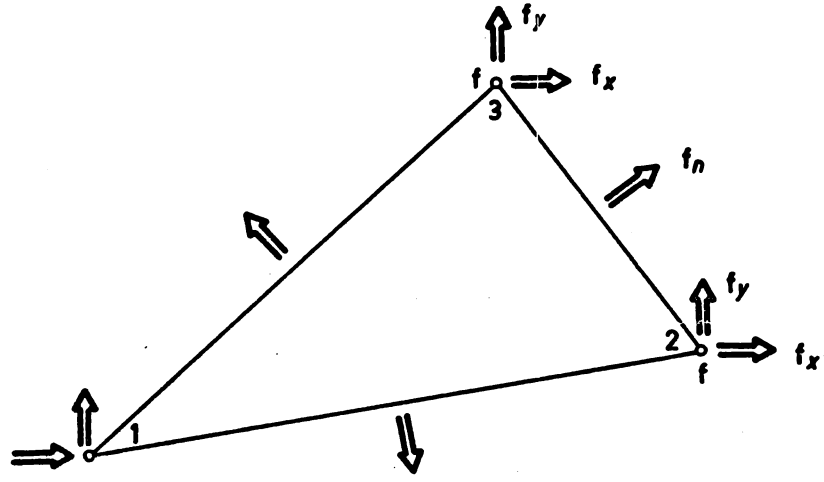


c)

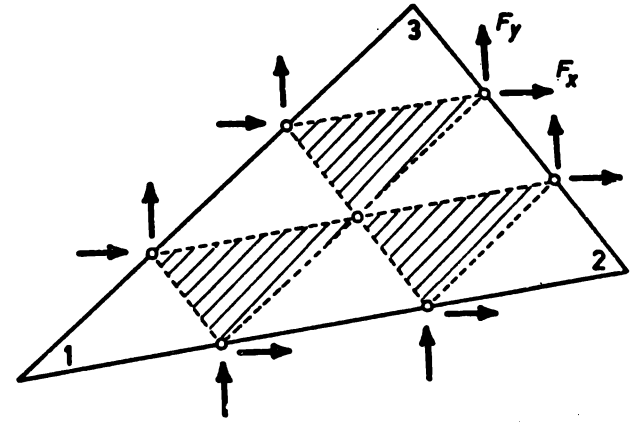


d)

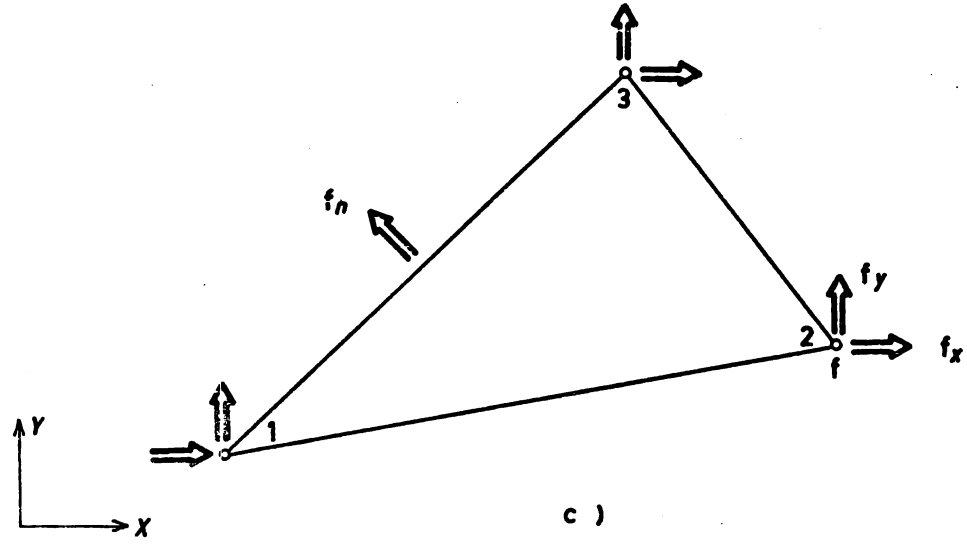
FIGURE 2.5



a) CUBIC AIRY STRESS FUNCTION

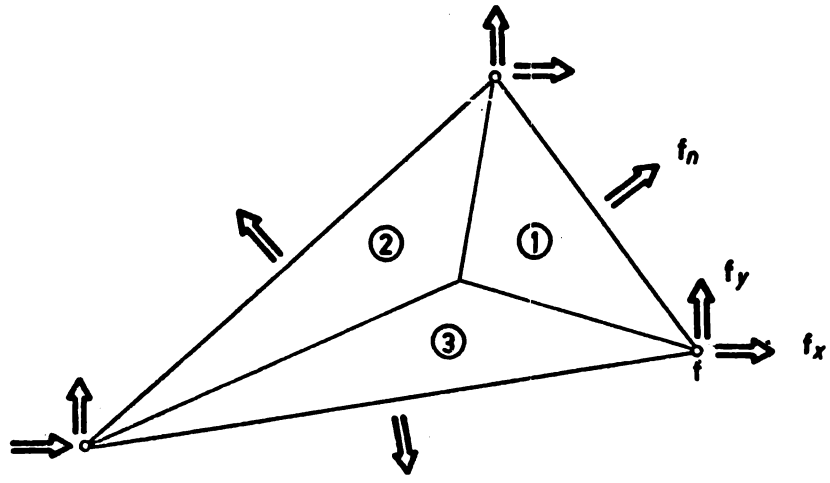


b) LINEAR STRESS FIELD

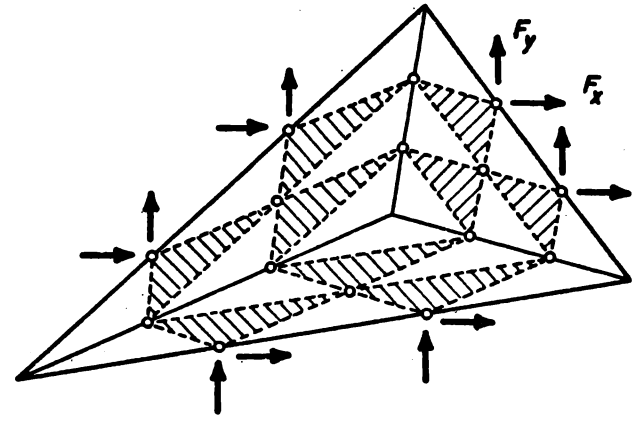


c)

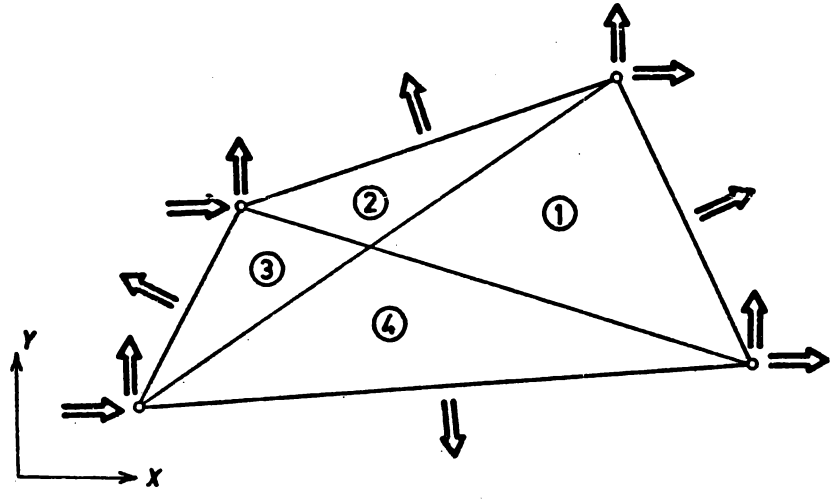
FIGURE 2.6



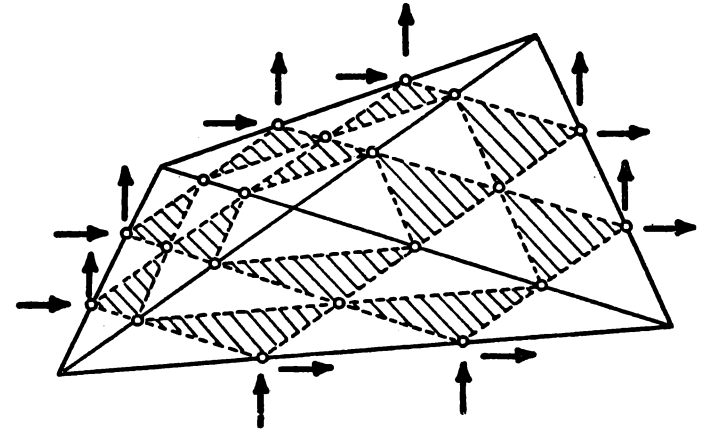
a)



b)

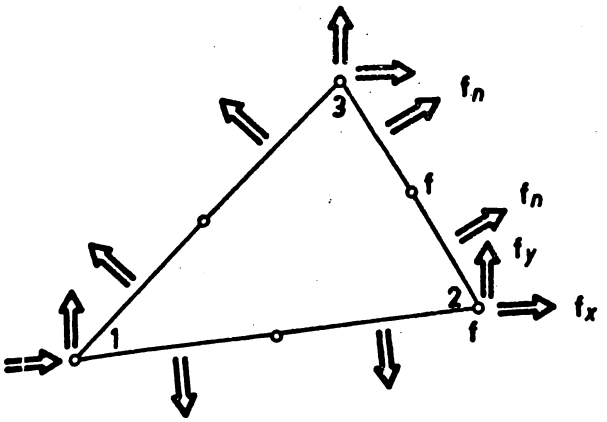


c)

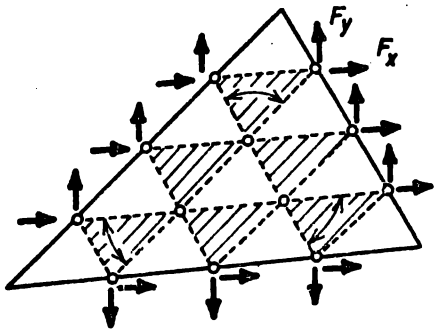


d)

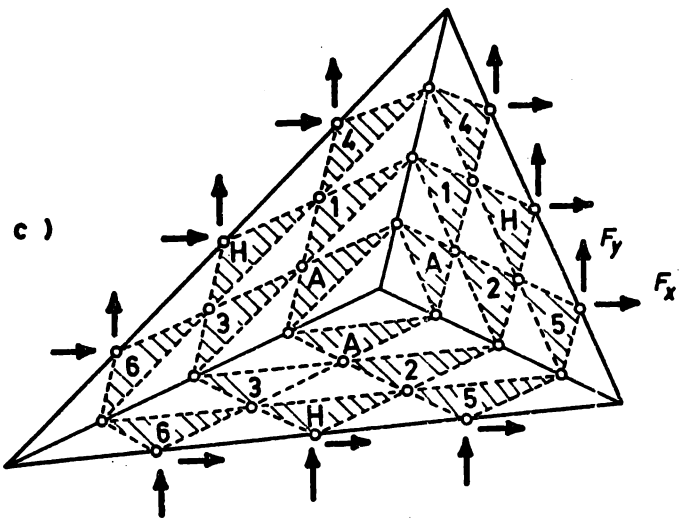
FIGURE 2.7



a) QUARTIC AIRY STRESS FUNCTION



b) QUADRATIC STRESS FIELD



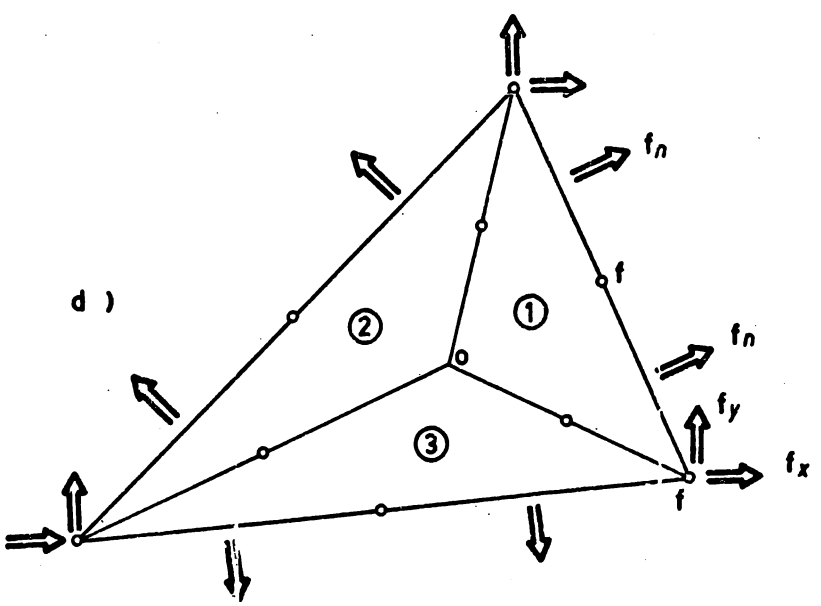
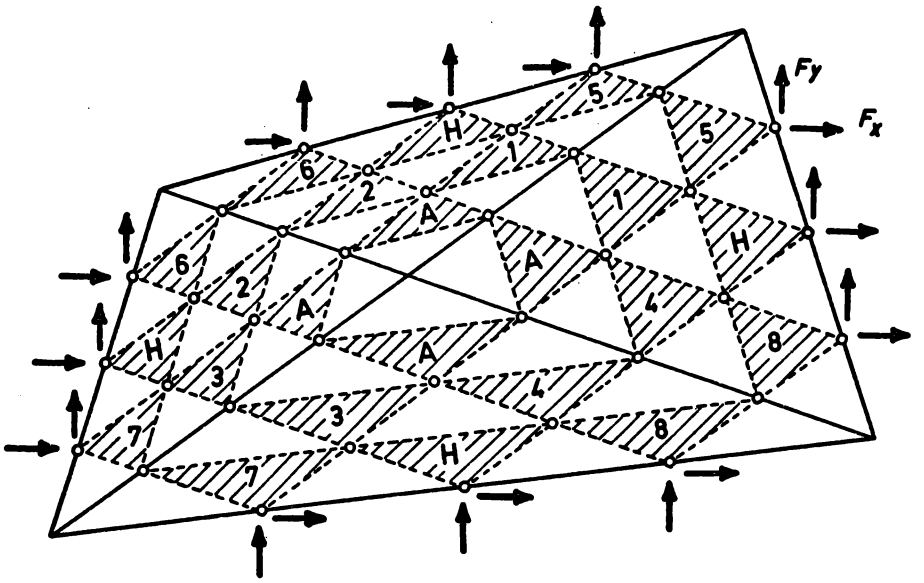
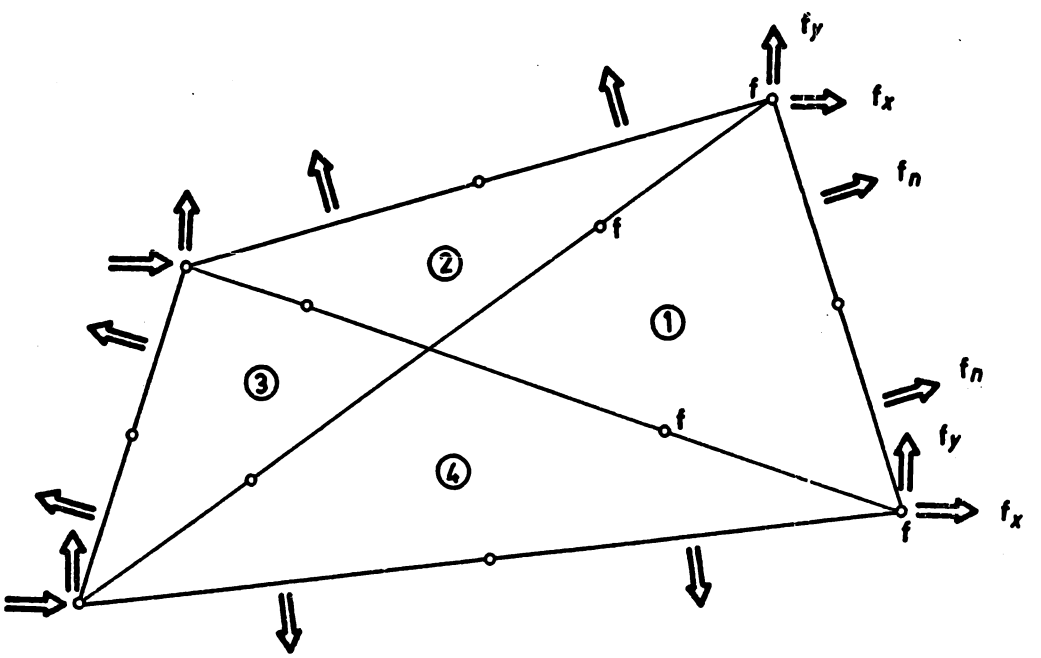


FIGURE 2.8

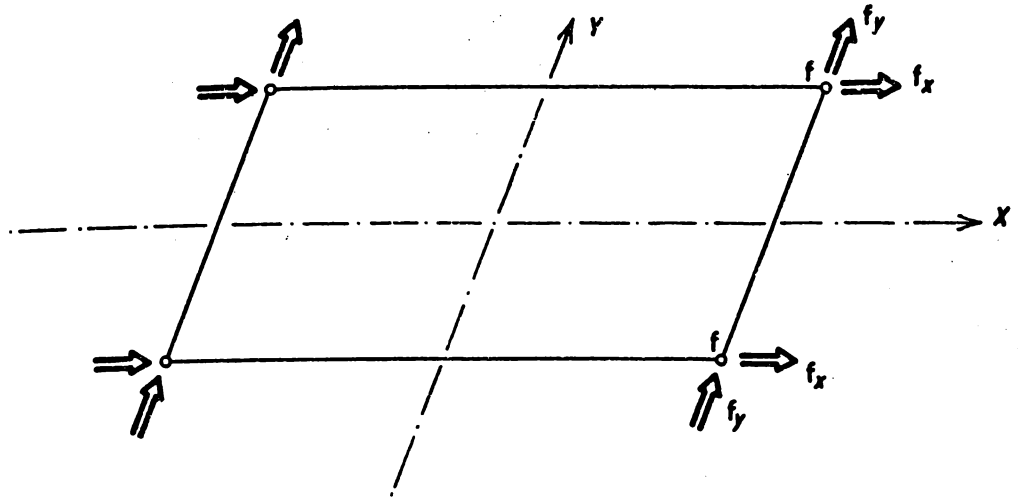


a) QUADRATIC STRESS FIELD



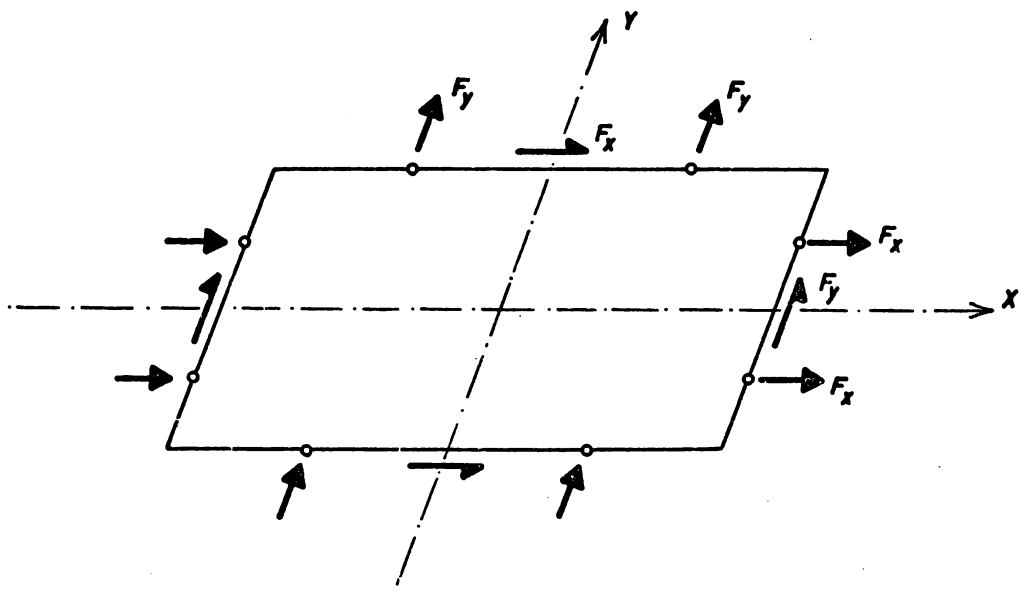
b) QUARTIC AIRY STRESS FUNCTION

FIGURE 2.9



Airy stress function

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3y + \alpha_{12} xy^3$$



Stress field

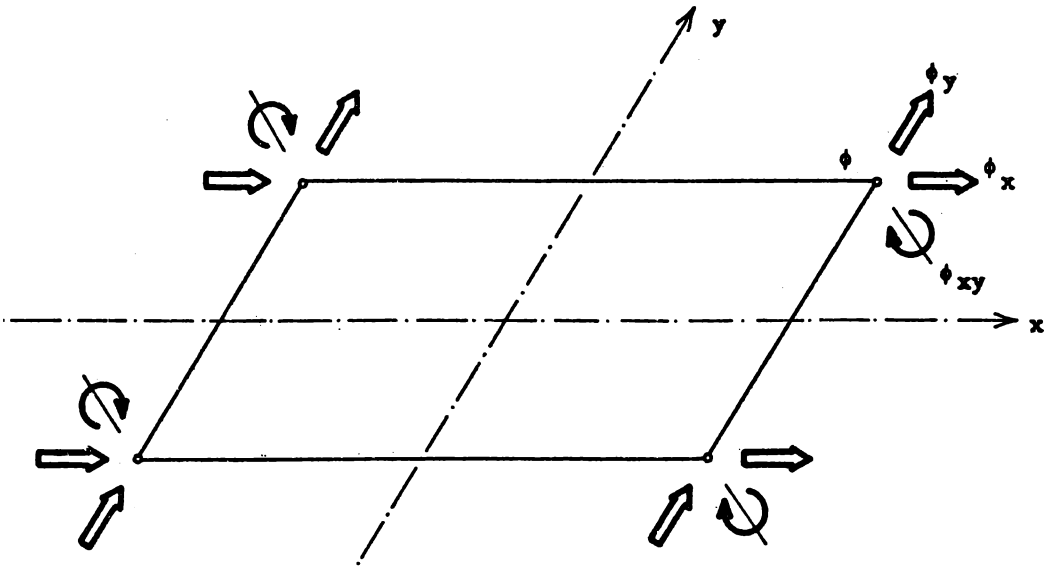
$$t_x = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

$$t_y = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy$$

$$t_{xy} = \beta_9 - \beta_7 x - \beta_2 y - \frac{1}{2}\beta_4 y^2 - \frac{1}{2}\beta_8 x^2$$

SUB EQUILIBRIUM MEMBRANE ELEMENT

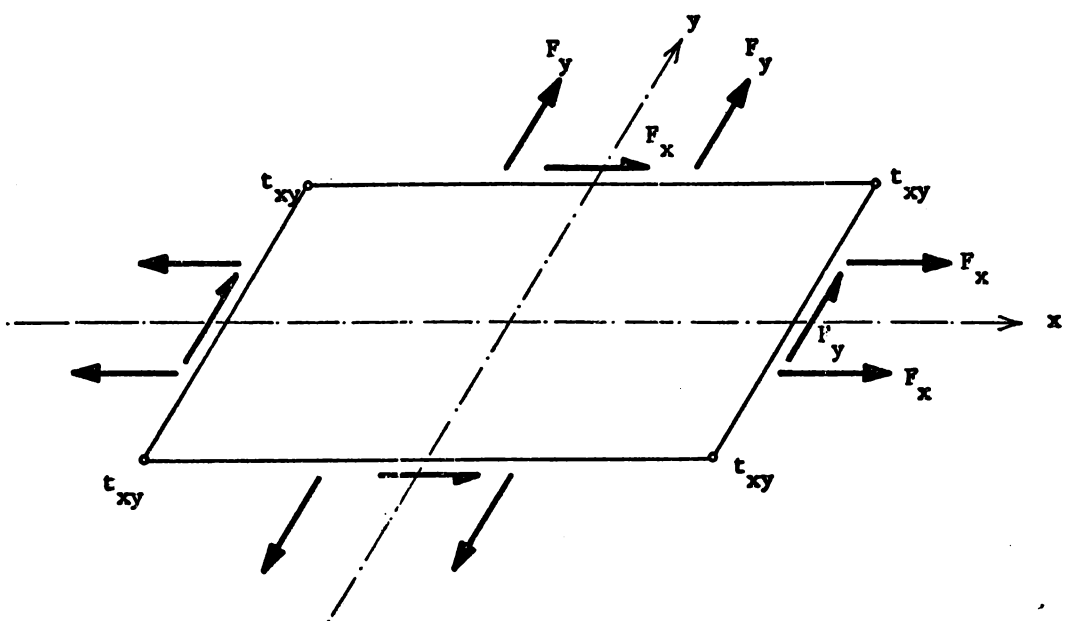
FIGURE 2.10



Bi cubic airy stress function

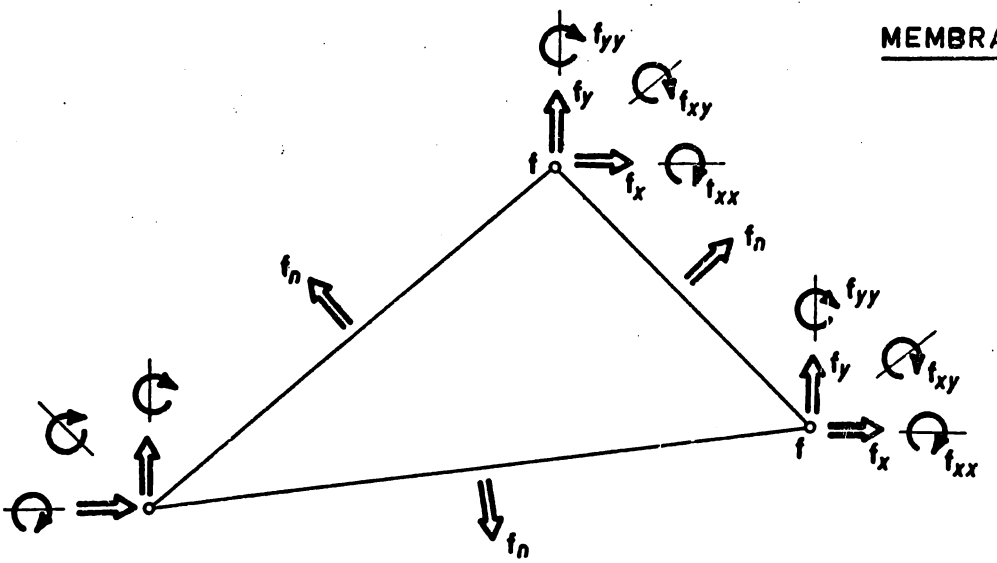
$$\phi = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3$$

$$a_{11} x y^3 + a_{12} x^2 y^2 + a_{13} x^3 y + a_{14} x^2 y^3 + a_{15} x^3 y^2 + a_{16} x^3 y^3$$



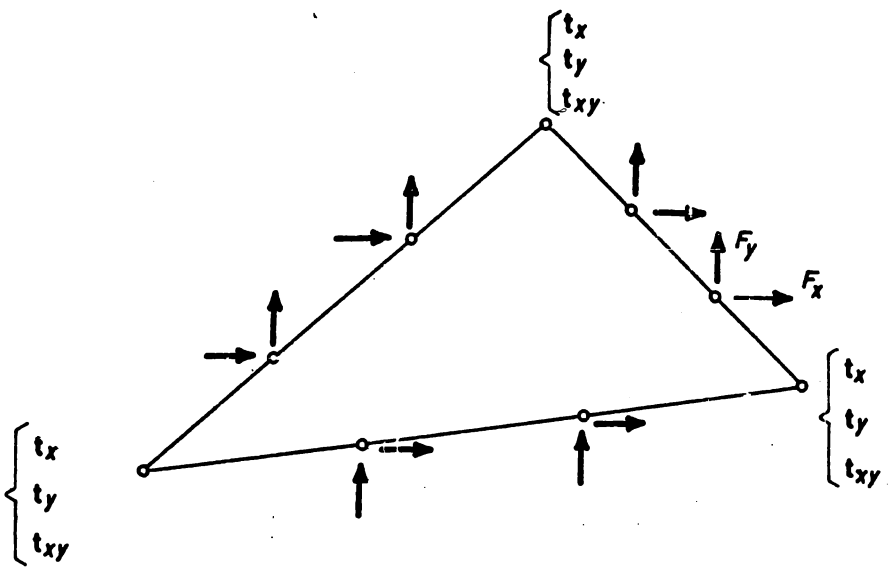
HYPER-EQUILIBRIUM MEMBRANE ELEMENT

Figure 2.11



Airy stress function

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^4 + \alpha_{12} x^3y + \alpha_{13} x^2y^2 + \alpha_{14} xy^3 + \alpha_{15} y^4 + \alpha_{16} x^5 + \alpha_{17} x^4y + \alpha_{18} x^3y^2 + \alpha_{19} x^2y^3 + \alpha_{20} xy^4 + \alpha_{21} y^5$$



Stress field

$$t_x = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^3 + \beta_8 x^2y + \beta_9 xy^2 + \beta_{10} y^3$$

$$t_y = \beta_{11} + \beta_{12} x + \beta_{13} y + \beta_{14} x^2 + \beta_{15} xy + \beta_4 y^2 + \beta_{16} x^3 + \beta_{17} x^2y + \beta_7 xy^2 + \beta_8 y^3$$

$$t_{xy} = \beta_{18} - \beta_{13} x - \beta_2 y - \tfrac{1}{2}\beta_{15} x^2 - 2\beta_4 xy - \tfrac{1}{2}\beta_5 y^2 - \tfrac{1}{3}\beta_{17} x^3 - 3\beta_7 x^2y - 3\beta_8 xy^2 - \tfrac{1}{3}\beta_9 y^3$$

FIGURE 2.12

TRIANGLE

Degree of the displacements

Degree of internal stress field σ	1	2	3	4	5	u
0	$n_i = 0$	$n_k = 6$	$n_k = 12$	$n_k = 18$	$n_k = 24$	
1	$n_i = 4$	$n_k = 2$	$n_k = 8$	$n_k = 14$	$n_k = 20$	
2	$n_i = 9$	$n_i = 3$	$n_k = 3$	$n_k = 9$	$n_k = 15$	
3	$n_i = 15$	$n_i = 9$	$n_i = 3$	$n_k = 3$	$n_k = 9$	
4	$n_i = 22$	$n_i = 16$	$n_i = 10$	$n_i = 4$	$n_k = 2$	

PARALLELOGRAM

Degree of the displacements

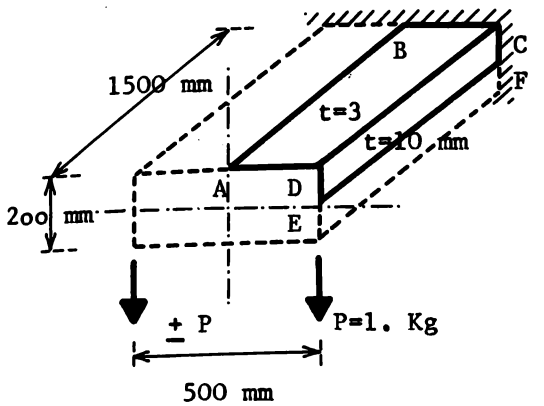
Degree of internal stress field σ	1	2	3	4	5	u
0	$n_k = 2$	$n_k = 10$	$n_k = 18$	$n_k = 26$	$n_k = 34$	
1	$n_i = 2$	$n_k = 6$	$n_k = 14$	$n_k = 22$	$n_k = 30$	
2	$n_i = 7$	$n_k = 1$	$n_k = 9$	$n_k = 17$	$n_k = 25$	
3	$n_i = 13$	$n_i = 5$	$n_k = 3$	$n_k = 11$	$n_k = 19$	
4	$n_i = 20$	$n_i = 12$	$n_i = 4$	$n_k = 4$	$n_k = 12$	

n_i = number of internal stress modes (bubble modes)

n_k = number of kinematic deformation modes

Figure 2.13

STRAIN ENERGY BOUNDS IN A BOX BEAM

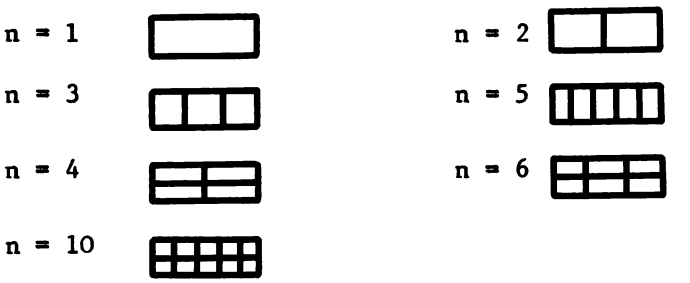


$E = 7.500\ Kg/mm^2$

$\nu = .3$

Idealization by quadrilaterals

Mesh sizes identical in ABCD and CDEF



NT = number of unknowns for $\frac{1}{4}$ of the box beam by the displacement method.

CONFORMING DISPLACEMENT ELEMENTS - LOWER BOUNDS TO THE STRAIN ENERGY

	n	NT	Strain energy	
			Bending	Torsion
Linear displacement field	2	21	1.057	1.099
	3	28	1.730	2.088
	5	42	2.602	3.566
	4	55	2.245	2.859
	6	77	2.862	4.140
	10	121	3.336	5.380
Quadratic displacement field	1	29	3.133	5.278
	2	47	3.609	6.306
	3	65	3.662	6.431
	5	101	3.697	6.520

Figure 2.14.A.

STRESS DIFFUSING EQUILIBRIUM ELEMENTS - UPPER BOUNDS TO THE STRAIN ENERGY

	n	NT	Strain energy	
			Bending	Torsion
Constant stress field	2	26	4.438	9.052
	3	37	4.224	8.915
	5	59	4.136	8.845
	4	84	3.948	7.106
	6	122	3.848	7.012
	10	198	3.803	6.971
Linear stress field	1	30	4.106	6.980
	2	52	3.792	6.658
	3	74	3.749	6.620
	5	118	3.731	6.602
Quadratic stress field	1	44	3.820	6.685
	2	78	3.736	6.607
	3	110	3.726	6.595

Figure 2.14.B

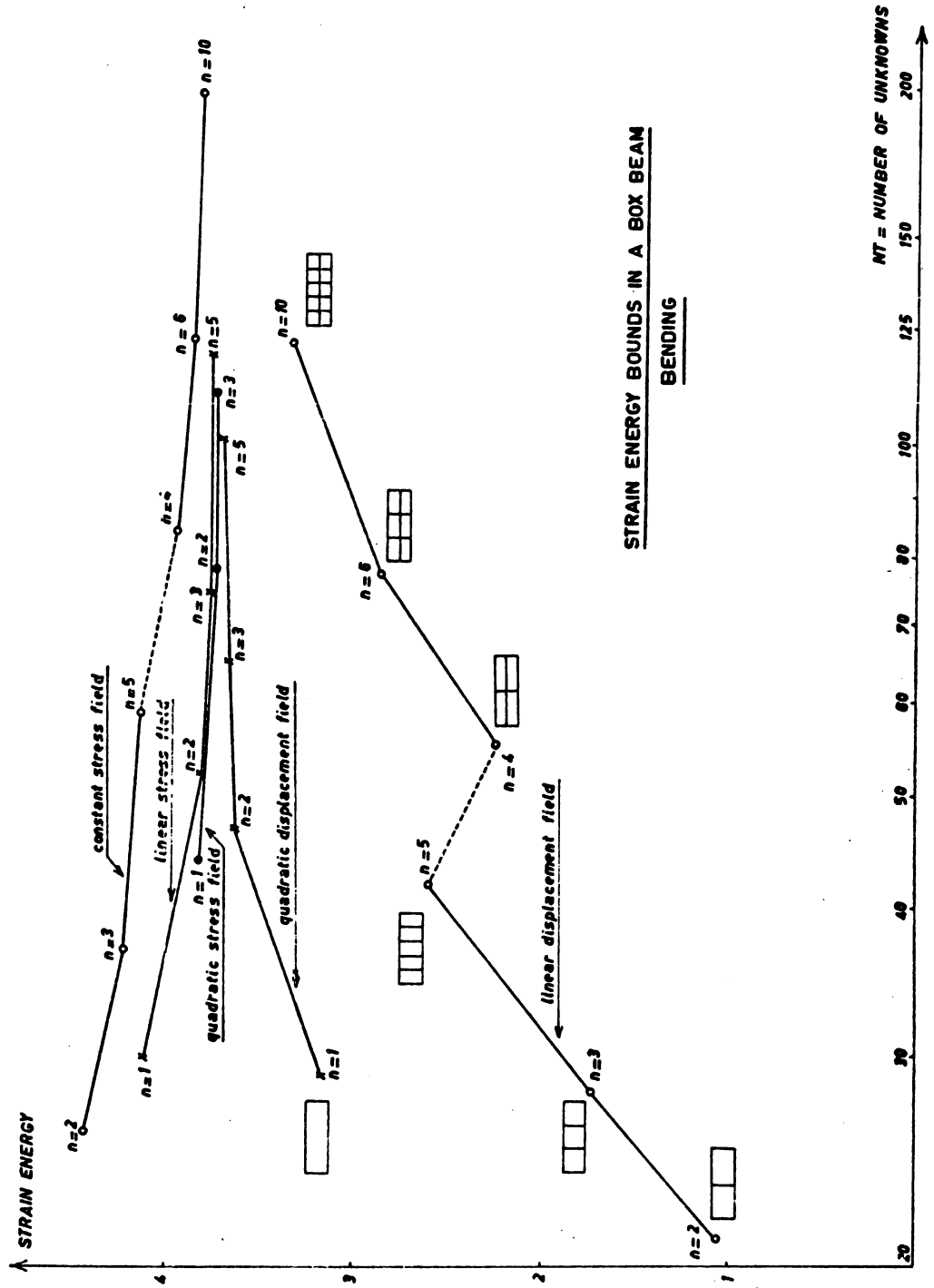

STRAIN ENERGY
NT = NUMBER OF UNKNOWNS
STRAIN ENERGY BOUNDS IN A BOX BEAM
BENDING
constant stress field
linear stress field
quadratic stress field
quadratic displacement field
linear displacement field
n=1
n=2
n=3
n=4
n=5
n=6
n=10
1
2
3
4
20
30
40
50
60
70
80
100
125
150
200

FIGURE 2.15

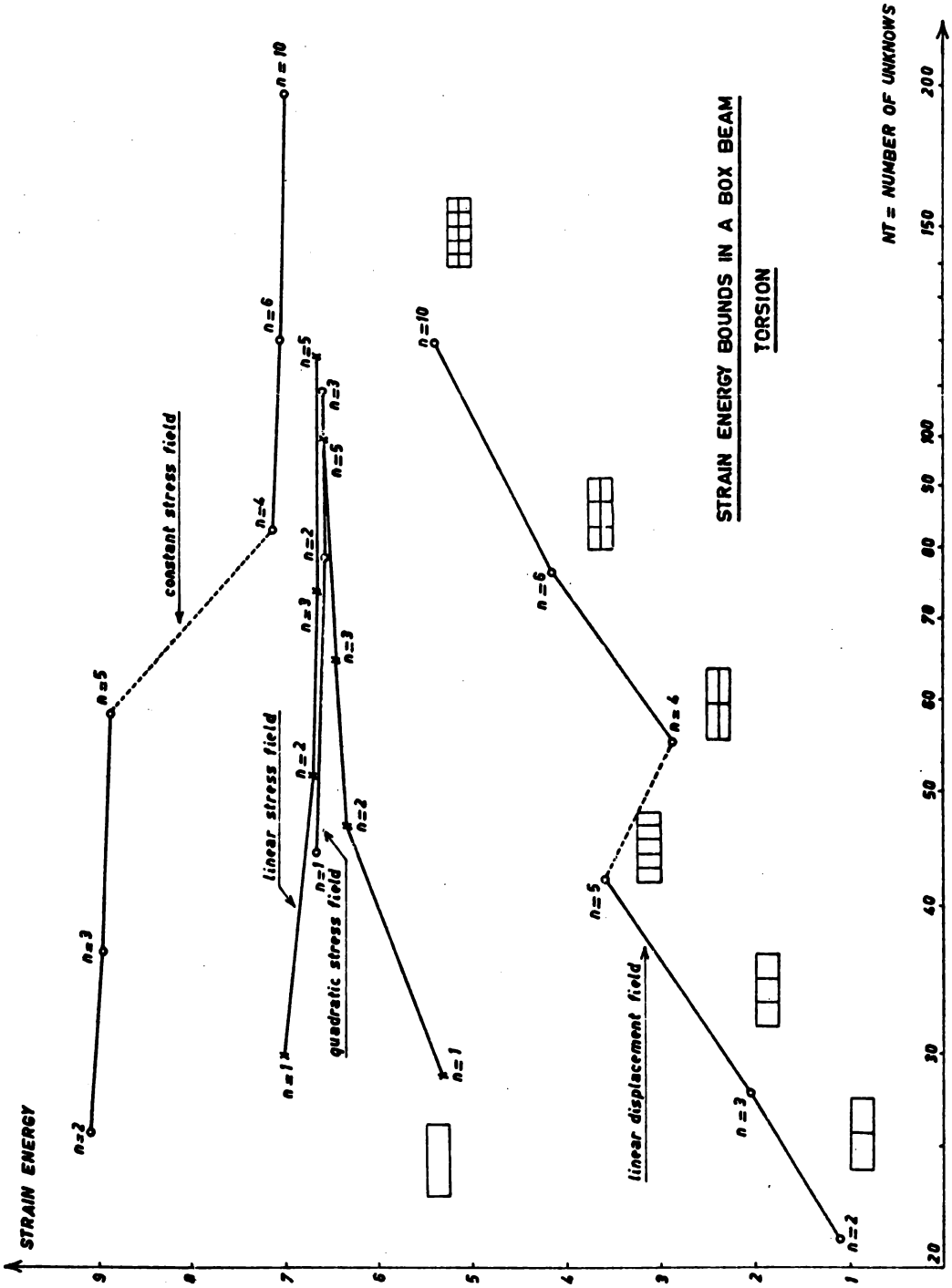
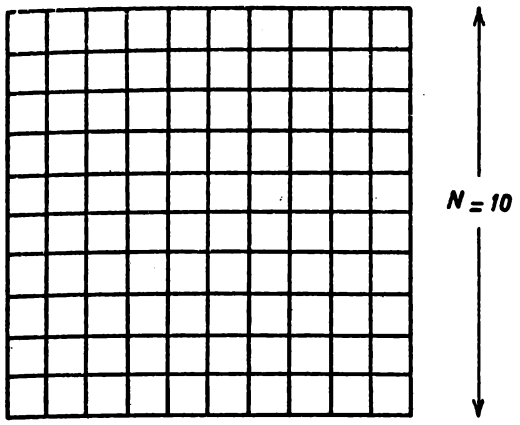

STRAIN ENERGY
NT = NUMBER OF UNKNOWS
STRAIN ENERGY BOUNDS IN A BOX BEAM
TORSION
constant stress field
linear stress field
quadratic stress field
linear displacement field
n=1
n=2
n=3
n=4
n=5
n=6
n=10
1
2
3
4
5
6
7
8
9
20
30
40
50
60
70
80
90
100
150
200

FIGURE 2.16

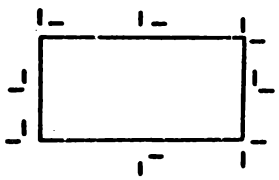


COMPARISON OF THE NUMBER OF UNKNOWNS (NT) USING DIFFERENT TYPES OF ELEMENTS.

(without application of the boundary conditions)

Membrane stretching problem (linear stress or linear strain assumptions)

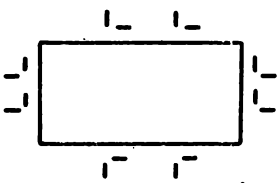
Conforming displacement element :



NT = 682

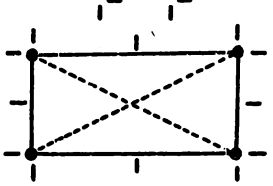
Stress diffusing equilibrium element :

generalized forces



NT = 880

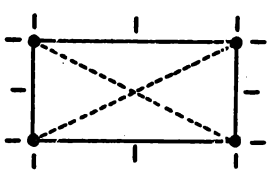
stress functions



NT = 583

Plate bending problem (linear moments or linear curvatures)

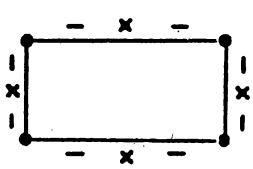
Conforming displacement element :



NT = 583

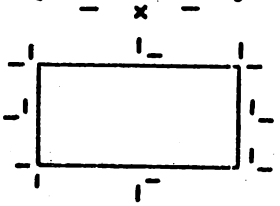
Stress diffusing equilibrium element :

generalized forces



NT = 731

stress functions



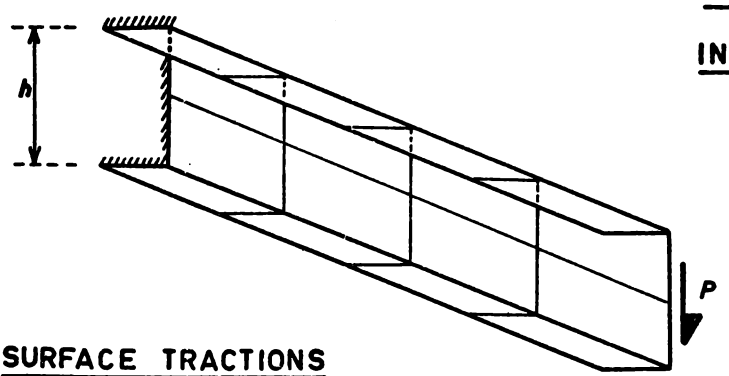
NT = 682

FIGURE 2.17

CANTILEVER U BEAM

BOUNDARY CONDITIONS

IN EQUILIBRIUM ANALYSIS



a) SURFACE TRACTIONS

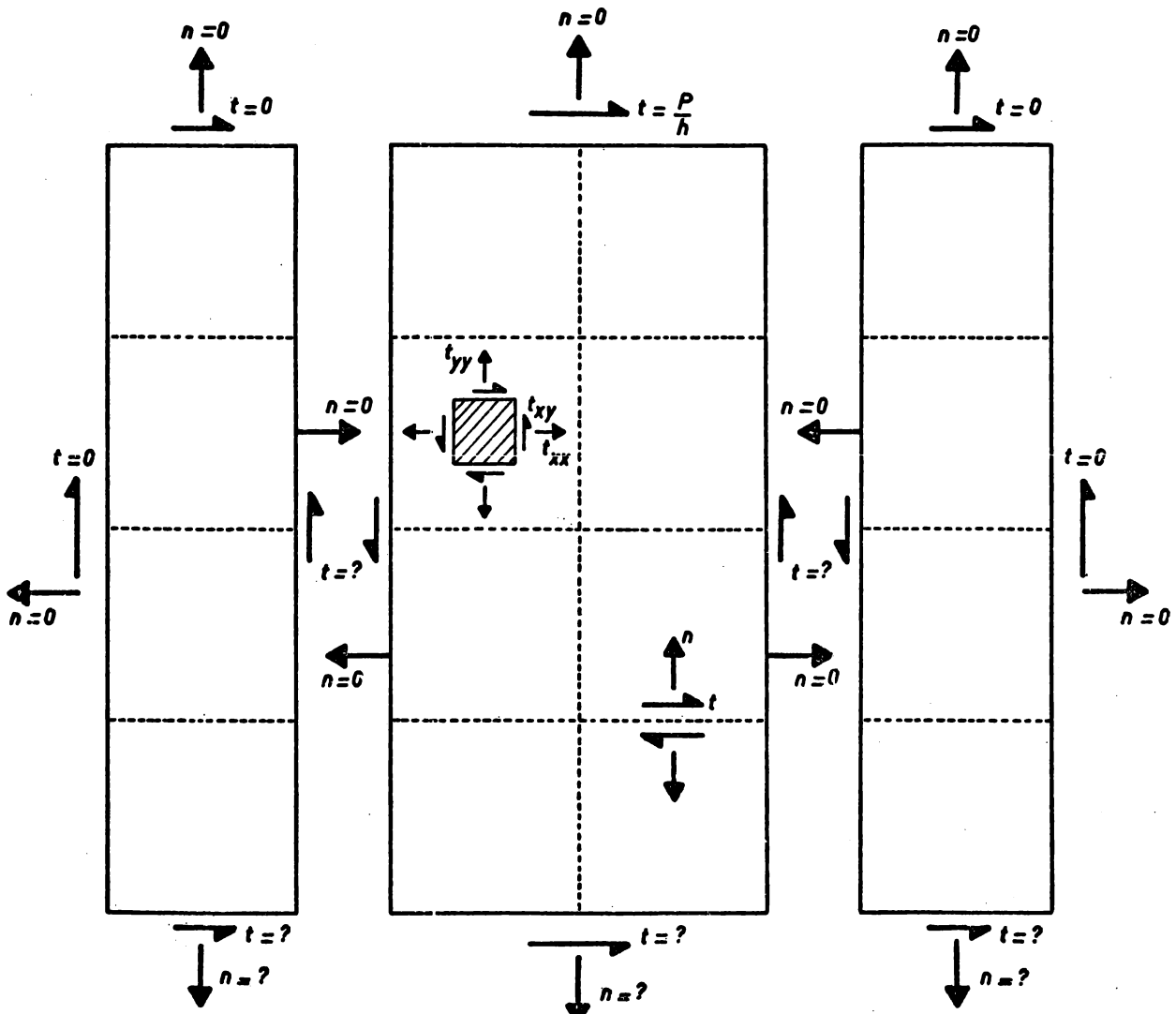
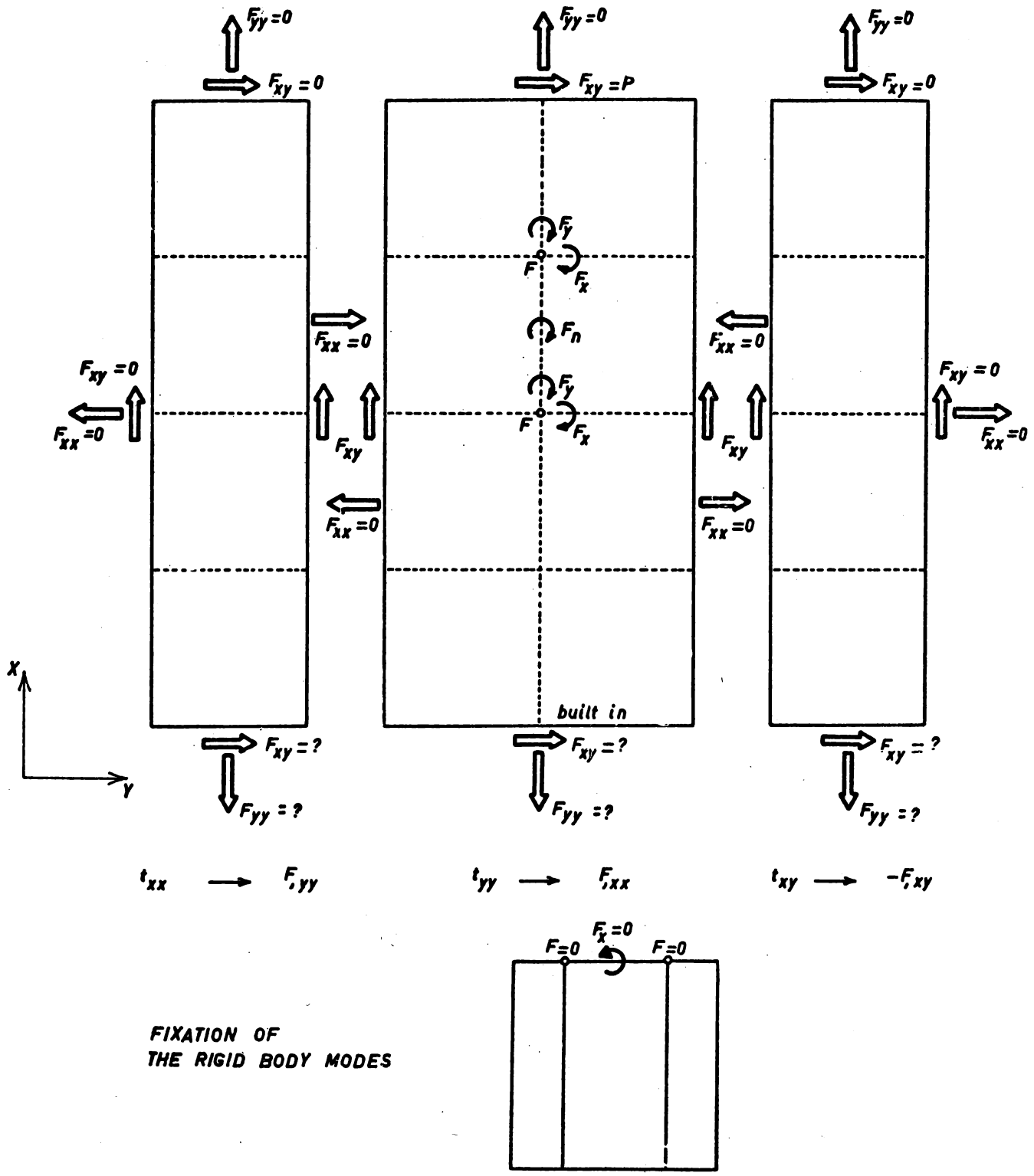


FIGURE 2 18

b) BOUNDARY CONDITIONS IN TERMS OF STRESS FUNCTION F(x.y)



FIXATION OF THE RIGID BODY MODES

FIGURE 2.19

c) CONSTRAINTS IMPOSED TO THE STRESS FUNCTION F(x.y)

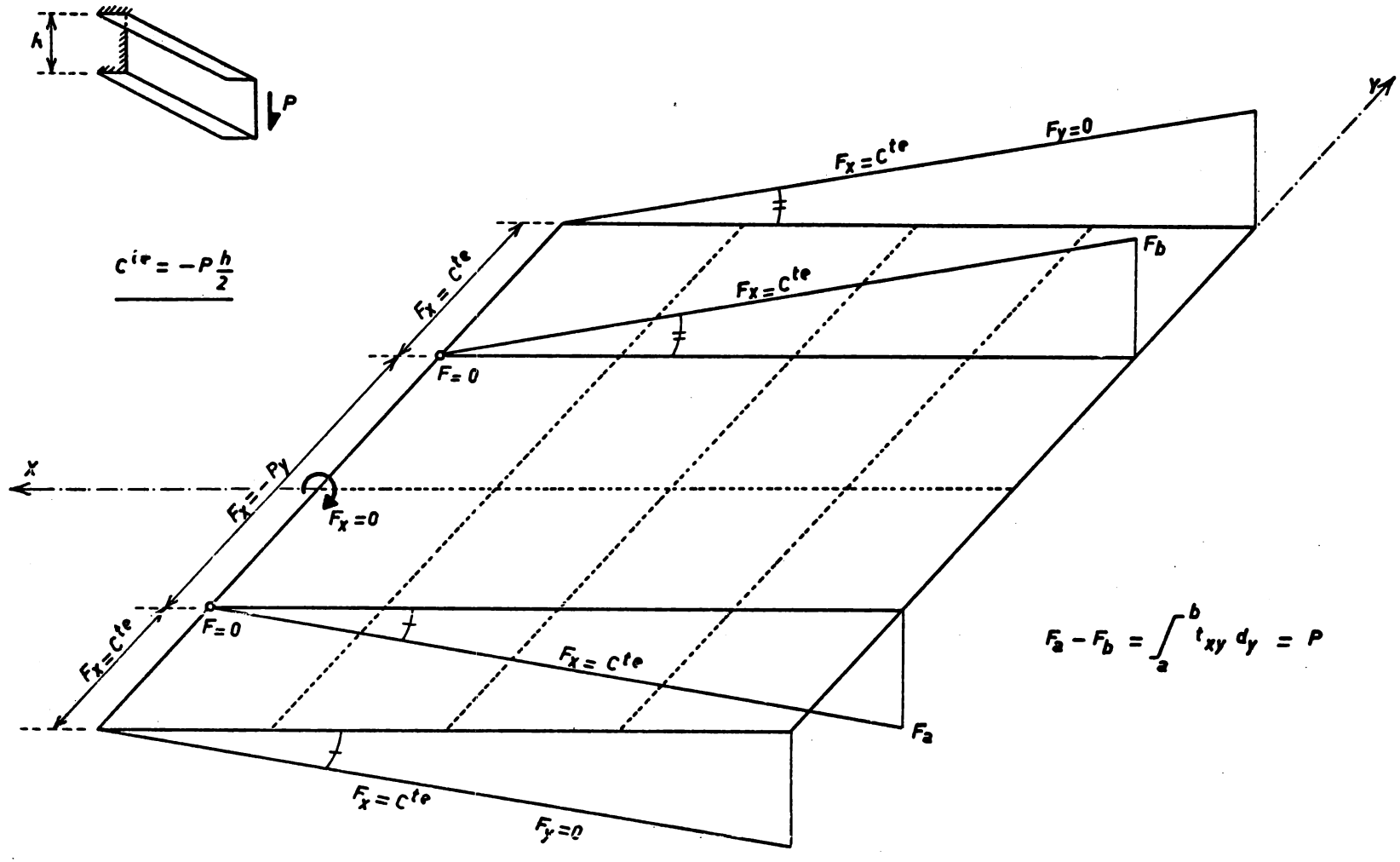
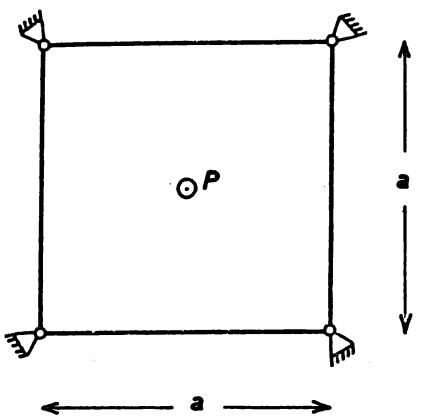


FIGURE 2.20

POINT SUPPORTED SQUARE PLATE

CENTRAL CONCENTRATED LOAD



$$\frac{\partial M_x}{\partial X} + \frac{\partial M_{xy}}{\partial Y} = V_x$$

$$\frac{\partial M_y}{\partial Y} + \frac{\partial M_{xy}}{\partial X} = V_y$$

$$\frac{\partial V_x}{\partial X} + \frac{\partial V_y}{\partial Y} = -p = 0$$

$$K_n = V_n + \frac{\partial M_{sn}}{\partial S}$$

$$Z_i = M_{sn_{i+\varepsilon}} - M_{sn_{i-\varepsilon}}$$

a) BOUNDARY CONDITIONS IN TERMS OF SURFACE TRACTIONS

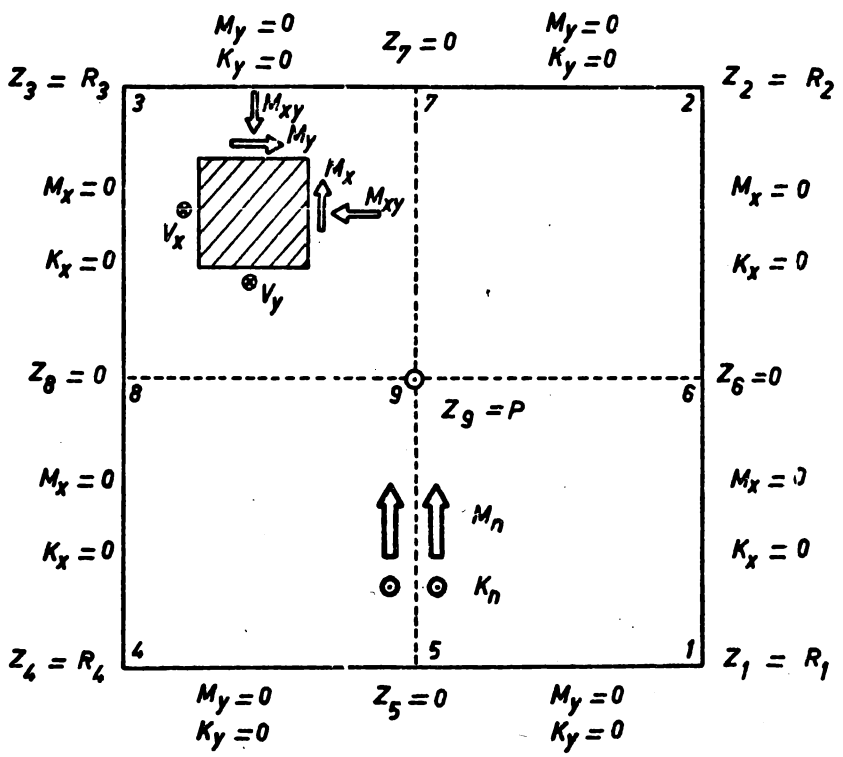


FIGURE 2.21

b) BOUNDARY CONDITIONS IN TERMS OF STRESS FUNCTIONS U AND V

$$M_x = \frac{\partial V}{\partial Y} = V_y \qquad K_x = -U_{yy}$$

$$M_y = \frac{\partial U}{\partial X} = U_x \qquad K_y = -V_{xx}$$

$$M_{xy} = -\frac{1}{2}(U_y + V_x) \qquad Z_{j_{xy}} = 2M_{xy_{j-\epsilon}} = U_{y_{x=ij}} + V_{x_{y=jk}}$$

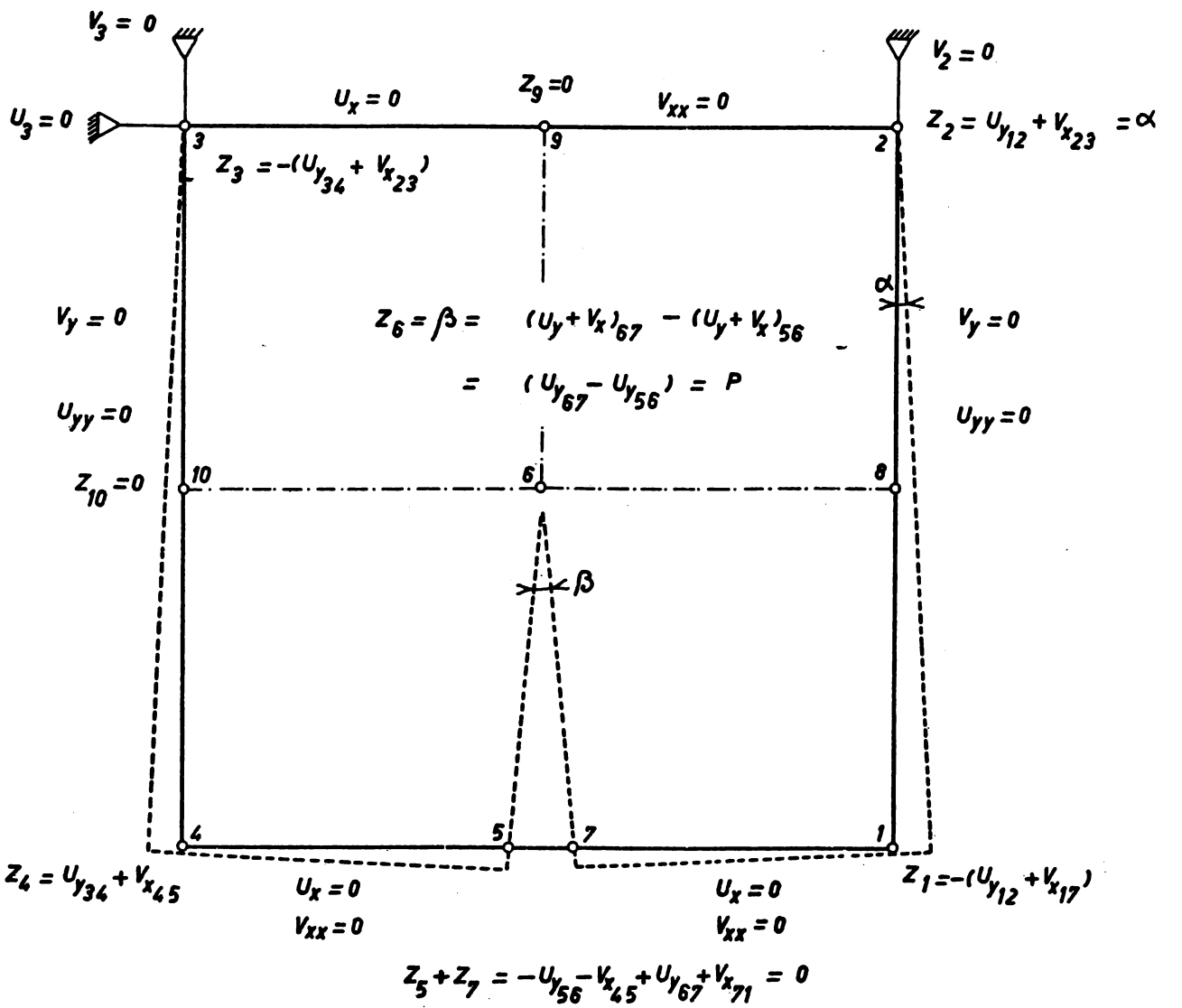
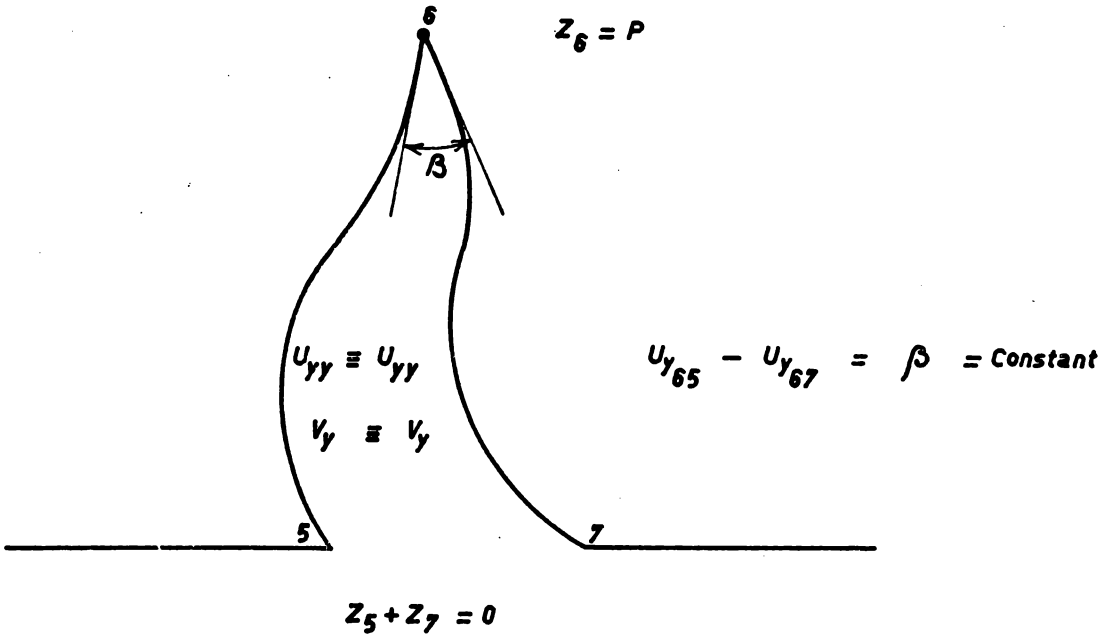
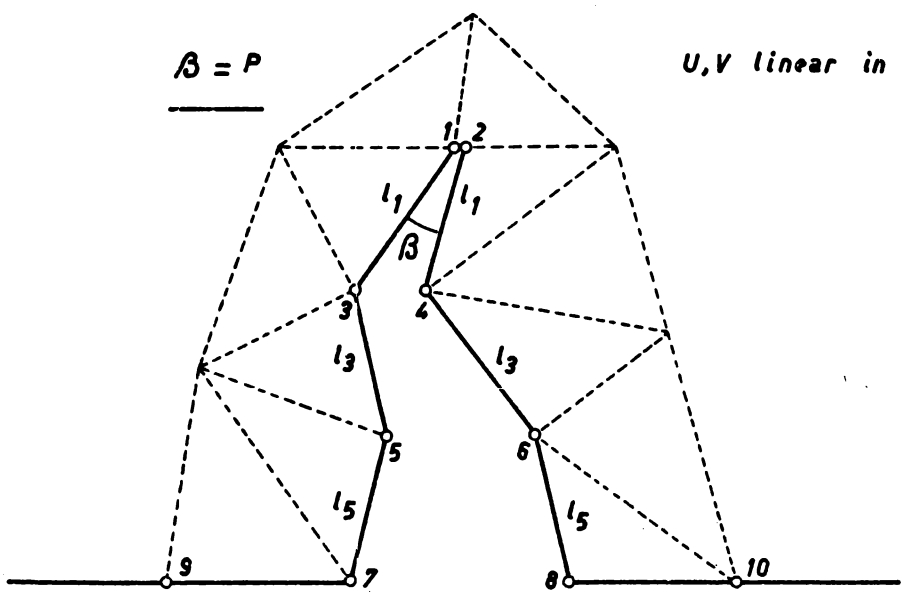


FIGURE 2.22





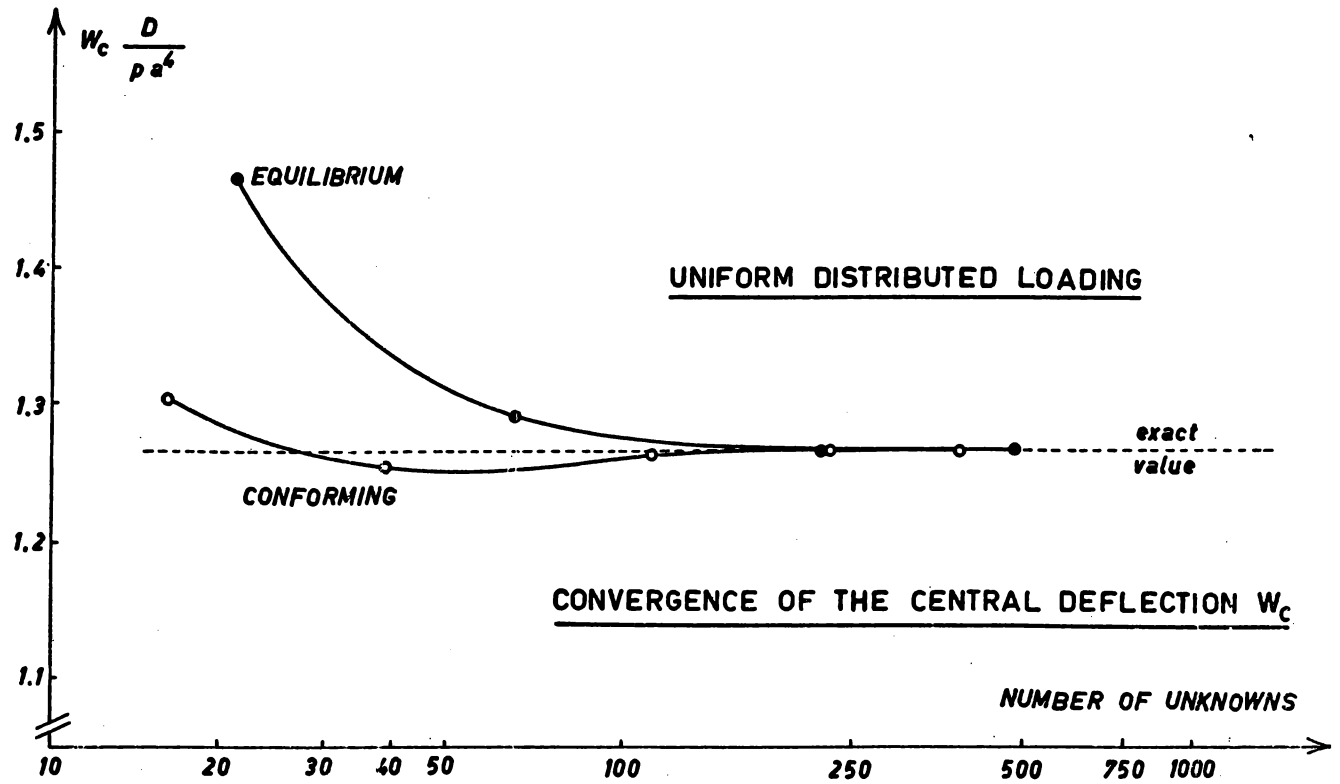
CONSTRAINTS

$$\text{I} \begin{cases} U_i = U_{i+1} + l_i \beta \\ V_i = V_{i+1} \end{cases} \qquad i = 1, 3, 5$$

$$\text{II} \begin{cases} \dfrac{\Delta V_{97}}{l_{97}} - \dfrac{\Delta U_{57}}{l_{57}} = \dfrac{\Delta V_{8.10}}{l_{8.10}} - \dfrac{\Delta U_{68}}{l_{68}} \quad \text{or} \quad Z_7 = Z_8 \\ \text{etc...} \quad \text{for} \quad (5,6) \quad (3,4) \quad (1,2) \end{cases}$$

FIGURE 2.23

CLAMPED SQUARE PLATE



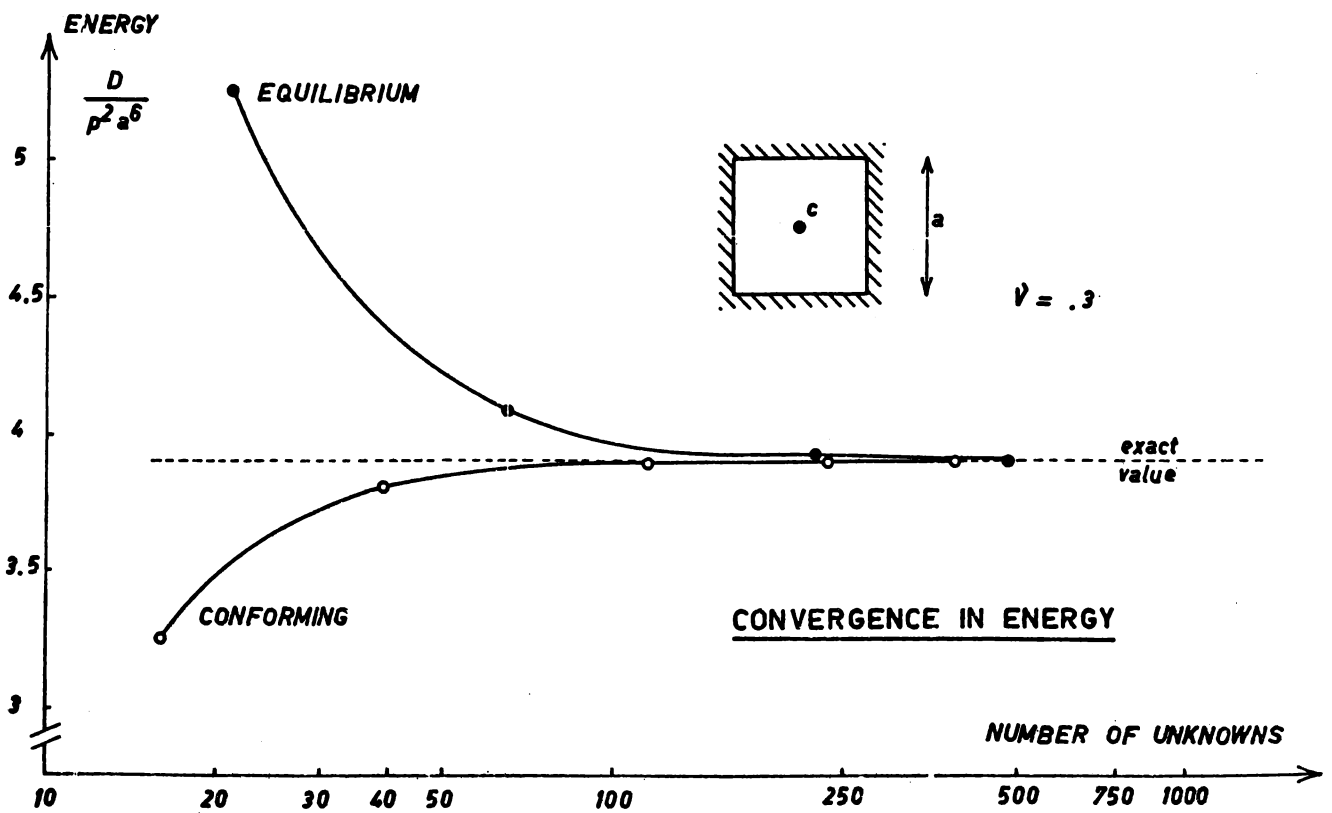
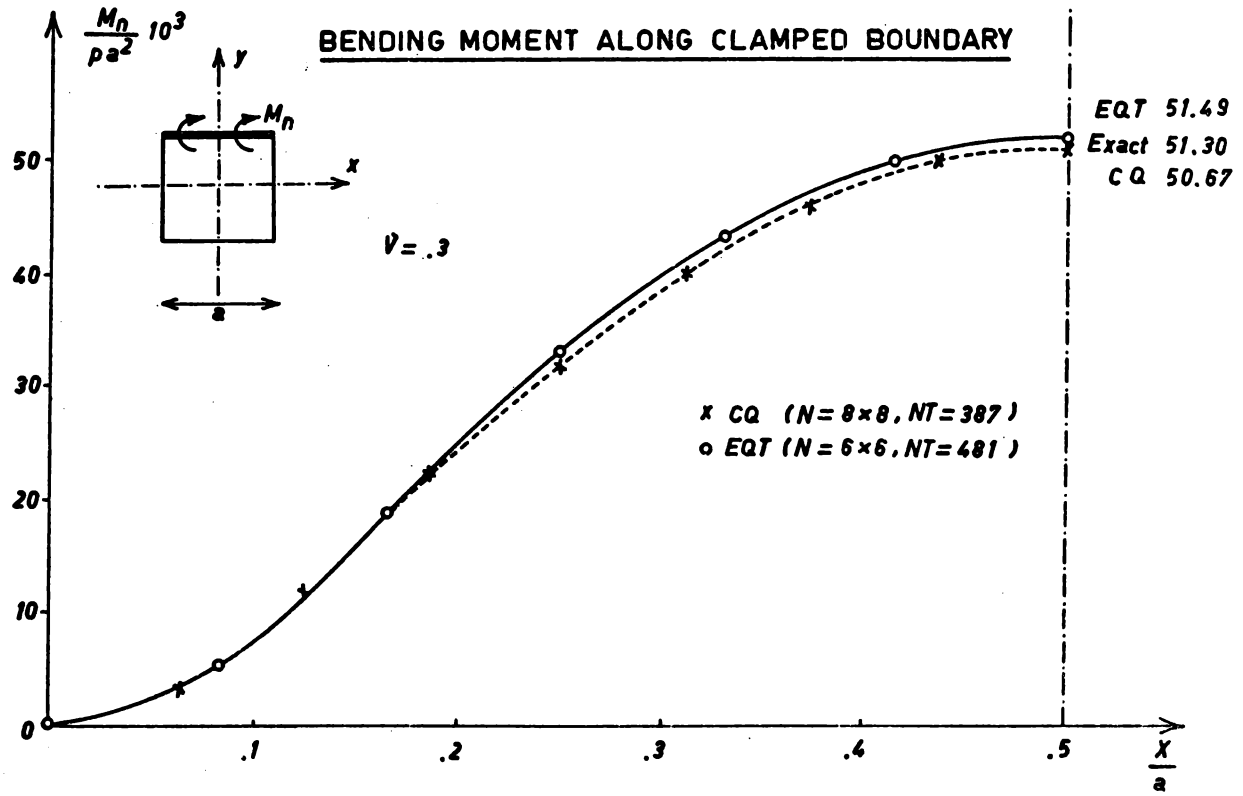


FIGURE 2.24

CLAMPED SQUARE PLATE: UNIFORM LOAD p



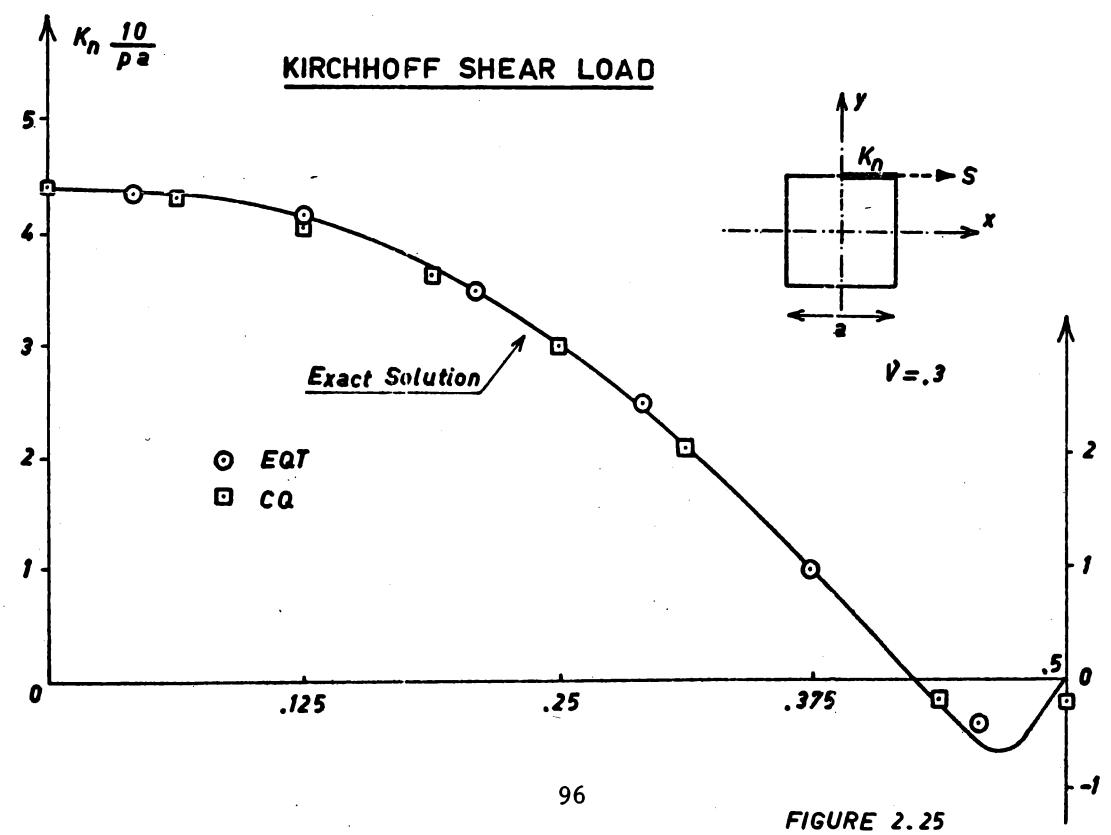


FIGURE 2.25

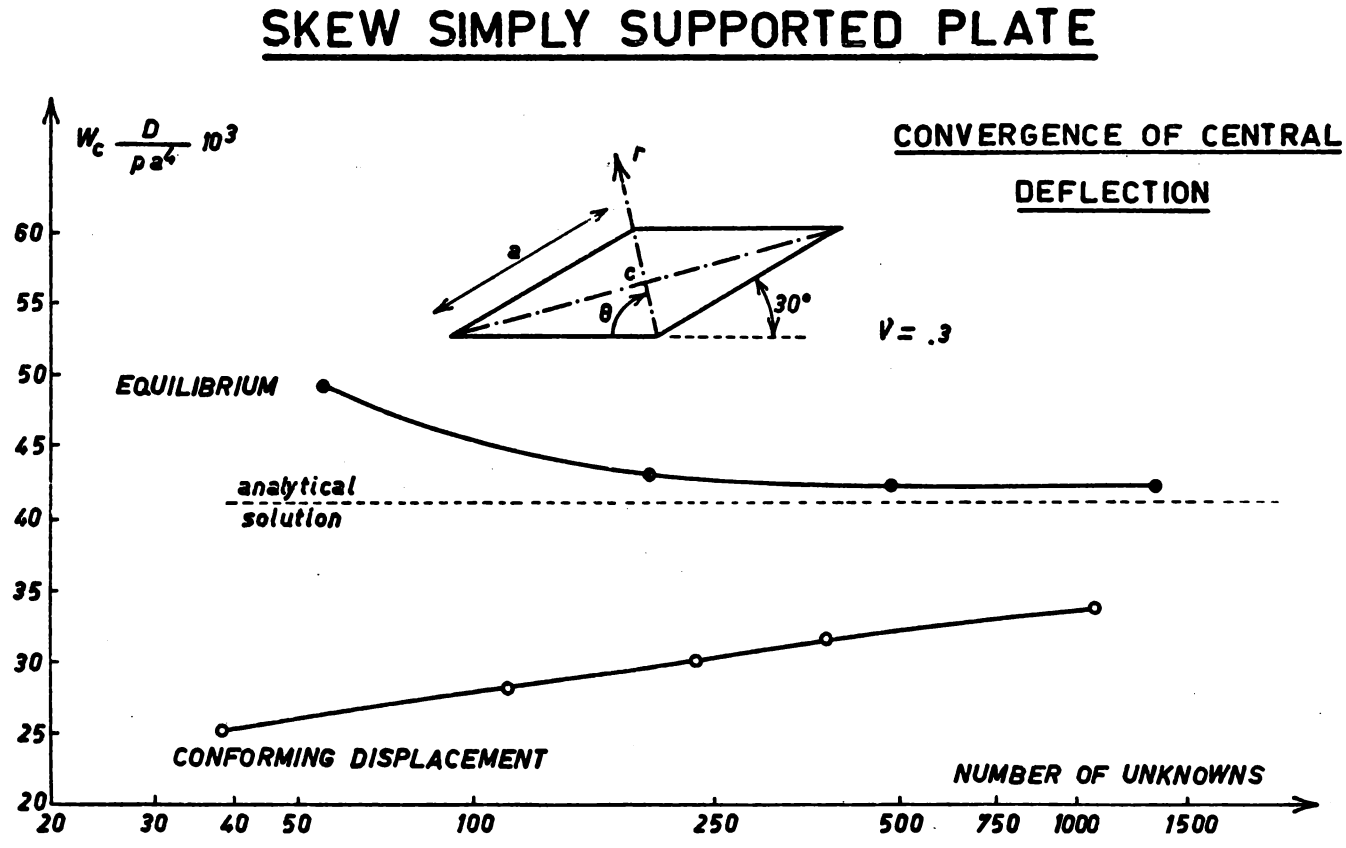

SKEW SIMPLY SUPPORTED PLATE
CONVERGENCE OF CENTRAL DEFLECTION
$W_c \frac{D}{pa^4} 10^3$
r
a
c
θ
30°
ν = .3
EQUILIBRIUM
analytical solution
CONFORMING DISPLACEMENT
NUMBER OF UNKNOWNS
60
55
50
45
40
35
30
25
20
20
30
40
50
100
250
500
750
1000
1500

FIGURE 2.26

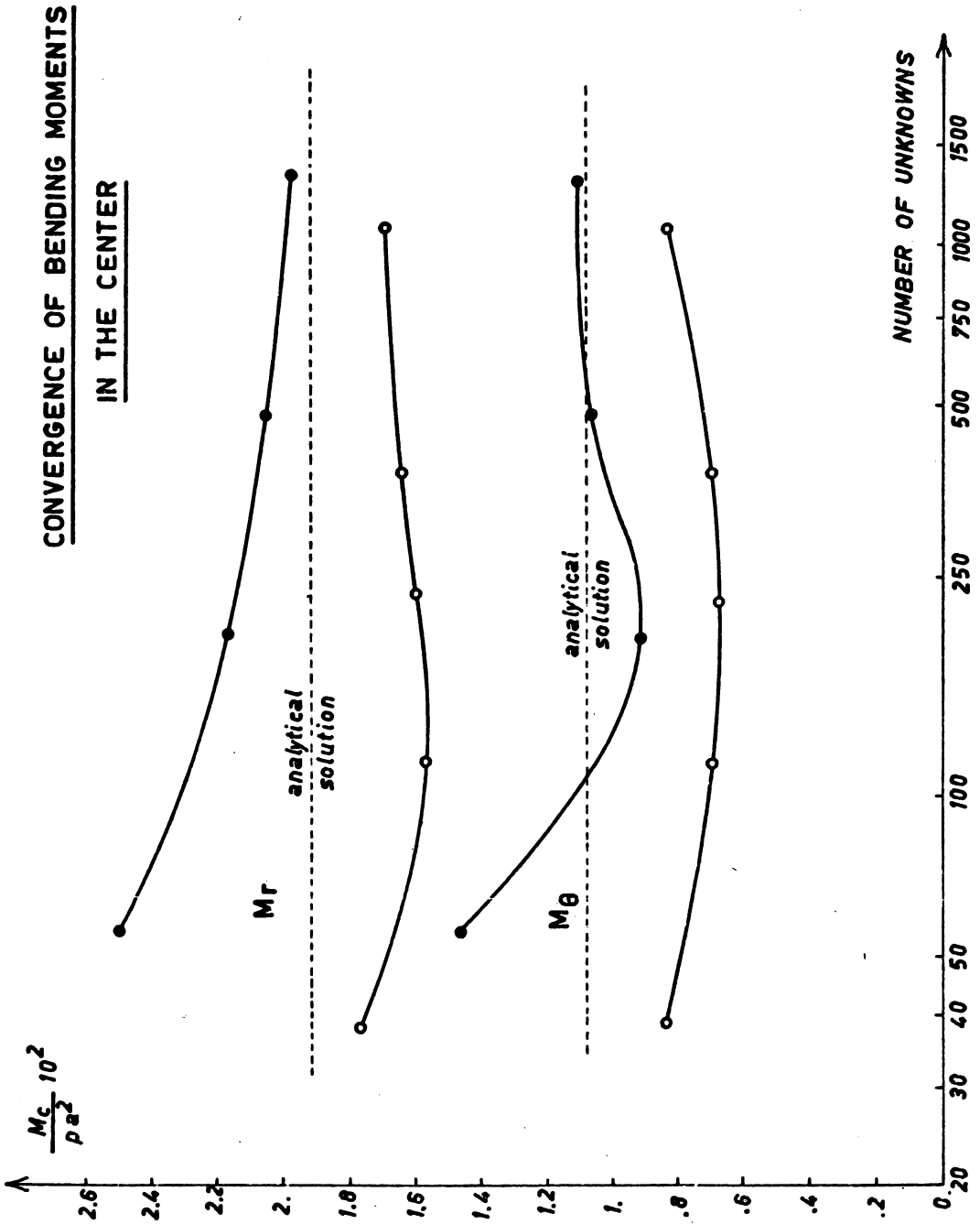
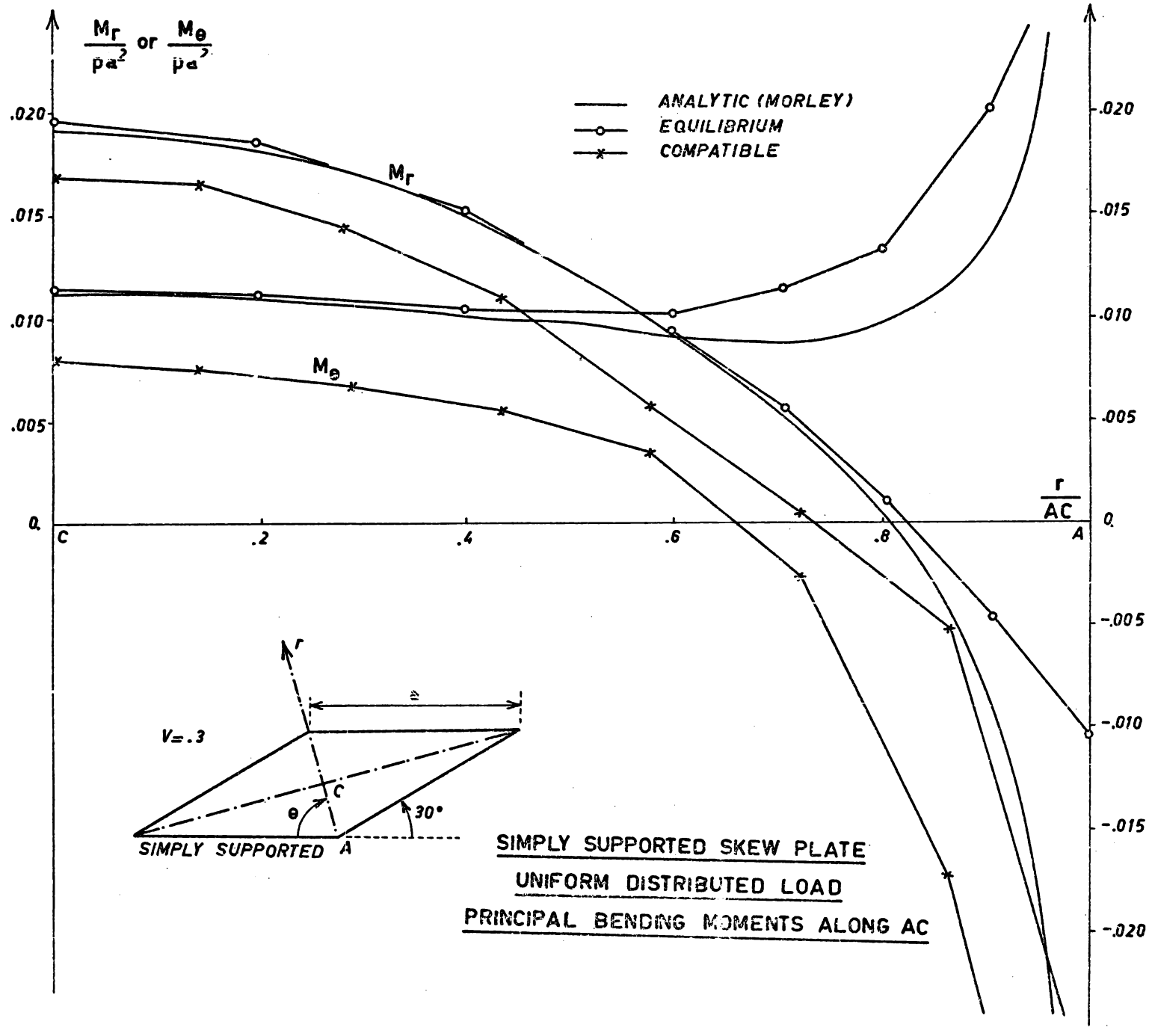

CONVERGENCE OF BENDING MOMENTS
IN THE CENTER
$\frac{M_c}{pa^2}10^2$
2.6
2.4
2.2
2.
1.8
1.6
1.4
1.2
1.
.8
.6
.4
.2
0.
Mr
analytical solution
M_θ
analytical solution
20
30
40
50
100
250
500
750
1000
1500
NUMBER OF UNKNOWNS

FIGURE 2.27



SIMPLY SUPPORTED SKEW PLATE

UNIFORM DISTRIBUTED LOAD

PRINCIPAL BENDING MOMENTS ALONG AC

FIGURE 2.28

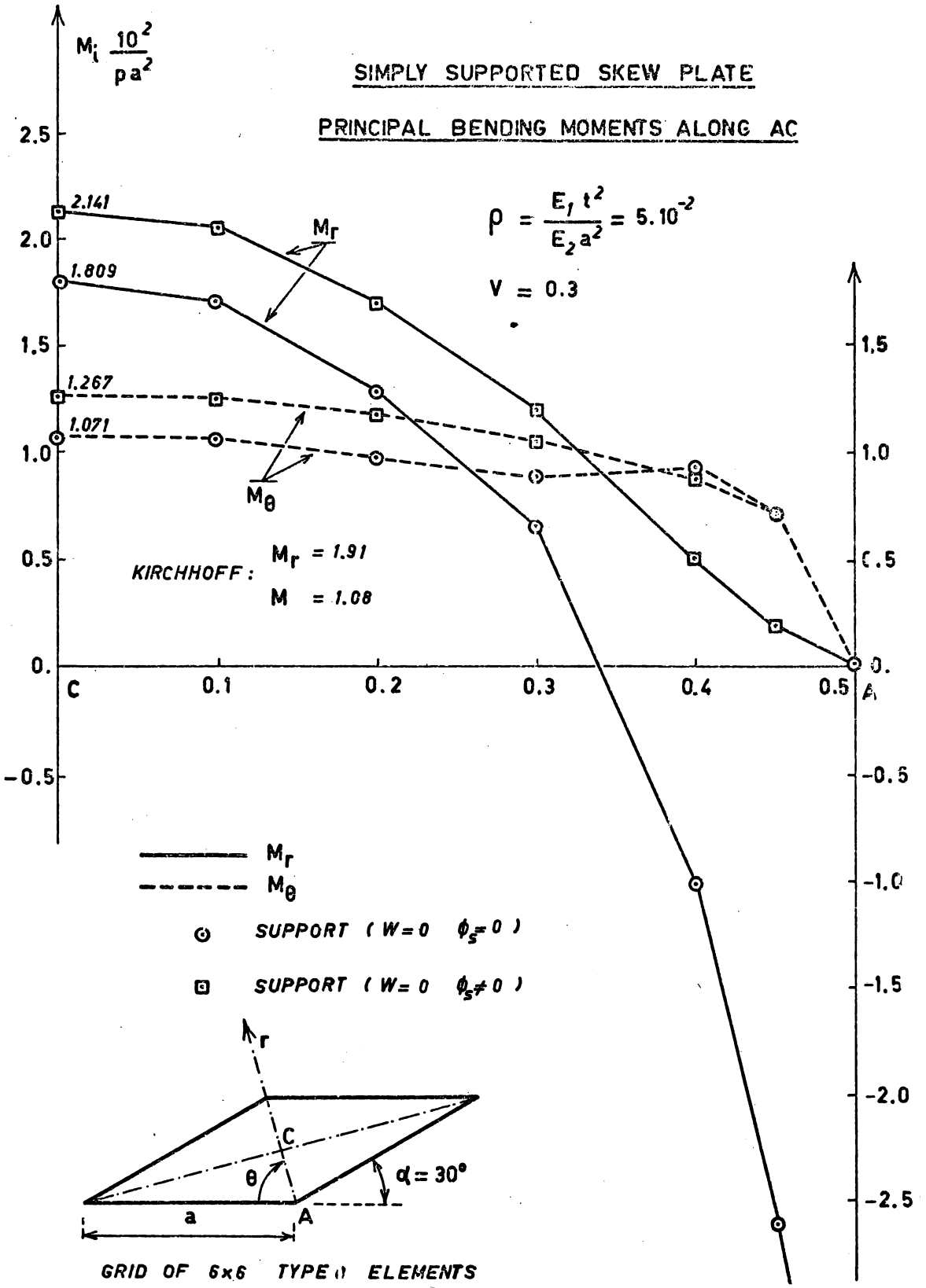

$M_i \frac{10^2}{pa^2}$
SIMPLY SUPPORTED SKEW PLATE
PRINCIPAL BENDING MOMENTS ALONG AC
$\rho = \frac{E_1 t^2}{E_2 a^2} = 5.10^{-2}$
V = 0.3
2.141
1.809
1.267
1.071
M_r
M_θ
KIRCHHOFF : M_r = 1.91
M = 1.08
2.5
2.0
1.5
1.0
0.5
0.
-0.5
C
0.1
0.2
0.3
0.4
0.5
A
M_r
M_θ
SUPPORT (W = 0 ϕ_s = 0)
SUPPORT (W = 0 $\phi_s \neq 0$)
r
C
θ
α = 30°
a
A
GRID OF 6x6 TYPE I ELEMENTS

FIGURE 2.29

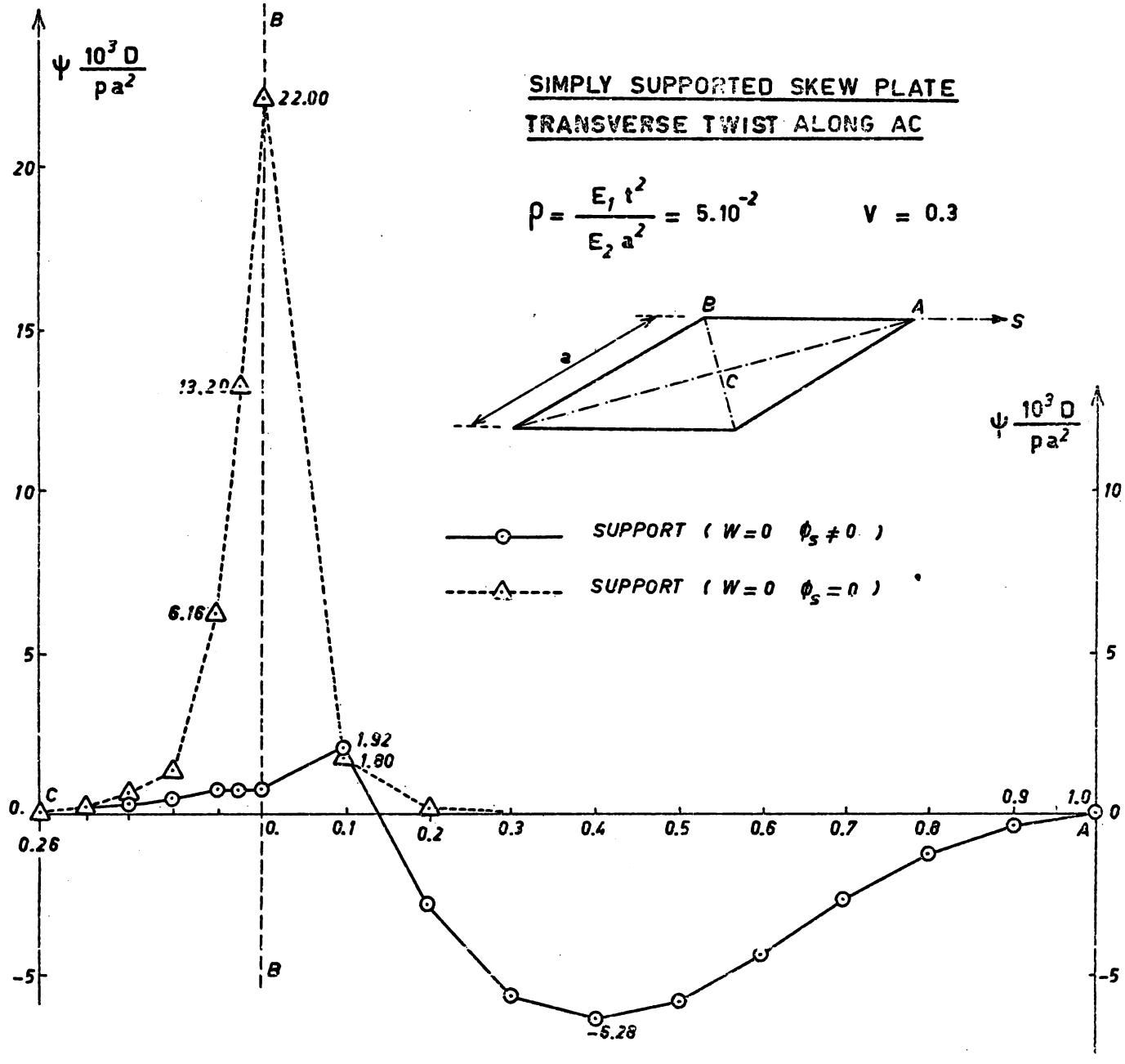

SIMPLY SUPPORTED SKEW PLATE
TRANSVERSE TWIST ALONG AC
$\rho = \frac{E_1 t^2}{E_2 a^2} = 5.10^{-2}$
V = 0.3
$\psi \frac{10^3 D}{pa^2}$
SUPPORT (W = 0 $\phi_s \neq 0$)
SUPPORT (W = 0 $\phi_s = 0$)
22.00
13.20
6.16
1.92
1.80
-6.28
0.26
A
B
C
S
a

FIGURE 2.31

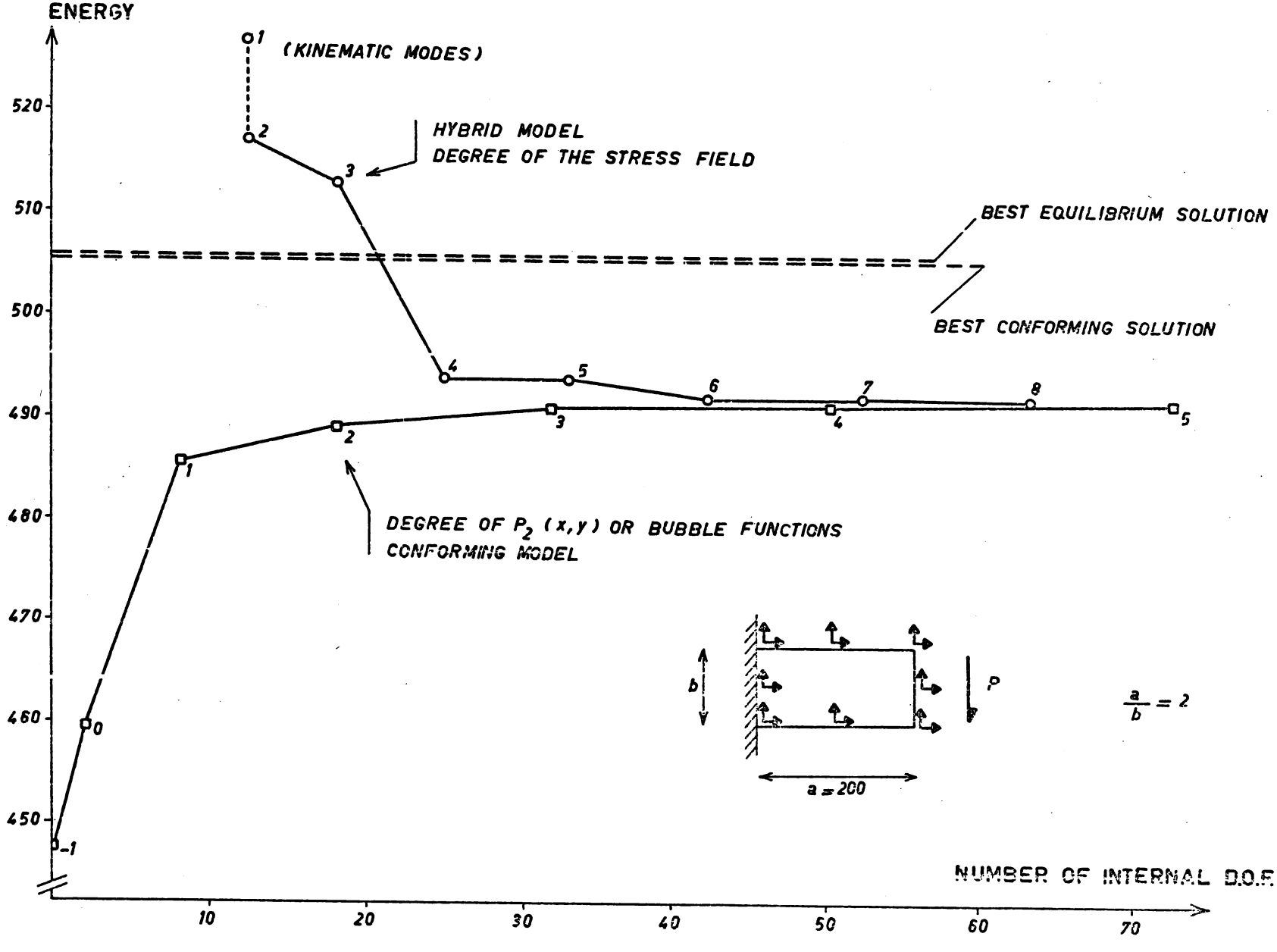

ENERGY
(KINEMATIC MODES)
HYBRID MODEL
DEGREE OF THE STRESS FIELD
BEST EQUILIBRIUM SOLUTION
BEST CONFORMING SOLUTION
DEGREE OF $P_2(x,y)$ OR BUBBLE FUNCTIONS
CONFORMING MODEL
520
510
500
490
480
470
460
450
10
20
30
40
50
60
70
NUMBER OF INTERNAL D.O.F
b
P
a = 200
$\frac{a}{b} = 2$

FIGURE 2.32

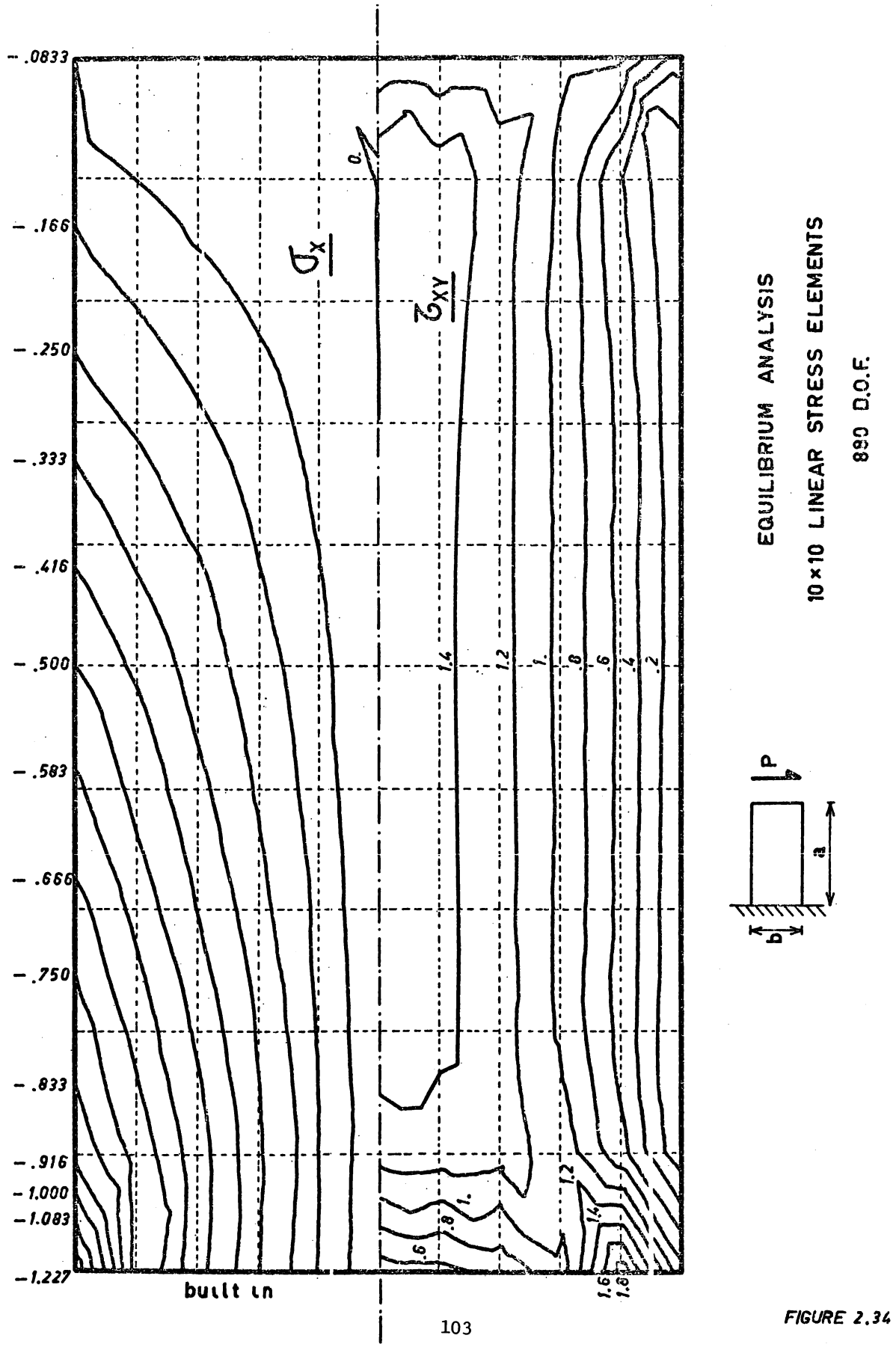

-.0833
-.166
-.250
-.333
-.416
-.500
-.583
-.666
-.750
-.833
-.916
-1.000
-1.083
-1.227
built in
σ_x
τ_{xy}
EQUILIBRIUM ANALYSIS
10×10 LINEAR STRESS ELEMENTS
880 D.O.F.
P
a
b

FIGURE 2.34

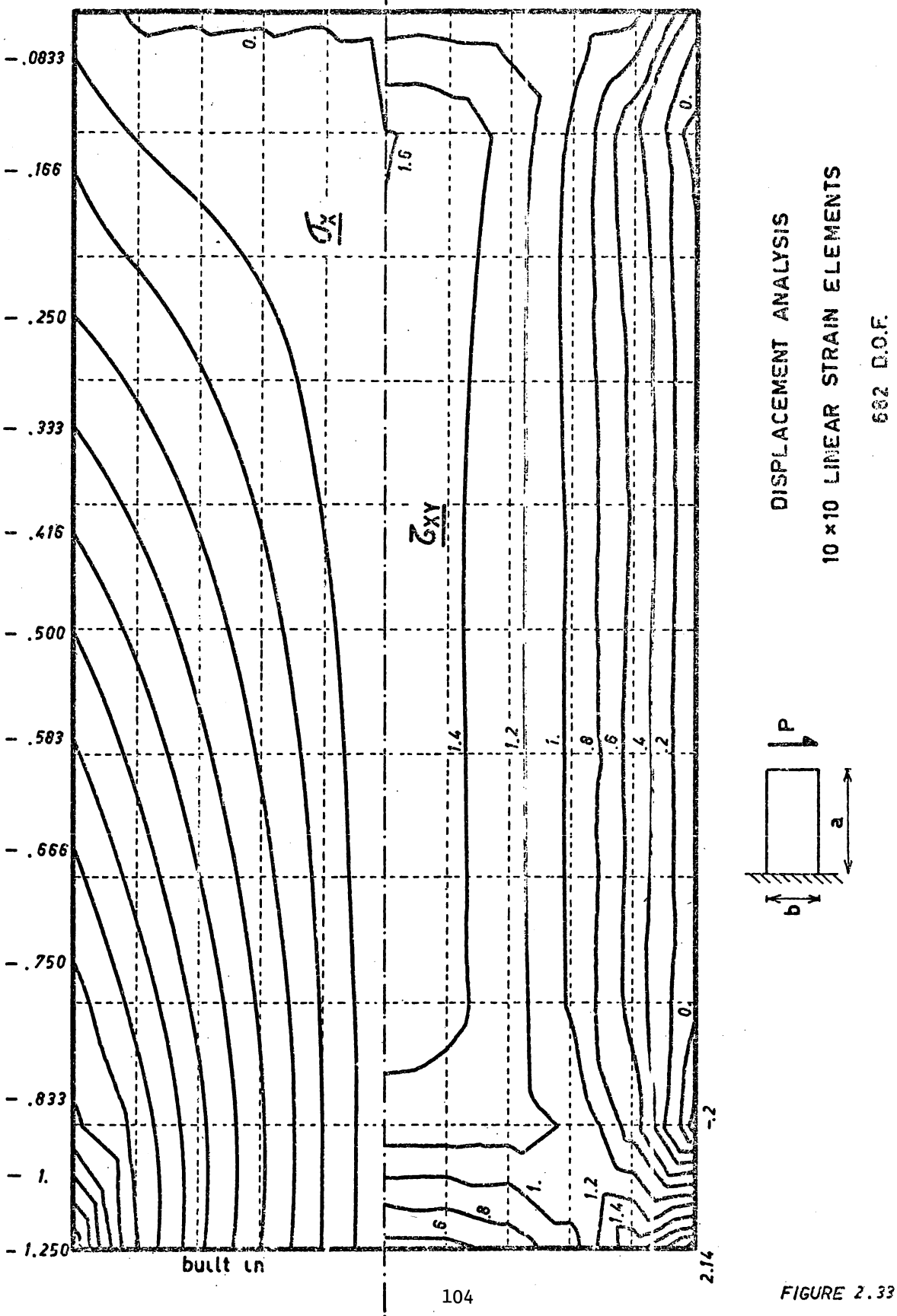

DISPLACEMENT ANALYSIS
10 x10 LINEAR STRAIN ELEMENTS
682 D.O.F.
σ_x
τ_{xy}
built in
P
a
b
-.0833
-.166
-.250
-.333
-.416
-.500
-.583
-.666
-.750
-.833
-1.
-1.250
2.14
-.2

FIGURE 2.33

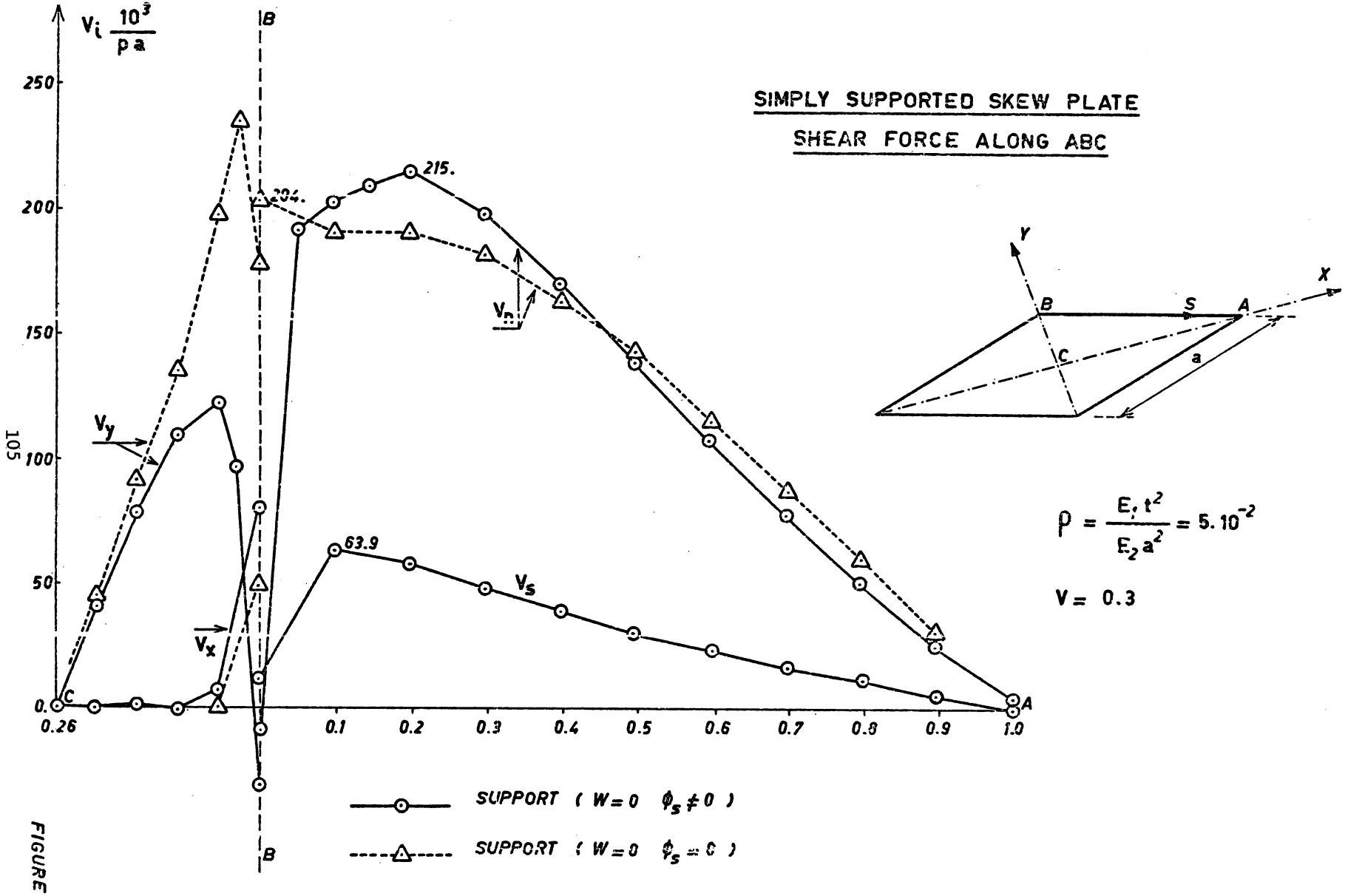

SIMPLY SUPPORTED SKEW PLATE
SHEAR FORCE ALONG ABC
$V_i \frac{10^3}{pa}$
B
250
200
150
100
50
0.
215.
204.
63.9
V_n
V_y
V_x
V_s
C
A
0.26
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Y
X
B
S
A
C
a
$\rho = \frac{E_1 t^2}{E_2 a^2} = 5.10^{-2}$
V = 0.3
SUPPORT (W=0 $\phi_s \neq 0$)
SUPPORT (W=0 $\phi_s = 0$)

FIGURE 2.30

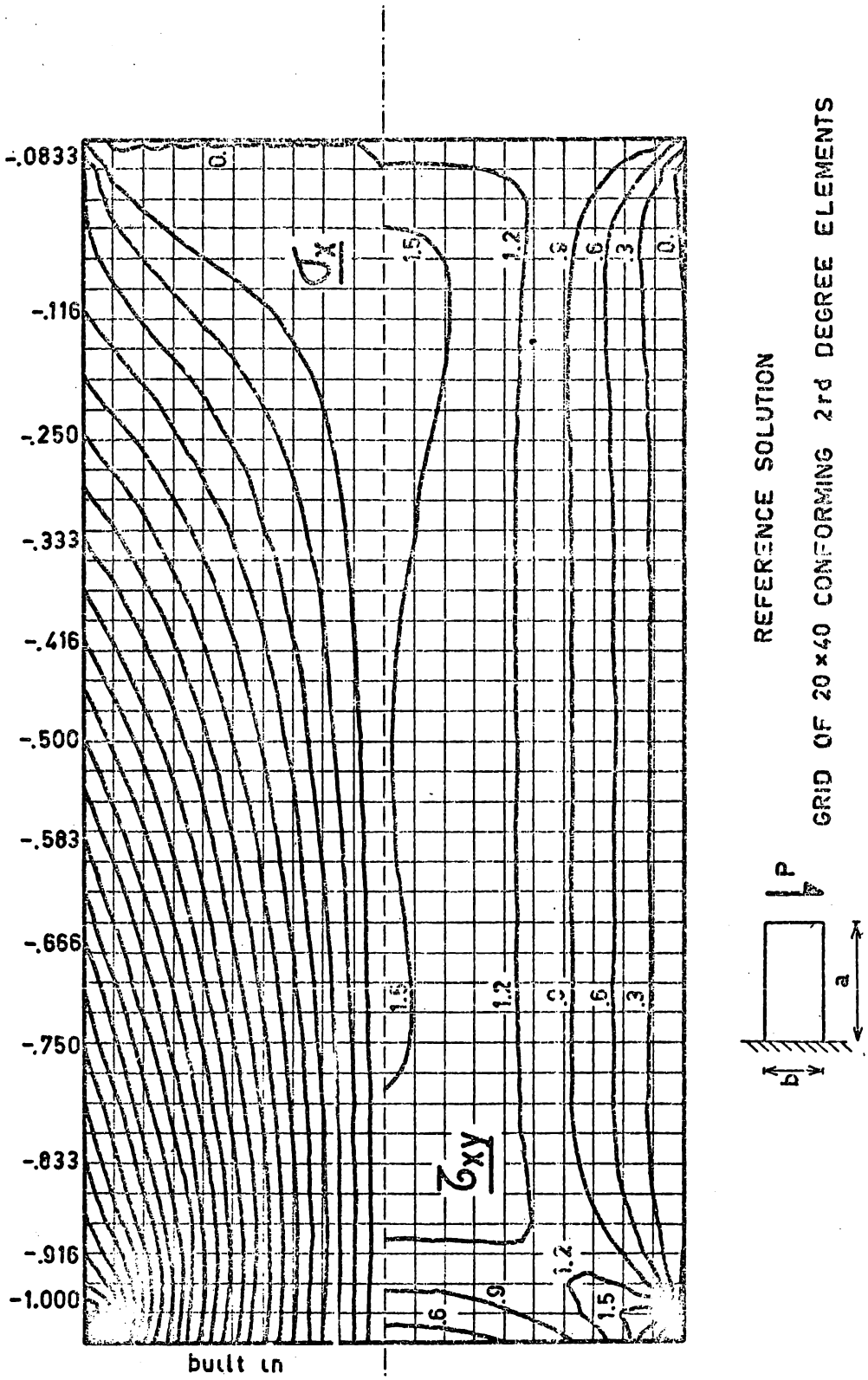
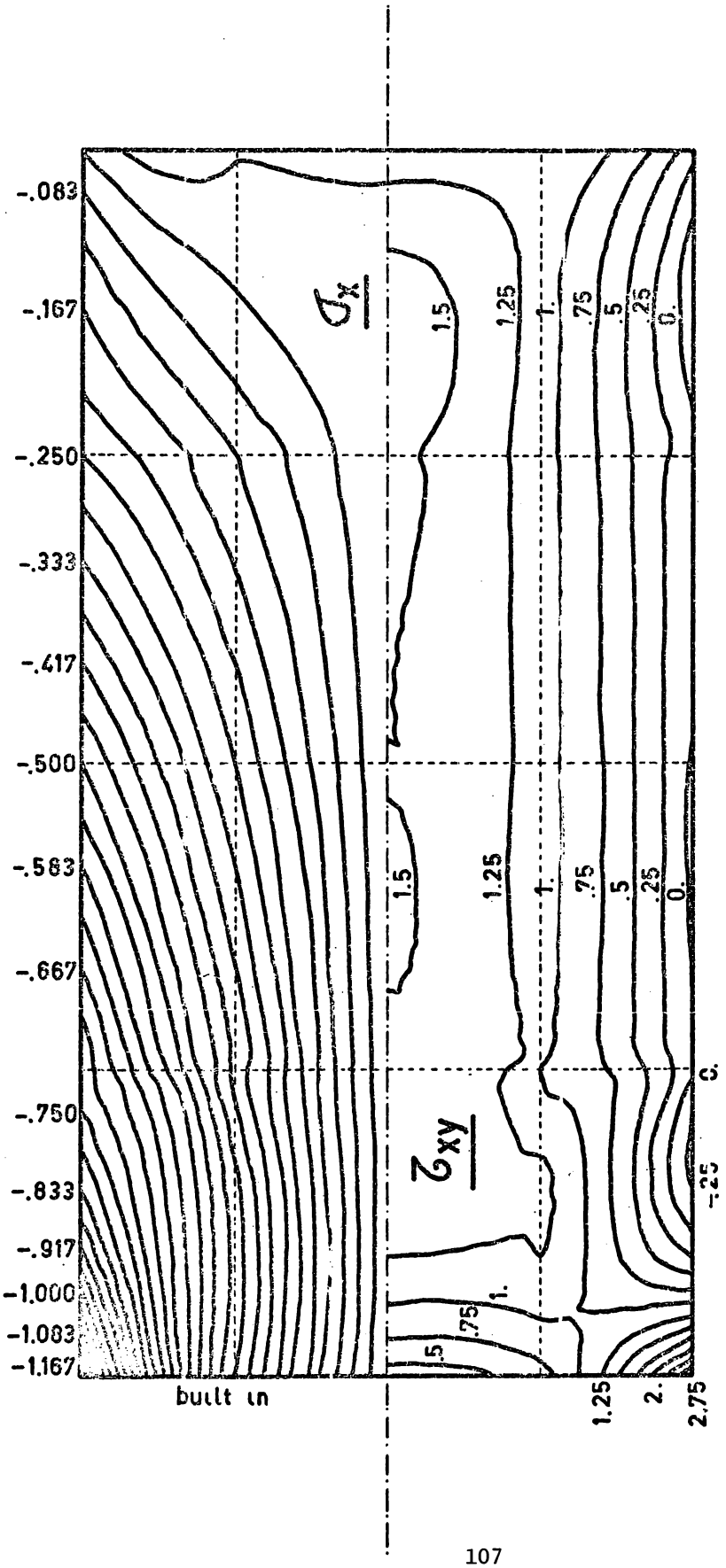

REFERENCE SOLUTION
GRID OF 20×40 CONFORMING 2rd DEGREE ELEMENTS
5042 D.O.F.
P
a
b
built in
-.0833
-.116
-.250
-.333
-.416
-.500
-.583
-.666
-.750
-.833
-.916
-1.000

FIGURE 2.35



GRID OF 4×4 CONFORMING 3rd DEGREE ELEMENTS

BUBBLE MODES WITH n=0

200 INTERFACE D.O.F.

32 INTERNAL D.O.F.

FIGURE 2.36

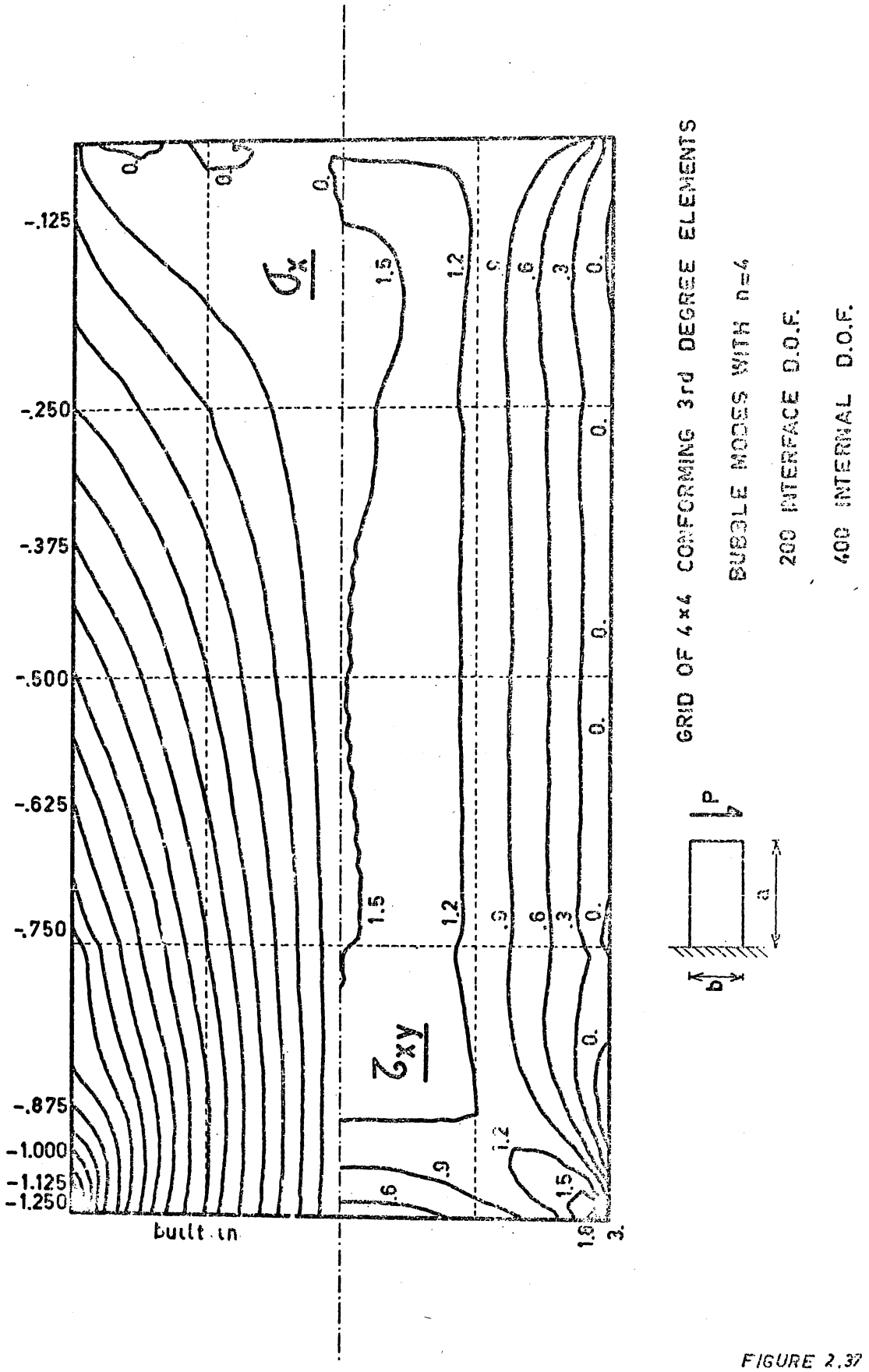

GRID OF 4×4 CONFORMING 3rd DEGREE ELEMENTS
BUBBLE MODES WITH n=4
200 INTERFACE D.O.F.
400 INTERNAL D.O.F.
P
a
b
built-in
-.125
-.250
-.375
-.500
-.625
-.750
-.875
-1.000
-1.125
-1.250

FIGURE 2.37

CHAPTER III

NON-LINEAR THEORIES

3.1. Large displacement formulations

All the difficulties inherent to large displacement formulations are due to the existence of finite material rotations. The strains will be kept of a very small order of magnitude (10^{-3} for example) in order to deal with purely elastic material behavior. Furthermore cartesian reference frames only will be used since this restriction does not affect any of the physical principles involved in the theory. The additional complications stemming from the use of general curvilinear coordinates, as necessary for instance in shell theory (see section 4), can be separately introduced afterwards.

It is for large displacements that the distinction between lagrangian and eulerian coordinates becomes effective. The lagrangian coordinates are here the cartesian coordinates a_i ($i = 1, 2, 3$) of a material point in a so-called "initial" or "reference" configuration of the elastic body. The eulerian coordinates x_i ($i = 1, 2, 3$) are the cartesian coordinates of particles in the "final" or "displaced" configuration.

The cartesian components of the displacement vector are then

$$u_i = x_i - a_i$$

From the lagrangian point of view one writes

$$x_i = x_i\,(a_j) = a_i + u_i\,(a_j) \qquad (3.1).$$

From the eulerian point of view

$$a_i = a_i\,(x_j) = x_i - u_i\,(x_j) \qquad (3.2).$$

The reference configuration will be assumed to coincide with a stress free configuration and the stress-strain characteristics to be defined in this configuration from a knowledge of the material properties and, when the case may be, its oriented fiber type reinforcements. It must be stressed that, in case of anisotropy, the stress-strain characteristics in the final configuration do depend on the local finite rotations and cannot be deduced without solving the field equations and boundary conditions of the problem. This rules out in practice the formulation in eulerian coordinates as initiated by eqs. (3.2). It is only for isotropic relations, which are then independant on the rotations, that such a formulation becomes possible.

Because of this limited usefulness the treatment will be here restricted to the lagrangian formulation; for considerations on the eulerian point of view reference can be made to (3.1).

3.1.1. Lagrangian measures of strain

Differentiating (3.1) we obtain the geometrical relationship between a neighborhood (da_i) of a point P in the reference configuration and its neighborhood ($d\,x_i$) in the displaced configuration :

$$dx_i = D_j\, x_i\, da_j = da_i + D_j\, u_i\, da_j \qquad (3.3)$$

where summation is understood for repeated subscripts and

$$D_j = \partial/\partial a_j$$

In matrix notation, the column matrices $da = \{ da_i \}$ and $dx = \{ dx_i \}$ are thus related by

$$dx = J\, da = (E + A)\, da \qquad (3.4)$$

where E denotes the identity matrix and J and A the jacobian matrices

$$J = \{ D_j\, x_i \} \qquad A = \{ D_j\, u_i \}$$

i row subscript

j column subscript

The elementary rectangular volume element $da_1\, da_2\, da_3$ in the reference configuration becomes in the displaced configuration an oblique parallelopiped with volume equal to $| J |\, da_1\, da_2\, da_3$, where $| J |$ denotes the determinant of the jacobian matrix. The square of the distance between two neighboring points is da'da in the reference and dx'dx in the final configuration. According to (3.4) this alteration can be expressed as a quadratic form in the da_i

$$dx'dx - da'da = da'\,(J'J - E)\, da = 2\, da'\, G_a\, da \qquad (3.5)$$

introducing the cartesian representation of the Green strain tensor

$$G_a = \frac{1}{2}\,(J'J - E) = G'_a \qquad (3.6)$$

which is the most commonly used lagrangian measure of local deformation. Using the jacobian matrix A = J - E in place of J, there follows

$$G_a = \frac{1}{2} (A + A' + A'A) \tag{3.7}$$

G_a is a measure of local strain since, when the neighborhood transformation (3.4) is of a rigid body type, the distances between neighboring points is not altered and the components of G_a are necessarily zero. Conversely, if the components of G_a are all zero, the distances are unaltered and the neighborhood transformation is either of rigid body type or consists of a rigid body displacement followed (or preceded) by a reflection with respect to a plane. Hence it must still be shown that reflections are ruled out by physical arguments. Polar decomposition of the transformation (3.4) is one of the methods to explain this; it has furthermore the advantage of relating the Green strain tensor to another lagrangian measure of strain, which is closer to the usual engineering definition. In the polar decomposition the transformation is split into two successive operations, first a pure strain transformation

$$dy = (E + H)\, da \qquad H = H' \tag{3.8}$$

followed by a finite rotation

$$dx = U\, dy \qquad UU' = U'U = E \qquad |U| = 1 \tag{3.9}$$

The restriction to orthogonal transformation matrices U of determinant equal to unity implies that one has a "true rotation" and not a rotation accompanied by a reflection, as would be the case for $|U| = -1$.
From (3.8) and (3.9) follows, by comparison with (3.4)

$$J = U\,(E + H) \text{ and } |J| = |E + H| \tag{3.10}$$

and it will be shown that this decomposition is always possible and unique. We first obtain from (3.10) and (3.6) that

$$G_a = \frac{1}{2} (E + H)\, U'U (E + H) - \frac{1}{2} E = \frac{1}{2} (E + H)^2 - \frac{1}{2} E$$

$$= H + \frac{1}{2} H^2 \tag{3.11}$$

Hence, wathever be the rotation U, the strain tensor H determines uniquely the Green strain tensor. Conversely G_a determines H uniquely as can be seen by a principal axis transformation where H reduces to diagonal form with diagonal elements h_i, whence, by virtue of (3.11), G_a is reduced to diagonal form with diagonal elements

$$\gamma_i = h_i + \frac{1}{2} h_i^2 \qquad (i = 1, 2, 3) \qquad (3.12)$$

Given the γ_i, the solutions for h_i are real and unique

$$h_i = -1 + \sqrt{1+2\gamma_i} \qquad (3.13)$$

provided one can justify the restrictions

$$1 + 2\gamma_i > 0 \qquad 1 + h_i > 0 \qquad (i = 1, 2, 3) \qquad (3.14)$$

Indeed, from (3.6) and (3.10) we have

$$| J |^2 = | 2 G_a + E | = (1+2\gamma_1)(1+2\gamma_2)(1+2\gamma_3) \qquad (3.15)$$

$$| J | = | E + H | = (1+h_1)(1+h_2)(1+h_3) \qquad (3.16)$$

Physically the transformation from initial to final configuration takes place continuously. At any intermediate stage none of the factors appearing in the right hand sides of (3.15) and (3.16) may vanish without making the volume element vanish, a physical impossibility. Hence since they satisfy conditions (3.14) at the start ($\gamma_i = 0$, $h_i = 0$) they must keep their sign and still verify those conditions at the end. In particular we have

$$| 2 G_a + E | > 0 \qquad | J | = | E + H | > 0 \qquad (3.17)$$

The argument is valid for finite strains; in the case of very small strains equations (3.13) are immediatly obvious. Given J satisfying (3.17) and once H uniquely determined as shown above by the relation

$$\frac{1}{2} (J'J - E) = H + \frac{1}{2} H^2$$

the orthogonal matrix in the polar decomposition (3.10) is uniquely deter-

mined by

$$U = J\,(E + H)^{-1}$$

and is that of a pure rotation.

In a principal axis system the principal strains of H have the usual engineering definitions

$$\gamma_i = \frac{dx_i^2 - da_i^2}{2da_i^2} = \frac{dx_i - da_i}{da_i} \cdot \frac{dx_i + da_i}{2da_i} \qquad (3.19)$$

The advantage of the Green strain measure is that it can be expressed immediatly in terms of the jacobian matrices J or A as shown by (3.6) and (3.7). The effective computation of the engineering strain measure H requires a cumbersome principal axis transformation . Both measures become however equivalent in the case of very small strains as shown by the relation (3.11) where the components of H^2 become negligible by comparison with those of H.

If we add the assumption of very small rotations to that of small strains, the rotation matrix can be written as

$$U = E + K$$

where the components of K are of the same small order of magnitude as those of H. The orthogonality condition U'U = E becomes

$$(E + K')(E + K) = E \quad \text{or} \quad K' + K = 0$$

if we neglect the second order quantities represented by K'K. Hence to this approximation K is skew symmetrical. Then from (3.10)

$$J = E + A = (E + K)(E + H)$$

which, after neglecting again the second order quantities represented by KH, yields the usual approximations of linear elasticity theory

$$A = K + H \qquad H = \frac{1}{2}(A + A') \qquad K = \frac{1}{2}(A - A') \quad (3.20).$$

Inversion of the order of polar decomposition is simple

$$J = U\,(E + H)\,U'U = (E + UHU')U = (E + \hat{H})U \qquad (3.21)$$

The rotation, which comes now first, is unchanged; the straining, which comes second, involves a matrix $\hat{H}$ related to H by

$$\hat{H} = U\,H\,U' = \hat{H}' \qquad (3.21)$$

This suggests that H and $\hat{H}$ are in fact representations of the same tensor in two different reference frames. It is clear from the first polar decomposition that, if an observer uses the initial configuration as a frame to describe the distribution of strains incurred by the configuration change, the matrix H is the representation of the local strain tensor for the actual cartesian axes. If he uses the final configuration as frame, H is the representation of the local strain tensor for cartesian axes as turned by the material rotation; $\hat{H}$ would be the representation of the same tensor for the original orientation of the cartesian axes. Using the Green measure for strains, the same statements apply respectively to G_a and to

$$\hat{G}_a = U\,G_a\,U' = \hat{H} + \frac{1}{2}\hat{H}^2 = \frac{1}{2}(JJ'-E) = \frac{1}{2}(A+A'+AA') \qquad (3.22)$$

(3.11) (3.21)

From (3.7) and (3.22) we deduce the following expressions of the Green tensor components

$$G_a = \{\,\gamma_{mn}\,\} \quad \text{and} \quad \hat{G}_a = \{\,\hat{\gamma}_{mn}\,\}$$

$$2\gamma_{mn} = D_m x_i D_n x_i - \delta_{mn} = D_m u_n + D_n u_m + D_m u_i D_n u_i \qquad (3.23)$$

$$2\hat{\gamma}_{mn} = D_i x_m D_i x_n - \delta_{mn} = D_m u_n + D_n u_m + D_i u_m D_i u_n \qquad (3.24)$$

3.1.2. The a_i as convective coordinates

Let $\vec{e}_j$ denote the unit vectors of the cartesian reference frame. Then, for the position vector of a point P in the reference configuration

$$\vec{r} = a_j\,\vec{e}_j \qquad (3.25)$$

and for the same point in the displacement configuration

$$\vec{R} = x_j \vec{e}_j \tag{3.26}$$

In this last configuration the surfaces a_i = constant define a set of convected curvilinear coordinates with local base vectors

$$\vec{G}_i = D_i \vec{R} = (D_i x_j) \vec{e}_j \tag{3.27}$$

The fundamental metric tensor of this set of curvilinear coordinates is

$$G_{mn} = \vec{G}_m \cdot \vec{G}_n = (D_m x_i)\vec{e}_i \cdot (D_n x_j)\vec{e}_j = D_m x_i D_n x_i = \delta_{mn} + 2\gamma_{mn} \tag{3.28}$$

This relationship with the components of the Green tensor further justifies their use as a measure of deformation. Let us now investigate the Christoffel symbols of the second kind defined by

$$D_m \vec{G}_i = \Gamma_{m\,i}^{\;\;n} \vec{G}_n$$

Or, preferably, taking the scalar product of both sides with the local base vectors, the symbols of the first kind

$$\vec{G}_p \cdot D_m \vec{G}_i = \Gamma_{m\,i}^{\;\;n} \vec{G}_n \cdot \vec{G}_p = \Gamma_{m\,i}^{\;\;n} G_{np} = \Gamma_{mpi} \tag{3.29}$$

Using (3.27) to evaluate the left-hand side

$$\Gamma_{mpi} = (D_p x_j)\vec{e}_j \cdot (D_m D_i x_k)\vec{e}_k = D_p x_j D_m D_i x_j \tag{3.30}$$

where the symmetry with respect to the subscripts m and i is noticeable. As one can easily verify, (3.30) can also be written as

$$\Gamma_{mpi} = D_m \gamma_{pi} + D_i \gamma_{mp} - D_p \gamma_{im} \tag{3.31}$$

3.1.3. Gradients of the rotation and compatibility equations

Introduce the elements α_{im} of the finite rotation operator U

$$U = \{ \alpha_{im} \} \quad \begin{matrix} i \text{ row index} \\ m \text{ column index} \end{matrix} \qquad \alpha_{im}\alpha_{jm} = \delta_{ij} \quad \alpha_{im}\alpha_{in} = \delta_{mn} \tag{3.32}$$

and those of the engineering strain tensor $H = \{ h_{jm} \}$

$$h_{jm} = h_{mj} \tag{3.33}$$

The polar decomposition (3.10) can then be expressed by

$$J_{ij} = D_j x_i = \alpha_{im} (\delta_{mj} + h_{mj}) \tag{3.34}$$

Our purpose is to analyse the gradient of the neighborhood transformation (3.4) as we proceed from a particle P, of coordinates a_q, to a nearby particle P' of coordinates $a_q + da_q$. Beginning with the rotation

$$U (a_q + da_q) = U (a_q) + D_q U \, da_q$$

where $D_q U$ is the gradient of the rotation operator at P. This we write in the form

$$U (a_q + da_q) = U (a_q)(E + U'(a_q)D_q U \, da_q) \tag{3.35}$$

so that the rotation in the nearby point P' is effected in two successive steps : first the small incremental rotation existing between P and P', then the finite rotation in P.
According to the known structure of a small rotation operator, $U'D_q U$ must be skew symmetrical and this obviously results from

$$D_q(U'U) = 0 \quad \rightarrow \quad U'D_q U + (D_q U')U = 0$$

Introduce the following notation for the elements of $U'D_q U$, the rotation gradients

$$\alpha_{in} D_q \alpha_{im} = {}^{q}\omega_{nm} \qquad {}^{q}\omega_{mn} = - {}^{q}\omega_{nm} \tag{3.36}$$

Take now the gradient of the complete neighborhood transformation operator (3.34)

$$D_q J_{ij} = D_q D_j x_i = D_q \alpha_{im} (\delta_{mj} + h_{mj}) + \alpha_{im} D_q h_{mj}$$

and look at it, as was done in (3.35), from the viewpoint of a reference frame in P turned by the material rotation.

$$\alpha_{in} D_q J_{ij} = \alpha_{in} D_q D_j x_i = {}^q\omega_{nm} (\delta_{mj} + h_{mj}) + D_q h_{nj} \qquad (3.37)$$

It is obvious from the middle term that the right-hand side is symmetrical in the subscripts q and j; this provides a means of eliminating x_i:

$${}^q\omega_{nm} (\delta_{mj} + h_{mj}) + D_q h_{nj} = {}^j\omega_{nm} (\delta_{mq} + h_{mq}) + D_j h_{nq}$$

(3.38)

This relation may be considered as the result of eliminating x_i by cross differentiation between equations (3.34), together with an interpretation in terms of rotational gradients.
The particular form

$${}^q\omega_{nj} + D_q h_{nj} = {}^j\omega_{nq} + D_j h_{nq} \qquad (3.39)$$

that approximates (3.38) for very small strains, is similar to the Beltrani type of equations one obtains from the linear elasticity approximation (3.20) or

$$D_j x_n = \omega_{nj} + h_{nj} \qquad K = \{ \omega_{nj} \}$$

By cross differentiation

$$D_q \omega_{nj} + D_q h_{nj} = D_j \omega_{nq} + D_j h_{nq} \qquad (3.40).$$

Both sets (3.39) and (3.40) can be manipulated to yield the independant components of $D_q \omega_{nj}$ or ${}^q\omega_{nj}$ in terms of the strain gradients. Considering first (3.39), then rewriting it with the cyclic change of subscripts (qnj) to (jqn), then a third time with (n j q), adding the two first and substracting the third, there comes, with due consideration to the symmetry of h_{nj} and skew symmetry of ${}^q\omega_{nj}$,

$${}^j\omega_{qn} = D_n h_{qj} - D_q h_{nj} \qquad (3.41).$$

Similarly for the set (3.40)

$$D_j \omega_{qn} = D_n h_{qj} - D_q h_{nj} \qquad (3.42).$$

The Beltrami equations (3.42) yield the partial derivatives of the small rotations of linear elasticity theory in terms of the derivatives of strains. A second elimination process by cross differentiation of the rotations produces the linear local compatibility conditions to be satisfied by the strains :

$$D_p (D_n h_{qj} - D_q h_{nj}) = D_j (D_n h_{qp} - D_q h_{np}) \qquad (3.43)$$

They are trivially satisfied for $p = j$, or $n = q$; in fact only six of them are known to be distinct.

The situation is different if one tries to eliminate the rotational gradients in the set (3.31). Returning to the definition (3.36), and applying (3.32)

$$\alpha_{jn}\alpha_{in} D_q \alpha_{im} = D_q \alpha_{jm} = \alpha_{jn}\, {}^{q}\omega_{nm}$$

Differentiate the last equality :

$$D_p D_q \alpha_{jm} = {}^{q}\omega_{nm} D_p \alpha_{jn} + \alpha_{jn} D_p ({}^{q}\omega_{nm}) ,$$

observe that the left-hand side is symmetrical in p and q :

$${}^{q}\omega_{nm} D_p \alpha_{jn} + \alpha_{jn} D_p ({}^{q}\omega_{nm}) = {}^{p}\omega_{nm} D_q \alpha_{jn} + \alpha_{jn} D_q ({}^{p}\omega_{nm})$$

multiply this by α_{jr} and use definition (3.36) and properties (3.32) to obtain

$${}^{q}\omega_{nm}\, {}^{p}\omega_{rn} + D_p ({}^{q}\omega_{rm}) = {}^{p}\omega_{nm}\, {}^{q}\omega_{rn} + D_q ({}^{p}\omega_{rm}) \qquad (3.44).$$

This is the equation that, for gradients of finite rotations, replaces the simple relation $D_p D_q \omega_{rm} = D_q D_p \omega_{rm}$ in linear elasticity. A substitution of (3.41) into (3.44) yields the compatibility conditions for very small strains in the presence of finite rotations :

$$(D_n\, h_{qm} - D_m\, h_{qn})(D_r\, h_{pn} - D_n\, h_{pr}) + D_p\,(D_r\, h_{qm} - D_m\, h_{qr}) =$$

$$(D_n\, h_{pm} - D_m\, h_{pn})(D_r\, h_{qn} - D_n\, h_{qr}) + D_q\,(D_r\, h_{pm} - D_m\, h_{pr})$$

(3.45).

They differ from (3.43) by the non linear terms in the first derivatives of strains; the same equations hold for the γ_{mn} measures of strain, which do not differ practically from the h_{mn}.

The general compatibility equations for finite rotations and strains will be derived here for the sake of completeness. Some preparatory work will be necessary. We introduce a convenient notation for the element of the inverse jacobian matrix

$$J^{-1} = \{\, \partial_j\, a_m \,\}^{\text{m rows}}_{\text{j columns}} \qquad \partial_j = \partial/\partial x_j$$

and note the corresponding properties

$$\partial_j\, a_m\, D_n\, x_j = \delta_{mn} \quad \text{and} \quad \partial_j\, a_m\, D_m\, x_i = \delta_{ji} \tag{3.46}.$$

They are used to express the reciprocal of the fundamental metric tensor given by (3.28). First

$$\partial_j\, a_m\, G_{mn} = (\partial_j\, a_m\, D_m\, x_i)\, D_n\, x_i = D_n\, x_j \quad ,$$

next $$\partial_j\, a_p\, \partial_j\, a_m\, G_{mn} = \partial_j\, a_p\, D_n\, x_j = \delta_{pn} \quad ,$$

from which we conclude that the reciprocal metric tensor can be written

$$G^{pn} = \partial_j\, a_p\, \partial_j\, a_n \tag{3.47}.$$

Note that since from (3.28) the elements of G_{mn} are expressible in terms of the strains γ_{mn}, the same is true of the elements of the reciprocal G^{pn}.

We now use (3.30) to obtain an expression for the second derivatives of x_r :

$$\partial_r\, a_p\, \Gamma_{mpi} = (\partial_r\, a_p\, D_p\, x_j)\, D_m\, D_i\, x_j = D_m\, D_i\, x_r \tag{3.48}.$$

Differenciating (3.30)

$$D_q \Gamma_{mpi} = (D_q D_p x_j)(D_m D_i x_j) + (D_p x_j)(D_q D_m D_i x_j) \quad ,$$

exchanging the subscripts q and m

$$D_m \Gamma_{qpi} = (D_m D_p x_j)(D_q D_i x_j) + (D_p x_j)(D_m D_q D_i x_j) \quad ,$$

and substracting the results to eliminate the third derivatives, while substituting all second derivatives by means of (3.48)

$$D_q \Gamma_{mpi} - D_m \Gamma_{qpi} = \partial_j a_r \partial_j a_s \Gamma_{qrp} \Gamma_{msi}$$

$$- \partial_j a_r \partial_j a_s \Gamma_{mrp} \Gamma_{qsi} \quad ,$$

we can finally introduce the reciprocal metric tensor and obtain

$$D_q \Gamma_{mpi} - D_m \Gamma_{qpi} = G^{rs} (\Gamma_{qrp} \Gamma_{msi} - \Gamma_{mrp} \Gamma_{qsi}) \quad (3.49)$$

or $$D_q \Gamma_{mpi} - D_m \Gamma_{qpi} - \Gamma_q{}^s{}_p \Gamma_{msi} + \Gamma_m{}^s{}_p \Gamma_{qsi} = 0$$

In differential geometry this result expresses that the space referred to the convected (curvilinear) coordinates is Euclidean; its curvature tensor is zero. It also constitutes the local compatibility conditions for the strains since from (3.28) and (3.31) everything in them is expressible in terms of the γ_{mn} and their derivatives.

3.1.4. Stress tensors

From the viewpoint of an observer using the initial configuration as reference frame, the lagrangian stresses are defined per unit original area and referred to their cartesian components. For instance a facet of direction cosines (1, 0, 0) for the outward normal gives surface traction components ($t_{11}\ da_2\ da_3$, $t_{12}\ da_2\ da_3$, $t_{13}\ da_2\ da_3$). Their virtual work in a material infinitesimal variation δu_j of the displacement components is thus $(t_{1j}\ \delta u_j)\ da_2\ da_3$. For all the surface tractions on the facets of an elementary parallelopiped this virtual work becomes

$$D_i\ (t_{ij}\ \delta u_j)\ da_1\ da_2\ da_3$$

If external body forces, defined per unit volume in the initial configuration, and having cartesian components denoted by A_j, exist, their contribution to the virtual work is $A_j\ \delta u_j\ da_1\ da_2\ da_3$ and the energy balance equation reads

$$\delta W = D_i\ (t_{ij}\ \delta u_j) + A_j\ \delta u_j \tag{3.50}$$

where W is the elastic energy per unit initial volume.
In translation modes, where each δu_j is an arbitrary constant, there is no energy increase and (3.50) reduces to the translational equilibrium conditions

$$D_i\ t_{ij} + A_j = 0 \qquad (j = 1, 2, 3) \tag{3.51}.$$

The expanded form of (3.50)

$$\delta W = \delta u_j\ (D_i\ t_{ij} + A_j) + t_{ij}\ D_i\ \delta u_j$$

can thus be simplified; moreover, since the material variations δ are taken under constant values of the lagrangian variables, the operators D_i and δ commute and the simplified form of the energy balance can be

$$\delta W = t_{ij}\ \delta\ (D_i\ u_j) \tag{3.52}$$

From this it appears that the lagrangian stresses can be considered to be the partial derivatives of the energy W, considered as a function of the nine displacement gradients $D_i\ u_j$. This is made more precise by consideration of the rotational equilibrium conditions. Consider a displacement variation field, again of rigid body type, but representing now an infinitesimal rotation

$$\delta u_1 = x_3\ \delta w_2 - x_2\ \delta w_3$$

$$\delta u_2 = x_1\ \delta w_3 - x_3\ \delta w_1$$

$$\delta u_3 = x_2\ \delta w_1 - x_1\ \delta w_2 \quad .$$

Note that those additional displacements must be taken from the actual

coordinates x_i of the particles and that the energy increase must again vanish. Then, substituting this field into (3.52) and equating separately to zero the coefficients of the infinitesimal rotation angles δw_i, we obtain the rotational equilibrium conditions to be satisfied by the lagrangian stresses :

$$\begin{aligned} t_{i3}\, D_i\, x_2 - t_{i2}\, D_i\, x_3 &= 0 \\ t_{i1}\, D_i\, x_3 - t_{i3}\, D_i\, x_1 &= 0 \\ t_{i2}\, D_i\, x_1 - t_{i1}\, D_i\, x_2 &= 0 \end{aligned} \tag{3.53}$$

They clearly express that the tensor $t_{im}\, D_i\, x_n$ is symmetrical in the subscripts m and n and not the stress tensor itself. The substitution

$$t_{im} = s_{ip}\, D_p\, x_m \tag{3.54}$$

satisfies the rotational equilibrium equations

$$s_{ip}\, D_p\, x_m\, D_i\, x_n = s_{ip}\, D_p\, x_n\, D_i\, x_m \tag{3.55}$$

provided the new stress tensor be symmetrical :

$$s_{ip} = s_{pi} \tag{3.56}$$

This is immediatly seen by exchanging the dummy subscripts p and i in the right hand side of (3.55). This somewhat artificial definition receives a strong foundation through the following consequences : (3.52) becomes

$$\delta W = s_{im}\, D_m\, x_j\, \delta\, (D_i\, u_j) = s_{im}\, D_m\, x_j\, \delta\, (D_i\, x_j)$$

or, exchanging the dummy indices i and m,

$$\delta W = s_{mi}\, D_i\, x_j\, \delta\, (D_m\, x_j).$$

After adding the two formulas, using the symmetry property (3.56), the components of the Green strain tensor appear through (3.23) :

$$2\delta W = s_{im}\,\delta\,(D_m x_j\, D_i x_j) \quad \text{or} \quad \delta W = s_{im}\,\delta\gamma_{im} \tag{3.57}$$

This result shows that W is a function of the displacement gradients through the six independant components of the (symmetrical) Green tensor. The symmetrical "Kirchhoff-Trefftz" stress tensor is made up of the partial derivatives of the energy :

$$s_{11} = \frac{\partial W}{\partial \gamma_{11}} \qquad s_{22} = \frac{\partial W}{\partial \gamma_{22}} \qquad s_{33} = \frac{\partial W}{\partial \gamma_{33}}$$

$$s_{12} = \frac{\partial W}{\partial \eta_{12}} \qquad s_{23} = \frac{\partial W}{\partial \eta_{23}} \qquad s_{31} = \frac{\partial W}{\partial \eta_{31}} \tag{3.58}$$

where the shearing strains

$$\eta_{12} = \gamma_{12} + \gamma_{21} \qquad \eta_{23} = \gamma_{23} + \gamma_{32} \qquad \eta_{31} = \gamma_{31} + \gamma_{13} \tag{3.59}$$

were introduced for convenience. The compact formula

$$s_{ij} = \frac{\partial W}{\partial \gamma_{ij}} \tag{3.60}$$

for describing the stress-strain relations (3.58) is also acceptable, provided one distinguishes between γ_{ij} and γ_{ji} as in (3.59) and applies the chain differentiation rule.

The translational equilibrium equations satisfied by the Kirchhoff-Trefftz stresses were given by Signorini (3.2), from (3.54) and (3.51) they are

$$D_i\,(s_{im}\,D_m x_j) + A_j = 0 \tag{3.61}$$

A straightforward geometrical interpretation of the Kirchhoff-Trefftz stresses is obtained when transforming the surface tractions successively by (3.54) and (3.27)

$$\vec{t}_i = t_{im}\,\vec{e}_m = s_{ip}\,D_p x_m\,\vec{e}_m = s_{ip}\,\vec{G}_m \tag{3.62}$$

The Kirchhoff-Trefftz stresses result from the decomposition of the lagrangian surface tractions in the metric induced by the convection of the lagrangian coordinates.

We are now in a position to check on the validity of the following interpretation of eq. (3.52)

$$t_{ij} = \partial W / \partial (D_i\ u_j) \tag{3.63}$$

where the explicit dependance of the energy on the displacement gradients is to result from its dependance on the Green strain tensor and from (3.23).

$$t_{ij} = \frac{\partial W}{\partial \gamma_{mn}}\, \partial \gamma_{mn} / \partial (D_i\ x_j) = \frac{1}{2}\, s_{mn}\, (\delta_{mi}\ D_n\ x_j + \delta_{ni}\ D_m\ x_j)$$

$$= \frac{1}{2}\, (s_{in}\ D_n\ x_j + s_{mi}\ D_m\ x_j) = s_{in}\ D_n\ x_j$$

which is confirmed by (3.54). In the form

$$t_{ij} = D_n\ x_j\ \frac{\partial W}{\partial \gamma_{in}} \tag{3.64}$$

it can be considered as the constitutive equations for the lagrangian stresses incorporating rotational equilibrium. Since from (3.11)

$$\gamma_{mn} = h_{mn} + \frac{1}{2}\, h_{mp}\ h_{pn} \tag{3.65}$$

the energy can also be conceived as a function of the six components h_{mn} of the engineering strain tensor H and a corresponding symmetrical stress tensor defined :

$$r_{ij} = \partial W / \partial h_{ij} \tag{3.66}.$$

From the rule of chain differentiation we obtain readily

$$r_{ij} = s_{mn}\, \frac{\partial \gamma mn}{\partial h_{ij}} = s_{mn}\, (\delta_{mi}\ \delta_{nj} + \frac{1}{2}\, h_{jn}\ \delta_{mi} + \frac{1}{2}\, h_{mi}\ \delta_{nj})$$

$$= s_{ij} + \frac{1}{2}\, s_{in}\ h_{jn} + \frac{1}{2}\, s_{mj}\ h_{mi} \tag{3.67}$$

or, in matrix formulation,

$$R = S + \frac{1}{2}\, (S\ H + H\ S) = \frac{1}{2}\, S\ (E + H) + \frac{1}{2}\, (E + H) S \tag{3.68}.$$

It appears that the difficulty encountered in expressing H in terms of G_a has here its counterpart in the difficulty of expressing the Kirchhoff-Trefftz stresses S in terms of the stress tensor R.
Looking to the relationship between R and the lagrangian stresses, we can introduce into (3.54) the polar decomposition (3.34)

$$t_{im} = s_{ip}\, \alpha_{mq}\, (\delta_{qp} + h_{qp})$$

and, premultiplying by α_{mr}, obtain the components of S (E + H)

$$\alpha_{mr}\, t_{im} = s_{ip}\, (\delta_{rp} + h_{rp})$$

whence follows

$$r_{ij} = \frac{1}{2}\, \alpha_{mj}\, t_{im} + \frac{1}{2}\, \alpha_{mi}\, t_{jm} \qquad (3.69)$$

Here again it appears difficult to solve for the lagrangian stresses. As a consequence there is no simple way to state the translational equilibrium equations satisfied by the R stress tensor. On the other hand it is obvious from the relations between the various stress tensors that they are practically equivalent for the case of very small strains.

3.1.5. Variational principles for large elastic displacements

The best known is that which states that the total energy, strain energy plus potential energy of dead loads, is stationary with respect to arbitrary, kinematically admissible, variations of the displacement field. It is important to note that the rotational equilibrium conditions are incorporated ab initio in the principle by considering that the energy can be expressed in terms of the Green strain tensor; otherwise the proof of the principle is merely a reformulation of our previous energy balance considerations.

$$\int_{R_a} (W - \overline{A}_j\, u_j)\, dR_a - \int_{\partial R_a} \overline{p}_j\, u_j\, d\partial R_a \quad \text{stationary} \qquad (3.70)$$

For simplicity the region R_a occupied by the elastic body is assumed to be simply connected. On its bounding surface ∂R_a either the surface tractions p_j are imposed, or the displacements u_j (the kinematical boundary

conditions). The surface integral only includes contributions of those areas where surface traction components are specified. The integrations are performed with respect to the lagrangian coordinates and dR_a stands for $da_1\ da_2\ da_3$ in our cartesian reference frame. In view of the assumptions made

$$\delta W = \frac{\partial W}{\partial \gamma_{im}}\,\delta\gamma_{im} = s_{im}\,\delta\left(\frac{1}{2}\,D_i\,x_j\,D_m\,x_j - \frac{1}{2}\delta_{im}\right)$$

$$= s_{im}\,\frac{1}{2}\,(D_i\,x_j\,\delta D_m\,x_j + D_m\,x_j\,\delta D_i\,x_j)$$

Or, in view of the symmetry of the stresses

$$\delta W = s_{im}\,D_m\,x_j\ \ \delta D_i\,x_j = s_{im}\,D_m\,x_j\,D_i\,(\delta u_j)$$

Hence, for the variation of (3.70),

$$\int_{R_a} (s_{im}\,D_m\,x_j\,D_i\,(\delta u_j) - \overline{A}_j\,\delta u_j)\,dR_a$$

$$- \int_{\partial R_a} \overline{p}_j\,\delta u_j\,d\,\partial R_a = 0$$

with $\delta u_j = 0$ there where u_j was specified. Integration by parts yields

$$\int_{\partial R_a} (l_i\,s_{im}\,D_m\,x_j - \overline{p}_j)\,\delta u_j\,d\partial R_a$$

$$- \int_{R_a} \{\,D_i\,(s_{im}\,D_m\,x_j) + \overline{A}_j\,\}\,\delta u_j\,dR_a = 0$$

The Signorini equations (3.61) are seen to be the Euler equations of this principle, while the natural boundary conditions

$$l_i\,s_{im}\,D_m\,x_j = \overline{p}_j \qquad (3.71);$$

where l_i denote the direction cosines of the outward normal in the initial configuration, express surface equilibrium between the stress field and the imposed tractions.

A generalization of the principle of variation of displacements to a two-field principle is possible by the Friedrichs method (3.7).

We will consider the energy density to be a function of the components of the Green strain tensor but, instead of expressing those a priori in terms of displacement gradients, we raise them to a status of independant variables by incorporating equations (3.23), considered as differential constraints, into the principle :

$$\int_{R_a} \{ W(\gamma) + s_{im} (\frac{1}{2} D_i x_j D_m x_j - \frac{1}{2} \delta_{im} - \gamma_{im}) - \bar{A}_j u_j \} dR_a$$

$$- \int_{\partial R_a} \bar{p}_j u_j \, d\partial R_a \text{ stationary} \quad (3.72)$$

Here, the quantities s_{im} denote independant lagrangian multipliers, whose Euler equations restore the differential constraints. It is already clear from the Euler equations (3.61) and natural boundary conditions (3.71) of this principle relative to variations on the displacements, that the set of multipliers has the nature of lagrangian stresses. This is further confirmes by the Euler equations (3.60) that are obtain from variations of the strains.

This very general three-field variational principle allows independant approximations to be made on strains, stresses and displacements. This considerable freedom makes it difficult to apply in practice. When the Euler equations (3.60) are accepted a priori, that is if the multipliers are already identified with the Kirchhoff-Trefftz stresses, the principle can be reduced to a two-field one by elimination of the strains. Assuming that the stress-strain relations (3.60) can be solved for the strains, one introduces the complementary energy density

$$\phi (s) = s_{im} \gamma_{im} - W \quad (3.73)$$

This Legendre transformation has the property

$$d \phi = (s_{im} - \frac{\partial W}{\partial \gamma_{im}}) d\gamma_{im} + \gamma_{im} ds_{im} = \gamma_{im} ds_{im}$$

or

$$\gamma_{im} = \frac{\partial \phi_a}{\partial s_{im}} \quad (3.74)$$

of inverting the stress-strain relations (3.60). It furnishes a two-field principle discovered independantly by E. Reissner and B. Fraeijs de Veubeke (3.5).

$$\int_{R_a} \{ \frac{1}{2} s_{im} (D_i x_j D_m x_j - \delta_{im}) - \phi(s) - \bar{A}_j u_j \} dR_a$$

$$- \int_{\partial R_a} \bar{p}_j u_j \, d\partial R_a \text{ stationary} \qquad (3.75)$$

While the Euler equations (3.61) and natural boundary conditions of this principle are unchanged the Euler equations due to variations on the stresses are now a combination of (3.74) and the original differential constraints (3.23)

$$\frac{1}{2} (D_i x_j D_m x_j - \delta_{im}) = \partial\phi / \partial s_{im} \qquad (3.76).$$

While in the linear elasticity case a variational principle for stresses only, the so-called complementary energy principle, is easily derivable from (3.75), finite rotations introduce serious difficulties.
To take the case of zero body forces for simplicity, the translational equilibrium equations can be solved for the lagrangian stresses by means of stress functions A_{sj} :

$$t_{ij} = e_{irs} D_r A_{sj} \qquad (e_{irs}, \text{ the permutation symbol}) \qquad (3.77)$$

and this indeed allows to modify principle (3.75) into one that does no more involve explicitly displacements. However the differential constraints

$$s_{im} D_m x_j = e_{irs} D_r A_{sj} \qquad (3.78)$$

derived from (3.77) are still implicitly coupling the choice of Kirchhoff-Trefftz stresses to the displacements. It is therefore generally recognized that there is no variational stress principle available in presence of finite rotations (3.6).
We shall now explore new variational principles based on the polar decomposition method (3.34). We start from the principle

$$\int_{R_a} \{ W (D_i u_j) - \bar{A}_j u_j \} dR_a - \int_{\partial R_a} \bar{p}_j u_j \, d\partial R_a \text{ stationary}$$

suggested by (3.52) and whose Euler equations and natural boundary conditions are $D_i t_{ij} + \bar{A}_j = 0$ and $l_i t_{ij} = \bar{p}_j$.

It contains also implicitly the rotational equilibrium conditions if the energy density is a function of the nine displacement gradients through the six Green strain tensor measures.

We now remove the nine differential constraints

$$D_j\, x_i - \alpha_{im}\,(\delta_{mj} + h_{mj}) = 0$$

equivalent to the polar decomposition (3.34), by means of nine lagrangian multipliers t_{ji}

$$\int_{R_a} \{ W(h) + t_{ji}\, D_j\, x_i - t_{ji}\, \alpha_{im}\,(\delta_{mj} + h_{mj}) - \bar{\Lambda}_i u_i \}\, dR_a$$

$$- \int_{\partial R_a} \bar{p}_j\, u_j\, d\partial R_a \quad \text{stationary} \qquad (3.79)$$

The strain energy density is naturally taken to be a function of the six liberated strain measures h_{ij}, while three additional degrees of rotational freedom (Euler angles for example) are contained in the liberated α_{im} to match the original nine displacement gradients. In taking variations with respect to the strain measures, we are not allowed to distinguish between h_{mj} and h_{jm}, so the corresponding Euler equations are

$$\frac{\partial W}{\partial h_{mj}} + \frac{\partial W}{\partial h_{jm}} - t_{ji}\, \alpha_{im} - t_{mi}\, \alpha_{ij} = 0 \quad , \text{ or}$$

$$\frac{\partial W}{\partial h_{mj}} = r_{mj} = \frac{1}{2}\,(t_{ji}\, \alpha_{im} + t_{mi}\, \alpha_{ij}) \qquad (3.80)$$

Comparison of this with (3.69) justifies the notation of lagrangian stresses utilized for the multipliers, and this is further confirmed by the translational equilibrium equations and natural boundary conditions they satisfy as a result of variations on the displacements. When taking variations on the rotations account must be taken of the property

$$\alpha_{im}\, \alpha_{in} = \delta_{mn}$$

from which follows

$$\alpha_{in}\, \delta\alpha_{im} + \alpha_{im}\, \delta\alpha_{in} = 0.$$

Hence we can introduce the skew symmetric variation

$$\delta\omega_{mn} = \alpha_{in}\ \delta\alpha_{im} = -\ \delta\omega_{nm}$$

or, equivalently, express the variation of the rotation matrix in terms of those three independant variations :

$$\delta\alpha_{im} = \alpha_{in}\ \delta\omega_{mn} \tag{3.81}$$

This furnishes as last Euler equations of principle (3.79)

$$\alpha_{in}\ t_{ji}\ (\delta_{mj} + h_{mj}) - \alpha_{im}\ t_{ji}\ (\delta_{nj} + h_{nj}) = 0 \tag{3.82}$$

One brings this back to the original form of the rotational equilibrium conditions when multiplying by $\alpha_{pn}\ \alpha_{qm}$ and using (3.34) :

$$t_{jp}\ D_j\ x_q - t_{jq}\ D_j\ x_p = 0$$

It can finally be concluded that the variational principle (3.79) is very general although somewhat complicated because of the 21 independant quantities that can he varied independantly. It contains the 3 translational equilibrium conditions through $\delta\alpha_{im}$, the polar decomposition constraints through variations on the 9 stresses and the 6 stress-strain relations (3.80) through variations on the strains.
Amongst the possible simplifications of the principle, consider the following group of terms in the volume integrand

$$W(h) - t_{ji}\ \alpha_{im}\ (\delta_{mj} + h_{mj})$$

Playing on the symmetry of the strains this can be written

$$W(h) - (\delta_{mj} + h_{mj})\ \frac{1}{2}\ (t_{ji}\ \alpha_{im} + t_{mi}\ \alpha_{ij})$$

$$= W(h) - (\delta_{mj} + h_{mj})\ r_{mj}$$

and suggests that we accept (3.80) a priori, solve for the strains in terms of the stresses r_{mj} and introduce the complementary energy density

$$\psi(r) = (\delta_{mj} + h_{mj})\, r_{mj} - W \tag{3.83}$$

This has the property

$$\frac{\partial \psi}{\partial r_{mj}} = \delta_{mj} + h_{mj} \tag{3.84}$$

We consider however the r_{mj} to act only as auxiliary quantities defined by the second equality in (3.80). The modified variational principle

$$\int_{R_a} (t_{ji}\, D_j\, x_i - \psi - \overline{A}_i\, u_i) dR_a - \int_{\partial R_a} \overline{p}_i\, u_i\, d\partial R_a \text{ stationary}$$

still contains as independant quantities subject to variations, the 3 u_i, the 9 t_{ji} and the 3 $\delta\omega_{mn}$ related to the rotations. We now substitute

$$D_j\, x_i = \delta_{ji} + D_j\, u_i$$

and integrate by parts with respect to the u_i. Then, assuming that we can manage to satisfy a priori the translational equilibrium equations and natural boundary conditions, the principle reduces to

$$\int_{R_a} (\psi - t_{jj})\, dR_a - \int_{\partial R_a} l_j\, t_{ji}\, \overline{u}_i\, d\partial R_a \text{ stationary} \tag{3.85}$$

and is very similar to the complementary energy principle of linear elasticity. Taking the simple case of no body forces, the translational equilibrium equations are easily satisfied by stress functions as in (3.77). The complementary energy density is then a function of

$$r_{mj} = \frac{1}{2} (e_{jrs}\, \alpha_{im}\, D_r\, A_{si} + e_{mrs}\, \alpha_{ij}\, D_r\, A_{si}) \tag{3.86}$$

While the term

$$t_{jj} = e_{jrs}\, D_r\, A_{sj}$$

goes immediatly to the boundaries by integration. The Euler equations obtained from variations of the stress functions are

$$e_{mrs}\, D_r \{ \alpha_{ij}\, (\delta_{jm} + h_{jm}) \} = 0 \tag{3.87}$$

and are clearly fulfilled when the polar decomposition is used :

$$e_{mrs} D_r D_m x_i = 0$$

This shows that, since (3.87) result from elimination of the displacements between the polar decomposition equations, they can be considered as integrability conditions for such displacements.

Variations on the α_{im} produce again a form of the rotational equilibrium conditions.

In practical applications the complementary energy form of this new principle allows the translational equilibrium equations to be rigorously satisfied; the weight of the approximations falls on the integrability conditions for displacements (explaining why displacements as such have disappeared from the principle) and on the rotational equilibrium conditions.

3.2. Elastic stability problems

This section discusses first the classical energy criterion of elastic stability, based on the second variation test of the total potential. It leads in general to a linear eigenvalue problem. The cases of perfectly straight plates and rods under perfectly centered loads are considered next with special emphasis on the mathematical justification of the usual approximations.

3.2.1. The second variation criterion

Let $$U = \int_{R_a} W(\gamma) dR_a$$

denote the strain energy expressed in terms of the Green measures of strain as given by (3.23). In the elastic range we take explicitly

$$W = \frac{1}{2} C^{pq}_{mn} \gamma_{mn} \gamma_{pq} \tag{3.88}$$

where the elastic moduli have the following symmetry properties

$$C^{pq}_{mn} = C^{pq}_{nm} = C^{qp}_{mn} = C^{mn}_{pq} \quad .$$

Application of (3.60) gives the stress-strain relations

$$s_{ij} = \frac{\partial W}{\partial \gamma_{ij}} = C^{pq}_{ij} \gamma_{pq} \tag{3.89}$$

As we postulate that the strains are small, in fact of the order of 10^{-3} in order to remain in the elastic range of usual materials, those stress-strain relations are the usual engineering ones. However, because the rotations can be large, the stresses are the Kirchhoff-Trefftz ones defined in section 3.1.4. They are defined per unit area in the initial (unstressed) configuration but decomposed in the metric induced by the deformation. The loads are assumed to be "dead" loads, they are not affected by the change in configuration and their potential energy is a linear functional of the displacement field :

$$P(u) = -\int_{R_a} \bar{A}_m u_m \, dR_a - \int_{\partial R_a} \bar{p}_m u_m \, d\partial R_a \tag{3.90}$$

Denote by u°_m a displacement field in a configuration of equilibrium with respect to the applied loads and consider a perturbed field

$$u_m = u^\circ_m + \varepsilon \hat{u}_m \tag{3.91}$$

where ε is a small parameter, that satisfies the kinematical boundary conditions for all values of ε. Hence $\hat{u}_m = 0$ there where the displacement u_m was prescribed on the boundary. Expanding the strains in powers of ε

$$\gamma_{mn} = \gamma^\circ_{mn} + \varepsilon \gamma^{(1)}_{mn} + \frac{1}{2} \varepsilon^2 \gamma^{(2)}_{mn} \tag{3.92}$$

gives

$$\gamma^\circ_{mn} = \frac{1}{2} (D_m u^\circ_n + D_n u^\circ_m + D_m u^\circ_i D_n u^\circ_i)$$

$$\gamma^{(1)}_{mn} = \frac{1}{2} (D_m \hat{u}_n + D_n \hat{u}_m + D_m u^\circ_i D_n \hat{u}_i + D_n u^\circ_i D_m \hat{u}_i) \tag{3.93}$$

$$\gamma^{(2)}_{mn} = D_m \hat{u}_i D_n \hat{u}_i \tag{3.94}.$$

Substitution of (3.92) into (3.88) gives

$$W = W^\circ + \varepsilon C^{pq}_{mn} \gamma^\circ_{pq} \gamma^{(1)}_{mn} + \frac{1}{2} \varepsilon^2 \{ C^{pq}_{mn} \gamma^{(1)}_{mn} \gamma^{(1)}_{pq} + C^{pq}_{mn} \gamma^\circ_{pq} \gamma^{(2)}_{mn} \}$$

where higher powers of ε were neglected and

$$W^\circ = \frac{1}{2} C^{pq}_{mn} \gamma^\circ_{mn} \gamma^\circ_{pq}$$

is the strain energy in the equilibrium configuration. Introducing the stresses at equilibrium

$$s^\circ_{mn} = C^{pq}_{mn} \gamma^\circ_{pq} \tag{3.95}$$

and the strain energy density calculated from the strains $\gamma^{(1)}_{mn}$ only

$$W^{(1)} = \frac{1}{2} C^{pq}_{mn} \gamma^{(1)}_{mn} \gamma^{(1)}_{pq} \tag{3.96}$$

the total potential, expanded in powers of ε up to second degree, reads

$$U + P = \int_{R_a} W^\circ \, dR_a + P\,(u^\circ)$$

$$+ \varepsilon \left\{ \int_{R_a} s^\circ_{mn} \gamma^{(1)}_{mn} dR_a + P\,(\hat{u}) \right\}$$

$$+ \varepsilon^2 \left\{ \int_{R_a} \left(W^{(1)} + \frac{1}{2} s^\circ_{mn} \gamma^{(2)}_{mn} \right) dR_a \right\} \qquad (3.97)$$

A necessary condition for a minimum of the total potential is the vanishing of the first degree term under all kinematically admissible perturbations $\hat{u}_m$, or

$$\int_{R_a} (s^\circ_{mn} \gamma^{(1)}_{mn} - \bar{A}_i \hat{u}_i) \, dR_a - \int_{\partial R_a} \bar{p}_i \hat{u}_i \, d\partial R_a = 0$$

In view of the symmetry of the stress tensor, the result of a substitution of (3.93) can be written

$$\int_{R_a} \{ s^\circ_{mn} (\delta_{ni} + D_n u^\circ_i) D_m \hat{u}_i - \bar{A}_i \hat{u}_i \} \, dR_a$$

$$- \int_{\partial R_a} \bar{p}_i \hat{u}_i \, d\partial R_a = 0$$

After an integration by parts this produces the field and boundary conditions

$$D_m \{ s^\circ_{mn} (\delta_{ni} + D_n u^\circ_i) \} + \bar{A}_i = 0$$

$$l_m s^\circ_{mn} (\delta_{ni} + D_n u^\circ_i) = \bar{p}_i$$

stating that the field u°_i is one that satisfies equilibrium as was assumed. It is observed that the vanishing of the first degree term in the expansion (3.97) is equivalent to the vanishing of the first variation of the total potential as discussed in section 3.1.5. on variational principles. A sufficient condition for the minimum of the total potential in that the coefficient of ε^2, the second variation of the total potential, be positive definite under all kinematically admissible perturbations

$$\int_{R_a} (W^{(1)} + \frac{1}{2} s^{\circ}_{mn} \gamma^{(2)}_{mn}) \, dR_a > 0 \qquad (3.98)$$

This positive definitness criterion will be adopted for the elastic stability of the equilibrium configuration. Clearly, if satisfied, external energy is required to maintain any kinematically admissible disturbance. Conversely, should the second variation be negative for some admissible disturbance, the energy release would have to be absorbed to hold it static. Inspection of the functional (3.98) shows it to be quadratic in the displacement disturbance field, hence, coupled with a linearized kinetic energy functional, one obtains a linear small oscillations problem. It has real eigenfrequencies if (3.98) is satisfied. Stability criterion (3.98) is valid for both bifurcation and snap-through phenomena (3.8, 3.9). This generality is one of its advantages.

3.2.2. The stability criterion as an eigenvalue problem

The stability criterion (3.98) is an inequality for the sum of two quadratic functionals.
The first

$$f_1 (\hat{u}) = \int_{R_a} \frac{1}{2} C^{pq}_{mn} \gamma^{(1)}_{mn} \gamma^{(1)}_{pq} dR_a \qquad (3.99)$$

is by nature positive definite. It vanishes if and only if the first order strains

$$\gamma^{(1)}_{mn} = \frac{1}{2} (\delta_{ni} + D_n u^{\circ}_i) D_m \hat{u}_i$$

$$+ \frac{1}{2} (\delta_{mi} + D_m u^{\circ}_i) D_n \hat{u}_i \qquad (3.100)$$

vanish everywhere. In that case the displacement disturbance field $\hat{u}_i$ only modifies the equilibrium strains by second order quantities; it may be suspected that this disturbance field is in the nature of a small rigid body motion. Confirmation of this is found by an interpretation of $\gamma^{(1)}_{mn} = 0$ in the metric induced by the equilibrium strains. Consider the relations

$$x^{\circ}_i = a_i + u^{\circ}_i (a_j)$$

as a change of lagrangian coordinates and introduce the notation

$$\partial_i = \partial / (\partial x^\circ_i)$$

One has then formulas such as

$$D_m \hat{u}_i = D_m x^\circ_p \, \partial_p \hat{u}_i$$

and the first order strains (3.100) can be written as

$$\gamma^{(1)}_{mn} = \frac{1}{2} (D_n x^\circ_i \, D_m x^\circ_p + D_m x^\circ_i \, D_n x^\circ_p) \, \partial_p \hat{u}_i \ .$$

Multiplying both sides by $\partial_r a_n \partial_s a_m$ and using the reciprocity formulas

$$\partial_r a_n D_n x^\circ_i = \delta_{ri}$$

we find $$\partial_r a_n \partial_s a_m \gamma^{(1)}_{mn} = \frac{1}{2} (\partial_s \hat{u}_r + \partial_r \hat{u}_s)$$

Hence the vanishing of first order strains entails that the structure of the disturbance field is that of a small rigid body motion expressed in the new lagrangian coordinates, as suits the fact that it follows the equilibrium straining.

$$\hat{u}_1 = \alpha_1 + \omega_2 x^\circ_3 - \omega_3 x^\circ_2$$

$$\hat{u}_2 = \alpha_2 + \omega_3 x^\circ_1 - \omega_1 x^\circ_3$$

$$\hat{u}_3 = \alpha_3 + \omega_1 x^\circ_2 - \omega_2 x^\circ_1$$

Of course the kinematical boundary conditions might be such as to prevent one or more of those six degrees of freedom.
The second functional

$$f_2 (\hat{u}) = \int_{R_a} \frac{1}{2} s^\circ_{mn} \gamma^{(2)}_{mn} \, dR_a \qquad (3.101)$$

has in general no definite sign. Let

$$\phi (\hat{u}) = f_1 (\hat{u}) + f_2 (\hat{u})$$

be the value of the complete functional. To prove its positive character under all admissible disturbances it is tempting to seek the sign of its minimum value. However, since the kinematical conditions imposed are homogeneous, if $\hat{u}_i$ is admissible, so is $\alpha\ \hat{u}_i$ and $\phi\ (\alpha\ \hat{u}) = \alpha^2\ \phi\ (\hat{u})$, where α is an arbitrary real scalar. This similarity rule shows that it is meaningless to look after an absolute "critical" disturbance that would give its minimum value to ϕ, however it shows also that the stable ($\phi > 0$) or unstable ($\phi < 0$) character of a disturbance is not altered by scaling. Hence it is appropriate to look after a critical disturbance "shape" by adding a norming constraint to the disturbance scale. Because of its positive definite character

$$f_1\ (\hat{u}) = \mu^2 \qquad \text{a fixed quantity} \tag{3.102}$$

is a suitable norming constraint. It insures that the disturbance will involve additional strain energy and not merely superposition of a small rigid body motion. The problem of finding the minimum of ϕ under the norming constraint is Jacobi's accessory problem (3.11). With (3.102) as norming constraint Jacobi's problem is clearly equivalent to that of finding the minimum of the ratio

$$\lambda = \frac{f_2\ (\hat{u})}{f_1\ (\hat{u})} \tag{3.103}$$

that is independant of scaling.

A necessary condition for a relative minimum is that the variation

$$\delta\ \{\ f_2\ (\hat{u}) - \lambda\ f_1\ (\hat{u})\ \} = 0 \tag{3.104}$$

for all kinematically admissible variations of the disturbance field. Expanding this condition produces

$$\int_{R_a} (\tfrac{1}{2}\ s^{\circ}_{mn}\ \delta\gamma^{(2)}_{mn} - \lambda\ s^{(1)}_{mn}\ \delta\ \gamma^{(1)}_{mn}\ )\ dR_a = 0$$

where $$s^{(1)}_{mn} = C^{pq}_{mn}\ \gamma^{(1)}_{pq}$$

Substitution of (3.94) and (3.100) and use of the symmetry of the stress tensors gives

$$\int_{R_a} \{ s^{\circ}_{mn} D_n \hat{u}_i - \lambda s^{(1)}_{mn} (\delta_{ni} + D_n u^{\circ}_i) \} D_m \delta\hat{u}_i \, dR_a = 0$$

Hence in addition to the homogeneous kinematical boundary conditions the critical disturbance shape must satisfy the field equations

$$D_m \{ s^{\circ}_{mn} D_n \hat{u}_i - \lambda s^{(1)}_{mn} (\delta_{ni} + D_n u^{\circ}_i) \} = 0 \qquad (3.105)$$

and, there where surface tractions are imposed, the boundary conditions

$$1_m \{ s^{\circ}_{mn} D_n \hat{u}_i - \lambda s^{(1)}_{mn} (\delta_{ni} + D_n u^{\circ}_i) \} = 0 \qquad (3.106)$$

All the equations are homogeneous of the first degree in the $\hat{u}_i$, hence non trivial solutions, defined except for scaling, will only exist for a spectrum of eigenvalues of λ. If $\hat{u}_i$ is an eigensolution and λ its eigenvalue, we find after scaling to satisfy the norming constraint (3.102) and using (3.103)

$$\phi(\hat{u}) = f_1(\hat{u}) + f_2(\hat{u}) = (1 + \lambda) \mu^2$$

Consequently the stability criterion will be satisfied if the smallest eigenvalue is larger than -1.

3.2.3. Stability boundaries

The previous analysis furnishes a way to test stability under a fixed set of dead loads. In many cases one wishes to evaluate the stability margin of a structure and therefore to increase the loading up to the appearance of the critical value $\lambda = -1$, where λ is the smallest eigenvalue. Topological investigations (3.9) show that the appearance of this critical value can have several meanings : loss of stability, asymmetric bifurcation to a stable continuous post-buckling equilibrium path, stable-symmetric bifurcation possibility and unstable-symmetric bifurcation. The equilibrium paths of the structure are followed by assuming a similarity law for the loading distribution that results in a single load scaling parameter σ. The general case is fairly complicated because the equilibrium stress distribution s°_{mn} depends non linearly on the load parameter σ and incremental analysis becomes necessary.

Direct access to the critical situation becomes possible when the initial equilibrium path can be dealt with by the linear elasticity approximation,

that is when both rotations and strains remain very small.
Then all displacement gradients remain very small at equilibrium

$$|D_m u^\circ_i| \ll 1 \tag{3.107}$$

and (3.93) can be approximated by the formulas

$$\gamma^{(1)}_{mn} = \frac{1}{2}(D_m \hat{u}_n + D_n \hat{u}_m) \tag{3.108}.$$

Consequently the equilibrium equations for the Kirchhoff-Trefftz stresses have valid linear approximations

$$D_m s^\circ_{mn} + \bar{A}_i = 0$$

$$l_m s^\circ_{mi} = \bar{p}_i$$

and the scaling operation on the loads becomes equivalent to a similar scaling on the equilibrium stress distribution :

$$s^\circ_{mn} = \sigma S_{mn} \qquad \sigma > 0 \tag{3.109}$$

where S_{mn} is a fixed distribution.
According to the analysis of the previous section the stability condition becomes

$$\sigma \mu > -1 \tag{3.110}$$

where

$$\mu = \min \frac{\frac{1}{2}\int_{R_a} S_{mn} \gamma^{(2)}_{mn} \, dR_a}{\frac{1}{2}\int_{R_a} C^{pq}_{mn} \gamma^{(1)}_{mn} \gamma^{(1)}_{pq} \, dR_a} \tag{3.111}$$

We now distinguish between two cases.

CASE 1 : $\mu > 0$ There is obviously unconditional stability; σ can be as large as the limitations on the elastic behavior of the material will permit.

CASE 2 : $\mu < 0$

we then set $$\mu = -\frac{1}{\sigma_{cr}} \qquad \sigma_{cr} > 0 \tag{3.112}$$

and the stability condition (3.110) becomes equivalent to

$$\sigma < \sigma_{cr} \tag{3.113}$$

$\sigma = \sigma_{cr}$ being the critical situation. Moreover with μ replaced by (3.112), (3.111) becomes equivalent to

$$\sigma_{cr} = \min \frac{\frac{1}{2}\int_{R_a} C^{pq}_{mn}\, \gamma^{(1)}_{mn}\, \gamma^{(1)}_{pq}\, dR_a}{-\frac{1}{2}\int_{R_a} S_{mn}\, \gamma^{(2)}_{mn}\, dR_a} \tag{3.114}$$

and the determination of the critical load parameter has been turned into an eigenvalue problem. The presence of the minus sign in the denominator accounts for the fact that positive values of σ_{cr} are normally associated with compressive stresses. If σ_{cr} as determined by (3.114) turns out to be negative, $\sigma_{cr} = -\alpha^2$, it is easily found that $\alpha^2 = \mu$ as given by (3.111). Hence, while (3.113) becomes meaningless, we are in a case of unconditional stability.

With due consideration to the simplified expressions (3.108), the Euler equations and natural boundary conditions to be satisfied by the critical disturbances of problem (3.114) are

$$D_m\, T_{mi} = 0 \qquad \text{in } R_a \tag{3.115}$$

$$l_m\, T_{mi} = 0 \qquad \text{where } \bar{p}_i \text{ specified on } \partial R_a \tag{3.116}$$

$$T_{mi} = C^{pq}_{mi}\, D_p\, \hat{u}_q + \sigma_{cr}\, S_{mn}\, D_n\, \hat{u}_i \tag{3.117}.$$

3.2.4. The equilibrium configuration as reference configuration

Up to now the unstressed configuration was used as initial or reference configuration. The stability analysis can however also be carried out by choosing the actual equilibrium configuration to be tested as reference. This may present some advantages when the prevailing state of stress is not due to external loading but to internal reasons like thermal effects or self-stressing. To be clear it is necessary to distinguish between the original cartesian coordinates a_i of the particles and their coordinates at equilibrium

$$x^{\circ}_i = a_i + u^{\circ}_i (a_j) \tag{3.118}$$

Note that x°_i is not to be confused here with an eulerian coordinate but represents a new lagrangian coordinate that will be used in formulating the stability analysis. Hence (3.118) is in the nature of a change of lagrangian coordinates. Then with the notation $\partial_i = \partial/\partial x_i$, the strains incurred in a disturbance $\varepsilon\, \hat{u}_i\, (x^{\circ}_j)$ from the equilibrium configuration are given by

$$\beta_{mn} = \varepsilon \frac{1}{2} (\partial_m \hat{u}_n + \partial_n \hat{u}_m) + \frac{\varepsilon^2}{2} \partial_m \hat{u}_i \partial_n \hat{u}_i$$

or, to prepare further calculations,

$$\beta_{mn} = \varepsilon \beta^{(1)}_{mn} + \frac{\varepsilon^2}{2} \beta^{(2)}_{mn} \tag{3.119}$$

with
$$\beta^{(1)}_{mn} = \frac{1}{2} (\partial_m \hat{u}_n + \partial_n \hat{u}_m) \tag{3.120}$$

$$\beta^{(2)}_{mn} = \partial_m \hat{u}_i \partial_n \hat{u}_i$$

The energy per unit volume in the new reference configuration is denoted by W_x and will be a function of the strains (3.119). From a Maclaurin expansion

$$W_x = W^{\circ}_x + \beta_{mn} \left(\frac{\partial W_x}{\partial \beta_{mn}}\right)_{\circ} + \frac{1}{2} \beta_{mn} \beta_{pq} \left(\frac{\partial^2 W_x}{\partial \beta_{mn} \partial \beta_{pq}}\right)_{\circ}$$

The stresses defined as partial derivatives of the energy will be

$$\sigma_{mn} = \left(\frac{\partial W_x}{\partial \beta_{mn}}\right)_{\circ} + \beta_{pq} \left(\frac{\partial^2 W_x}{\partial \beta_{mn} \partial \beta_{pq}}\right)_{\circ}$$

$$= \sigma^{\circ}_{mn} + E^{pq}_{mn} \beta_{pq}$$

Substitution of (3.119) into

$$W_x = W^{\circ}_x + \sigma^{\circ}_{mn} \beta_{mn} + \frac{1}{2} E^{pq}_{mn} \beta_{mn} \beta_{pq}$$

and grouping of powers in ε yields

$$W_x = W^\circ{}_x + \varepsilon\ (\sigma^\circ{}_{mn}\ \beta^{(1)}_{mn}) + \varepsilon^2\ (\tfrac{1}{2}\ E^{pq}_{mn}\ \beta^{(1)}_{mn}\ \beta^{(1)}_{pq} + \tfrac{1}{2}\ \sigma^\circ{}_{mn}\ \beta^{(2)}_{mn}) + \ldots$$

Since the new reference configuration is in equilibrium, the first variation of the total potential vanishes under all kinematically admissible disturbances and we easily obtain the equilibrium equations

$$\partial_m\ \sigma^\circ{}_{mn} + \bar{X}_n = 0$$

$$n_m\ \sigma^\circ{}_{mn} = \bar{q}_n$$

where $\bar{X}_n$ and $\bar{q}_n$ are the prescribed applied forces described in the new geometry and n_i the direction cosines of the outward normal to the new boundary surface ∂R_x. The stability criterion is now

$$\frac{1}{2} \int_{R_x} (E^{pq}_{mn}\ \beta^{(1)}_{mn}\ \beta^{(1)}_{pq} + \sigma^\circ{}_{mn}\ \beta^{(2)}_{mn})\ dR_x > 0$$

It bears very close ressemblance to the old form and can in fact be shown to be identical if due account is taken of all effects due to the change (3.118) in lagrangian coordinates. It has apparently the advantage that the expressions (3.120)of the first order strains are simpler than their counterpart (3.93) and analogous to their simplified values (3.108). In general however this advantage is more than offset by the necessity to apply the boundary conditions to a distorted configuration.
In the case discussed under 3.2.3. where the linear elasticity approximation is adequate to follow the equilibrium path up to a critical situation, both formulations of the stability problem are identical. In the first formulation the first order strains can be approximated by (3.108), in the second one may simply ignore the change in lagrangian coordinates and perform all operations on the unstressed configuration.

3.2.5. Stability of membrane loaded flat plates

The assumptions are that the plate structure is perfectly symmetrical with respect to the middle plane $a_3 = 0$ and similarly for the applied set of external loads and kinematical boundary conditions. Then we are in a case where, provided we remain in the elastic range of material behavior, (3.107) and (3.108) are valid to determine the set of equilibrium stresses. The energy density $W^{(1)}$ has the classical structure of linear elasticity in the perturbation displacements $\hat{u}_i$. Moreover, because of the assumed symmetry, it must be independant of a change of orientation of the a_3 axis. Only $\gamma^{(1)}_{13}$ and $\gamma^{(1)}_{23}$ do change sign with the substitution $-a_3$ for a_3, hence in the quadratic form $W^{(1)}$ those strains are not coupled with the other ones and the quadratic form is a sum of two independant positive definite forms in each group of variables :

$$W^{(1)} = B\,(\gamma^{(1)}_{11}, \gamma^{(1)}_{12}, \gamma^{(1)}_{22}, \gamma^{(1)}_{33}) + A\,(\gamma^{(1)}_{13}, \gamma^{(1)}_{23}) \qquad (3.121)$$

In order to establish with mathematical accuracy the considerable simplifications that can be introduced in the stability criterion, it will be sufficient to introduce the simplest assumptions on the structure of the perturbation displacement field :

$$\begin{aligned} \hat{u}_1 &= v_1\,(a_1, a_2) + a_3\,\alpha_1\,(a_1, a_2) \\ \hat{u}_2 &= v_2\,(a_1, a_2) + a_3\,\alpha_2\,(a_1, a_2) \\ \hat{u}_3 &= w\,(a_1, a_2) \end{aligned} \qquad (3.122)$$

They produce the following strains

$$\left.\begin{aligned} \gamma^{(1)}_{33} &= D_3\,\hat{u}_3 = 0 \\ \gamma^{(1)}_{11} &= D_1\,\hat{u}_1 = D_1\,v_1 + a_3\,D_1\,\alpha_1 \\ \gamma^{(1)}_{22} &= D_2\,\hat{u}_2 = D_2\,v_2 + a_3\,D_2\,\alpha_2 \\ 2\,\gamma^{(1)}_{12} &= D_1\,\hat{u}_2 + D_2\,\hat{u}_1 = (D_1\,\alpha_2 + D_2\,\alpha_1)a_3 \\ &\qquad + (D_1\,v_2 + D_2\,v_1) \end{aligned}\right\} \qquad (3.123)$$

$$\left.\begin{aligned} 2\gamma^{(1)}_{13} &= D_1\,\hat{u}_3 + D_3\,\hat{u}_1 = \alpha_1 + D_1\,w \\ 2\gamma^{(1)}_{23} &= D_2\,\hat{u}_3 + D_3\,\hat{u}_2 = \alpha_2 + D_2\,w \end{aligned}\right] \qquad (3.124)$$

Integration across the thickness $2h(a_1, a_2)$ of the plate, induces uncoupling of terms due to their even or odd dependance on a_3 :

$$\int_{-h}^{h} B\,da_3 = B_v\,(D_1v_1,\ D_2v_2,\ D_1v_2 + D_2v_1) + B_\alpha\,(D_1\alpha_1,\ D_2\alpha_2,\ D_1\alpha_2 + D_2\alpha_1) \qquad (3.125)$$

$$\int_{-h}^{h} A\,da_3 = K\,(\alpha_1 + D_1w,\ \alpha_2 + D_2w) \qquad (3.126)$$

Similarly we assume that the equilibrium membrane stresses have the following simple structure

$$s^\circ_{11}\,(a_1, a_2) \qquad s^\circ_{12}\,(a_1, a_2) \qquad s^\circ_{22}\,(a_1, a_2)$$

$$s^\circ_{13} = 0 \qquad s^\circ_{23} = 0 \qquad s^\circ_{33} = 0$$

whence that part of the stability criterion that depends on the equilibrium stresses reduces to the volume integral of

$$\frac{1}{2}\,s^\circ_{11}\,\gamma^{(2)}_{11} + s^\circ_{12}\,\gamma^{(2)}_{12} + \frac{1}{2}\,s^\circ_{22}\,\gamma^{(2)}_{22}$$

where, in accordance with the previous assumptions,

$$\gamma^{(2)}_{11} = (D_1v_1 + a_3\,D_1\,\alpha_1)^2 + (D_1\,v_2 + a_3\,D_1\,\alpha_2)^2 + (D_1w)^2$$

$$\gamma^{(2)}_{22} = (D_2v_1 + a_3\,D_2\,\alpha_1)^2 + (D_2\,v_2 + a_3\,D_2\,\alpha_2)^2 + (D_2w)^2$$

$$\gamma^{(2)}_{21} = (D_1v_1 + a_3\,D_1\,\alpha_1)(D_2v_1 + a_3\,D_2\,\alpha_1) + (D_1v_2 + a_3D_1\alpha_2)(D_2v_2 + a_3\,D_2\,\alpha_2) + (D_1w)(D_2w)$$

Again, integration across the thickness of the plate produces uncoupling of terms

$$\frac{1}{2}\int_{-h}^{h}\left[s^{\circ}_{11}\ \gamma^{(2)}_{11} + 2\, s^{\circ}_{12}\,\gamma^{(2)}_{12} + s^{\circ}_{22}\,\gamma^{(2)}_{22}\right] da_3$$

$$= S_v + S_\alpha + S_w \qquad (3.127)$$

where

$$S_v = h\, s^{\circ}_{11}\{(D_1 v_1)^2 + (D_1 v_2)^2\} + 2\, s^{\circ}_{12} h$$

$$\{D_1 v_1 D_2 v_1 + D_1 v_2 D_2 v_2\} + s^{\circ}_{22} h\{(D_2 v_1)^2 + (D_2 v_2)^2\}$$

$$S_\alpha = \frac{1}{3} h^3 s^{\circ}_{11}\{(D_1\alpha_1)^2 + (D_1\alpha_2)^2\} + \frac{2}{3} h^3 s^{\circ}_{12} \qquad (3.128)$$

$$\{D_1\alpha_1 D_2\alpha_1 + D_1\alpha_2 D_2\alpha_2\} + \frac{1}{3} h^3 s^{\circ}_{22}\{(D_2\alpha_1)^2 + (D_2\alpha_2)^2\}$$

$$S_w = h\, s^{\circ}_{11}(D_1 w)^2 + 2h\, s^{\circ}_{12} D_1 w\, D_2 w + h\, s^{\circ}_{22}(D_2 w)^2$$

The general stability criterion (3.98) thus assumes the form

$$\iint (B_v + S_v)\, da_1\, da_2 + \iint (B_\alpha + S_\alpha)\, da_1\, da_2$$

$$+ \iint (K + S_w)\, da_1\, da_2 > 0 \qquad (3.129)$$

where the integrals are extended over the area of the plate.
The first term depends only on the "in plane" perturbations v_1 and v_2 and is decoupled from the other two. It will now be shown that this first term can be canceled because its contribution is always positive.
We first examine the particular case of in plane perturbations only where

$$D_1 v_1 = 0 \qquad D_2 v_2 = 0 \qquad D_1 v_2 + D_2 v_1 = 0$$

and consequently $B_v = 0$. The in plane perturbation field is then of rigid body type

$$v_1 = b_1 - \omega a_2 \qquad v_2 = b_2 + \omega a_1$$

and the contribution of S_v to the criterion is, after computation, found to be

$$\iint S_v\, da_1\, da_2 = \omega^2 \iint h(s^{\circ}_{11} + s^{\circ}_{22})\, da_1\, da_2 \quad .$$

Should this be negative, the criterion would be violated at any load level and the problem would not make sense. This case must be ruled out, either because the kinematical boundary conditions do not allow the small rigid body rotation ω, or, as we shall see, by a condition imposed on the external loads. Indeed, the equilibrium equations

$$D_1 (2hs^\circ_{11}) + D_2 (2hs^\circ_{12}) + \int_{-h}^{h} \bar{A}_1 \, da_3 = 0$$

$$D_1 (2hs^\circ_{12}) + D_2 (2hs^\circ_{22}) + \int_{-h}^{h} \bar{A}_2 \, da_3 = 0$$

multiplied respectively by a_1 and a_2, added and integrated over the area of the plate, yield

$$2 \iint \{ a_1 D_1 (h s^\circ_{11}) + a_1 D_2 (h s^\circ_{12}) + a_2 D_1 (h s^\circ_{12}) + a_2 D_2 (h s^\circ_{22}) \} da_1 da_3 = - \iiint_{-h}^{h} (a_1 \bar{A}_1 + a_2 \bar{A}_2) da_1 da_2 da_3$$

Transforming the left-hand side by integration by parts and noting that

$$2h (\ell_1 s^\circ_{11} + \ell_2 s^\circ_{12}) = \int_{-h}^{h} \bar{p}_1 \, da_3$$

$$2h (\ell_1 s^\circ_{12} + \ell_2 s^\circ_{22}) = \int_{-h}^{h} \bar{p}_2 \, da_3$$

there comes

$$2 \iint h(s^\circ_{11} + s^\circ_{22}) \, da_1 da_2 = \iiint_{-h}^{h} (a_1 \bar{A}_1 + a_2 \bar{A}_2) da_1 da_2 da_3 + \oint_{-h}^{h} (a_1 \bar{p}_1 + a_2 \bar{p}_2) da_3 \, ds \tag{3.130}$$

and the stability condition in the absence of kinematical boundary conditions is that the imposed loads give a positive value to the right-hand side of this equation.

Returning now to the general case where all the in plane strains $D_1 v_1$, $D_2 v_2$ and $D_1 v_2 + D_2 v_1$ are not zero, the contribution from B_v is positive. It can be counterbalanced by a negative contribution from S_v only if there are equilibrium stresses of appropiate sign and of an order of magnitude equal to that of the elastic in plane moduli. Thus some of the corresponding

equilibrium strains would have to be of an order of magnitude equal to unity, contradicting assumptions (3.107). The contribution of the first term in (3.129) is thus positive in any case and the criterion can be reduced to the one

$$\iint (B_\alpha + S_\alpha)\, da_1\, da_2 + \iint (K + S_w)\, da_1\, da_2 > 0 \qquad (3.131)$$

involving only the skew symmetric disturbances α_1, α_2 and w.
In a similar manner it will now be shown that the contribution from S_α may be disregarded. Again consider first the special case where

$$D_1\, \alpha_1 = 0 \qquad D_2\, \alpha_2 = 0 \qquad D_1\, \alpha_2 + D_2\, \alpha_1 = 0$$

so that $B_\alpha = 0$. The disturbance field is now of the form

$$\hat{u}_1 = a_3\, (c_1 - \phi\, a_2) \qquad \hat{u}_2 = a_3\, (c_2 + \phi a_1)$$

and represents rigid body motions of sheets parallel to the middle plane. The contribution from S_α can be worked out as due to the twist ϕ between sheets only

$$\frac{1}{3}\, \phi^2 \iint h^3\, (s^\circ_{11} + s^\circ_{22})\, da_1\, da_2 \qquad (3.132)$$

and will be compared to the contribution from K. To this purpose we take a typical form of K

$$K = \frac{1}{2}\, G\, 2h\, (\alpha_1^2 + \alpha_2^2)$$

where G is an average shear modulus between the sheets. Then

$$\iint K\, da_1\, da_2 = G \iint \{\, h\, (c_1 - \phi\, a_2)^2 + h\, (c_2 + \phi\, a_1)^2 \,\}\, da_1\, da_2$$

The minimum of this for a twist ϕ is found by adjusting c_1 and c_2 so that the center of twist is located at the center of gravity of the middle plane. The cartesian axes have their origin at this center if

$$\iint h\, a_1\, da_1\, da_2 = 0 \qquad \iint h\, a_2\, da_1\, da_2 = 0 \quad .$$

The minimal contribution is then

$$\iint K\, da_1\, da_2 = \phi^2\, G \iint h\, (a_1^2 + a_2^2)\, da_1\, da_2$$

This is positive and much larger than (3.132). In fact, if s denotes the order of magnitude of an equilibrium stress, $\bar{h}$ the average half thickness of the plate and ρ the radius of gyration of the middle plane

$$\iint h\, (a_1^2 + a_2^2)\, da_1\, da_2 = \bar{h}\, \rho^2\, \text{Area}$$

the ratio of the two terms is

$$\frac{1}{3}\frac{s}{G}\left(\frac{\bar{h}}{\rho}\right)^2 << 1 \quad .$$

Thus the contribution from S_α is negligible compared to that of K.
In the case that the strains $D_1\, \alpha_1$, $D_2\, \alpha_2$ and $D_1\, \alpha_2 + D_2\, \alpha_1$ are not all zero, the same arguments as between S_v and B_v show that the contribution from S_α can be neglected before that of B_α.
The stability criterion thus finally reduces to

$$\iint (K + B_\alpha + S_w)\, da_1\, da_2 > 0 \qquad (3.133)$$

$K + B_\alpha$ is in fact any of the classical flexure energies per unit plate area including shearing deformations. Under the Kirchhoff hypothesis we would have furthermore

$$\alpha_1 + D_1\, w = 0 \qquad \alpha_2 + D_2\, w = 0$$

so that the shearing energy K would disappear also, and the flexure energy B_α would be classically expressed in terms of the second derivatives $D_1^2\, w$, $D_2^2\, w$, $D_1\, D_2\, w$ of the transverse disturbance.

3.2.6. Other derivations of the plate stability criterion

Another method for deriving the plate stability criterion makes use of the equilibrium configuration as reference configuration; our lagrangian coordinates are here the x°_i. The criterion states that the

increase in strain energy due to a kinematically admissible disturbance must be larger than the additional work performed by the external loads. Consider then a transverse disturbance field

$$u_1 = \varepsilon\, a_3\, \alpha_1\,(x^\circ_1, x^\circ_2) \qquad u_2 = \varepsilon\, a_3\, \alpha_2\,(x^\circ_1, x^\circ_2)$$

$$u_3 = \varepsilon\, w\,(x^\circ_1, x^\circ_2) \qquad \text{(note that } a_3 \equiv x^\circ_3\text{)}$$

The in plane components vanish in the middle plane $a_3 = 0$ so that no additional work is performed by the external in plane loads. It will however be seen that the additional strain energy is not neccessarily positive and that the stability criterion requiring it to be positive is meaning ful. A first part of the additional strain energy is the bending energy (it is not coupled to the membrane energy).
Neglecting terms of order ε^4 it can be calculated from the linearized strain measures

$$\varepsilon_{11} = \partial_1 u_1 = \varepsilon\, a_3\, \partial_1 \alpha_1 \qquad \varepsilon_{22} = \partial_2 u_2 = \varepsilon\, a_3\, \partial_2 \alpha_2$$

$$2\varepsilon_{12} = \partial_1 u_2 + \partial_2 u_1 = \varepsilon\, a_3\, (\partial_1 \alpha_2 + \partial_2 \alpha_1)$$

$$2\varepsilon_{13} = \partial_1 u_3 + \partial_3 u_1 = \varepsilon\, (\alpha_1 + \partial_1 w)$$

$$2\varepsilon_{23} = \varepsilon\, (\alpha_2 + \partial_2 w)$$

and will be denoted by ε^2 B. B is in fact the equivalent in the equilibrium configuration of our $\iint (B_\alpha + K)\, da_1\, da_2$ of section 3.2.5.
Although the linearized in plane strains vanish for $a_3 = 0$, the exact membrane strains induced by the disturbance field do not. In fact the membrane strains calculated from the Green strain measures are easily found to be

$$\gamma_{11} = \frac{1}{2}\varepsilon^2\,(\partial_1 w)^2 \qquad \gamma_{22} = \frac{1}{2}\varepsilon^2\,(\partial_2 w)^2$$

$$\gamma_{12} = \frac{1}{2}\varepsilon^2\,\partial_1 w\,\partial_2 w \qquad (a_3 = 0)$$

Those strains are easily seen to be additive to the equilibrium strains $\gamma^\circ_{\alpha\beta}$, because they are induced by a displacement component that is absent in the equilibrium strains. Thus the new value of the membrane energy

would be (the greek summation indices run from 1 to 2)

$$\frac{1}{2}\iint E^{\lambda\mu}_{\alpha\beta}\,(\gamma^{\circ}_{\alpha\beta} + \frac{1}{2}\,\epsilon^2\,\partial_\alpha w\,\partial_\beta w)(\gamma^{\circ}_{\lambda\mu} + \frac{1}{2}\epsilon^2\partial_\lambda w\partial_\mu w)\,dx_1\,dx_2$$

Or, neglecting again term of order ϵ^4, an additional membrane strain energy (interference energy) of order ϵ^2 appears equal to

$$\epsilon^2\,\frac{1}{4}\iint E^{\lambda\mu}_{\alpha\beta}\,(\gamma^{\circ}_{\alpha\beta}\,\partial_\lambda w\,\partial_\mu w + \gamma^{\circ}_{\lambda\mu}\,\partial_\alpha w\,\partial_\beta w)\,dx_1\,dx_2$$

or, using the symmetry properties of the elastic moduli,

$$\epsilon^2\,\frac{1}{2}\iint E^{\lambda\mu}_{\alpha\beta}\,\gamma^{\circ}_{\lambda\mu}\,\partial_\alpha w\,\partial_\beta w\,dx_1\,dx_2$$

$$=\epsilon^2\,\frac{1}{2}\iint \sigma^{\circ}_{\alpha\beta}\,\partial_\alpha w\,\partial_\beta w\,dx_1\,dx_2$$

Thus, if there are compressive equilibrium stresses, the transverse displacement w can actually relieve the membrane strain energy. If this energy release is higher than the bending energy, the stability criterion is violated. This means in general that the plate will "bifurcate" in a new stable equilibrium configuration involving bending. As a matter of fact the stability criterion that has been found is, after suppression of the factor ϵ^2

$$\mathcal{D} + \frac{1}{2}\iint \sigma^{\circ}_{\alpha\beta}\,\partial_\alpha w\,\partial_\beta w\,dx_1\,dx_2 > 0 \qquad (3.134)$$

completely analogous to (3.133). It becomes identical to it when, as was shown to be licit, one ignores the change in metric induced by the change (3.118) of lagrangian coordinates.

This new derivation of the simplified stability criterion for plates certain ly showns it to be neccessary. But, inasmuch as it ignores other types of disturbances, it does not prove it to be sufficient, although engineering intuition is perhaps enough to convince that the addition of displacement disturbances of membrane type is necessarily stabilizing.

It is rather amusing to record how, by adding in fact displacement disturbances of membrane type, Timoshenko (3.10) obtains the same criterion by a devious way. Since the transverse disturbance produces membrane strains, let us cancel them by suitable membrane displacements v_1 and v_2. We should then have that

$$\partial_1 v_1 + \frac{1}{2}\,(\partial_1 v_1)^2 + \frac{1}{2}\,(\partial_1 v_2)^2 + \frac{1}{2}\,\epsilon^2\,(\partial_1 w)^2 = 0$$

$$\partial_2 v_2 + \frac{1}{2} (\partial_2 v_1)^2 + \frac{1}{2} (\partial_2 v_2)^2 + \frac{1}{2} \varepsilon^2 (\partial_2 w)^2 = 0$$

$$\partial_1 v_2 + \partial_2 v_1 + \partial_1 v_1 \partial_2 v_1 + \partial_1 v_2 \partial_2 v_2 + \varepsilon^2 \partial_1 w \partial_2 w = 0$$

However it is accurate enough to write

$$\partial_1 v_1 = -\frac{1}{2} \varepsilon^2 (\partial_1 w)^2 \qquad \partial_2 v_2 = -\frac{1}{2} \varepsilon^2 (\partial_2 w)^2$$

$$\partial_1 v_2 + \partial_2 v_1 = -\varepsilon^2 \partial_1 w \partial_2 w \tag{3.135}$$

since it appears from this that the neglected terms are of order ε^4. To be quite precise it is necessary to recall that the material rotation $\partial_1 v_2 - \partial_2 v_1$ is of the same order of magnitude as the linearized strains, so that all gradients $\partial_\alpha v_\beta$ are of the same order.

So doing, the additional membrane energy is now canceled and the additional energy in the criterion consists of $\varepsilon^2 B$ alone. However we have now introduced an additional work performed by the external loads

$$T = \iint (\bar{X}_1 v_1 + \bar{X}_2 v_2) \, dx_1 \, dx_2 + \oint (\bar{p}_1 v_1 + \bar{p}_2 v_2) \, ds$$

and the stability criterion is now

$$\varepsilon^2 B > T \tag{3.136}$$

In the expression for T we substitute from the equilibrium equations

$$\bar{X}_1 = -\partial_1 \sigma^\circ_{11} - \partial_2 \sigma^\circ_{12} \qquad \bar{X}_2 = -\partial_1 \sigma^\circ_{12} - \partial_2 \sigma^\circ_{22}$$

$$\bar{p}_1 = n_1 \sigma^\circ_{11} + n_2 \sigma^\circ_{12} \qquad \bar{p}_2 = n_1 \sigma^\circ_{12} + n_2 \sigma^\circ_{22}$$

and perform an integration by parts in the first term, then

$$T = \iint \{ \sigma^\circ_{11} \partial_1 v_1 + \sigma^\circ_{12} (\partial_1 v_2 + \partial_2 v_1) + \sigma^\circ_{22} \partial_2 v_2 \} \, dx_1 \, dx_2$$

or, using (3.135)

$$T = -\frac{1}{2} \varepsilon^2 \iint \sigma_{\alpha\beta} \partial_\alpha w \partial_\beta w \, dx_1 \, dx_2 \tag{3.137}$$

Thus, when substituted into (3.136) we fall back again on (3.134). In this way Timoshenko is able to introduce a potential energy of the external loads in the second variation of the energy, although it is by nature absent in the general criterion (3.98).

3.2.7. Stability of axially loaded beams

Critical loads of space structures have been discussed in several other publications (3.12,3.13 , 3,14). Here the conditions of stability are first derived from the general energy criterion in an attempt to bring out clearly all the assumptions underlying the usual simplified version of the stability analysis. In particular the reduction of torsional rigidity due to axial stresses is taken into account; it is significant when the members of a frame are of thin-walled open sections. This feature has been investigated both analytically and experimentally (3.15,3.16 , 3.17,3.18 ,3.19 ,3.20) and can be incorporated easily in a finite element solution.

The first assumption is that a space frame member has a perfectly straight axis of centroids and is perfectly axially loaded in the equilibrium state. This restriction gives direct access to the critical point and avoids the iterative or incremental procedures that become necessary otherwise (3.20). Instead of the notations of the general theory we shall use here (x, y, z) for the lagrangian coordinates in the unstressed configuration. The x axis is the line of centroids, the y and z axes are principal of inertia in each cross section; thus

$$\iint y\, dy\, dz = 0 \qquad \iint z\, dy\, dz = 0 \qquad \iint yz\, dy\, dz = 0 \quad (3.138)$$

The disturbance field is resolved in terms of rigid body motions of each cross section (two translations $\varepsilon v(x)$ and $\varepsilon w(x)$ and a rotation $\varepsilon\phi(x)$ about the (convected) x axis plus an axial disturbance field $\varepsilon u(x, y, z)$ that is able to take into account axial translations, bending rotations and warping. From the total displacement field

$$u_t = u^\circ (x) + \varepsilon\, u\, (x, y, z)$$

$$v_t = \varepsilon\, v\, (x) + y\, (\cos \varepsilon\phi - 1) - z \sin \varepsilon\phi$$

$$w_t = \varepsilon\, w\, (x) + y \sin \varepsilon\phi + z\, (\cos \varepsilon\phi - 1)$$

follows after expansion in powers of ε

$$\hat{u} = u(x, y, z) \quad \hat{v} = v(x) - z\phi(x) \quad \hat{w} = w(x) + y\phi(x) \tag{3.139}$$

In contrast to the assumptions of the general theory, eq. (3.91), there are also ε^2 and higher order terms in the displacement disturbances for v_t and w_t; they have however no effect on the second variation.
We assume the following strain energy density

$$W = \frac{E}{2}\varepsilon_x^2 + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)$$

so that the stress-strain relations are

$$\sigma_x = E\,\varepsilon_x \qquad \sigma_y = 0 \qquad \sigma_z = 0$$

$$\tau_{xy} = G\,\gamma_{xy} \qquad \tau_{xz} = G\,\gamma_{xz} \quad \tau_{yz} = G\,\gamma_{yz}$$

and are related to an isotropic material together with the Saint Venant's semi-inverse assumptions $\sigma_y = 0 \quad \sigma_z = 0$.
Calculation of the first order strains

$$\dot{\varepsilon}_x^{(1)} = \partial\hat{u}/\partial x = \partial u/\partial x$$

$$\gamma_{xy}^{(1)} = \partial\hat{u}/\partial y + \partial\hat{v}/\partial x = \partial u/\partial y + \dot{v} - z\,\dot{\phi}$$

$$\gamma_{xz}^{(1)} = \partial\hat{u}/\partial z + \partial\hat{w}/\partial x = \partial u/\partial z + \dot{w} + y\,\dot{\phi}$$

$$\gamma_{yz}^{(1)} = \partial\hat{v}/\partial z + \partial\hat{w}/\partial y = 0$$

(a dot denotes the derivative of functions of x alone), gives as first term of the stability criterion

$$f_1 = \frac{1}{2}\int \{E\,(\frac{\partial u}{\partial x})^2 + G\,(\frac{\partial u}{\partial y} + \dot{v} - z\,\dot{\phi})^2 + G\,(\frac{\partial u}{\partial z} + \dot{w} + y\,\dot{\phi})^2\}\,dx\,dy\,dz \tag{3.139}$$

The second term requires the determination of the coefficient of $\frac{1}{2}\varepsilon^2$ in the axial Green strain measure

$$\frac{\partial u_t}{\partial x} + \frac{1}{2}\left(\frac{\partial u_t}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v_t}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w_t}{\partial x}\right)^2$$

this turns out to be

$$\varepsilon_x^{(2)} = \left(\frac{\partial u}{\partial x}\right)^2 + (\dot{v} - z\,\dot{\phi})^2 + (\dot{w} + y\,\dot{\phi})^2 \quad .$$

The second term of the stability criterion is then

$$f_2 = \frac{1}{2}\int \{\, s(x)(\dot{v} - z\,\dot{\phi})^2 + s(x)(\dot{w} + y\,\dot{\phi})^2 \,\}\, dx\, dy\, dz \tag{3.140}$$

where s(x) stands for the axial stress $\sigma^\circ_x(x)$ at equilibrium and where the term involving $(\partial u/\partial x)^2$ has been neglected by comparison with the similar one in f_1 because $|\, s \,| << E$.

3.2.8. The contribution of warping to the critical disturbance

We wish to solve the problem up to its formulation in terms of functions of x alone. To this effect the nature of the u(x, y, z) disturbance must be elucidated. Since it concerns only the functional f_1, we have by taking variation of f_1 with respect to this unknown function

$$\int \{\, E\,\frac{\partial u}{\partial x}\,\frac{\partial}{\partial x}\,\delta u + G\,(\frac{\partial u}{\partial y} + \dot{v} - z\,\dot{\phi})\,\frac{\partial}{\partial y}\,\delta u$$

$$+ G\,(\frac{\partial u}{\partial z} + \dot{w} + y\,\dot{\phi})\,\frac{\partial}{\partial z}\,\delta u \,\}\, dx\, dy\, dz = 0$$

After integration by parts it turns out that u should satisfy the following partial differential equation

$$E\,\frac{\partial^2 u}{\partial x^2} + G\,(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = 0 \tag{3.141}$$

the lateral boundary condition

$$\frac{\partial u}{\partial n} + \frac{\partial y}{\partial n}(\dot{v} - z\,\dot{\phi}) + \frac{\partial z}{\partial n}(\dot{w} + y\,\dot{\phi}) = 0 \tag{3.142}$$

where $(\partial y/\partial n,\ \partial z/\partial n)$ are the direction cosines of the outward normal to the cross-sectional boundary, and the end conditions

$$\frac{\partial u}{\partial x}\,\delta u = 0 \qquad \text{in } x = 0 \quad \text{and} \quad x = l \tag{3.143}$$

The lateral boundary condition, which expresses tangency of the shear stresses, suggests that we set

$$u = - y \dot{v} - z \dot{w} + \dot{\phi} X (y, z) + \omega (x, y, z) \qquad (3.144)$$

The warping function X would then satisfy

$$\frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2} = 0 \qquad (3.145)$$

with $$\frac{\partial X}{\partial n} = z \frac{\partial y}{\partial n} - y \frac{\partial z}{\partial n} \qquad (3.146)$$

while the "residual" warping function ω would satisfy

$$E \frac{\partial^2 \omega}{\partial x^2} + G \left(\frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}\right) = E y \ddot{v} + E z \ddot{w} - \dot{X} E \ddot{\phi} \qquad (3.147)$$

with $$\partial\omega/\partial n = 0 \qquad (3.148)$$

In (3.144) the first two terms represent rotation of the cross-sections that keep then orthogonal to the neutral fiber, X (y, z) will be later recognized as the free Saint-Venant warping under twist. From (3.147) it appears that the residual warping is induced by the rates of change of curvature and by the third derivative of the torsion.

At this stage it might be possible to further amplify the definition of warping by introducing transverse shear warping functions B (y, z) and C (y, z) and a secondary warping function under twist A (y, z) satisfying differential equations and boundary conditions like

$$G \left(\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial z^2}\right) = E y \qquad G \left(\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) = E z$$

$$G \left(\frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right) = E X$$

$$\partial B/\partial n = 0 \qquad \partial C/\partial n = 0$$

$$\partial A/\partial n = 0$$

Then setting $\omega = \ddot{v} B + \ddot{w} C - \ddot{\phi} A$

One would satisfy (3.147) and (3.148) except for terms involving the fifth derivative of v, w and ϕ.

In what follows we will be satisfied with the approximation that neglects the residual warping ω. Then introducing the moments of inertia

$$I_y = \iint y^2 \, dy \, dz \qquad I_z = \iint z^2 \, dy \, dz \quad ,$$

the polar moment of inertia

$$I_p = I_y + I_z \quad ,$$

the cross-sectional area

$$A = \iint dy \, dz \quad ,$$

the integrals over the torsional warping function

$$P = \iint X^2 \, dy \, dz \qquad Q = \iint Xy \, dy \, dz \qquad R = \iint Xz \, dy \, dz \tag{3.149},$$

and the free torsional rigidity

$$J = \iint \{ (\frac{\partial X}{\partial y} - z)^2 + (\frac{\partial X}{\partial z} + y)^2 \} \, dy \, dz \tag{3.150},$$

the stability criterion appears in a form involving only functionals over functions of x alone

$$\int_0^L \frac{E}{2} (I_y \ddot{v}^2 + I_z \ddot{w}^2 + P \ddot{\phi}^2 - 2 Q \ddot{v} \ddot{\phi} - 2 R \ddot{w} \ddot{\phi}) \, dx + \int_0^L \frac{G}{2} J \dot{\phi}^2 \, dx + \int_0^L \frac{s}{2} (A \dot{v}^2 + A \dot{w}^2 + I_p \dot{\phi}^2) \, dx > 0 \tag{3.151}$$

For $\phi \equiv 0$, $w \equiv 0$, $s(x) = s$ constant, $E I_y = E I$ constant and A constant, this turns into the classical stability condition for the Euler strut

$$\frac{EI}{2} \int_0^L \ddot{v}^2 \, dx + \frac{sA}{2} \int_0^L \dot{v}^2 \, dx > 0$$

For $v \equiv 0$, $w \equiv 0$, P, J and I_p constant, one has the purely torsional buckling problem

$$\frac{EP}{2}\int_0^L \ddot{\phi}^2\,dx + \frac{GJ}{2}\int_0^L \dot{\phi}^2\,dx + \frac{sI_p}{2}\int_0^L \dot{\phi}^2\,dx > 0$$

If one deals with a massive cross section or a box beam, J is of the same order of magnitude as I_p so that in the elastic range where $|s| << G$, the criterion for pure torsion is always satisfied. For the same reason the last term of the last integral in (3.151) can be dropped in the general problem. It is only for thin-walled open sections that the order of magnitude of J becomes so small that purely torsional buckling is a possibility.

The P term takes approximately into account the restraints against torsional warping due to end conditions and changes in the rate of twist presented by the critical disturbance. For instance if the section $x = 0$ is not allowed to warp, the kinematical boundary condition $\dot{\phi}(0) = 0$ must be applied. This condition is $\ddot{\phi}(0) = 0$ if it is free to warp.

We shall now investigate the significance of the Q and R terms that produce coupling between bending and torsion apart from the coupling that could accrue from the end conditions.

It will be necessary therefore to recapitulate briefly the properties of the torsional warping function X (y, z) by reference to the de Saint-Venant torsion problem (3.21).

3.2.9. The torsional warping function and the flexure-torsion center

In the Barré de Saint-Venant torsion problem all quantities are independant of x, all stresses are zero except for τ_{xy} and τ_{xz}. The twist rate θ being constant, one usually introduces the Prandtl torsion function $\Theta_{(y,z)}$ in order to free the equilibrium equations in volume

$$\tau_{xz} = - G\theta \frac{\partial \Theta}{\partial y} \qquad \tau_{xy} = + G\theta \frac{\partial \Theta}{\partial z} \tag{3.152}$$

The complementary energy is then

$$\frac{L}{2G} \iint (\tau^2_{xz} + \tau^2_{xy}) dy\, dz = \frac{LG}{2} \theta^2 \iint \left[(\frac{\partial \Theta}{\partial y})^2 + (\frac{\partial \Theta}{\partial z})^2 \right] dy\, dz$$

Setting the displacements $v = 0$ and $w = 0$ in section $x = 0$ and $v = - L\,\theta\, z \quad w = + L\,\theta\, y$ in the end section $x = L$, the complementary potential energy is

$$- \iint (-L\theta z \tau_{xy} + L\theta y \tau_{xz}) dydz = LG\theta^2 \iint (y \frac{\partial \Theta}{\partial y} + z \frac{\partial \Theta}{\partial z}) dydz$$

The total complementary energy (divided by the constant $LG\theta^2$)

$$\iint \left[\frac{1}{2} (\frac{\partial \Theta}{\partial y})^2 + \frac{1}{2} (\frac{\partial \Theta}{\partial z})^2 + y \frac{\partial \Theta}{\partial y} + z \frac{\partial \Theta}{\partial z} \right] dydz \quad \min$$

should be minimum under the additional assumption that the surface equilibrium condition

$$\ell\, \tau_{xy} + m \tau_{xz} = \frac{\partial y}{\partial n} \tau_{xy} + \frac{\partial z}{\partial n} \tau_{xz} = 0 \tag{3.153}$$

be satisfied along the outward limiting contour c_0 and along the inner contours c_i of inside cavities, if the section is multiply-connected.

Since $$\frac{\partial y}{\partial n} = + \frac{\partial z}{\partial s} \qquad \frac{\partial z}{\partial n} = - \frac{\partial y}{\partial s} \tag{3.154}$$

and in view of (3.152), this is equivalent to requiring

$$\frac{\partial \Theta}{\partial s} = 0 \qquad \text{along } c_0 \text{ and } c_i$$

Without loss of generality one can take

$$\theta = 0 \qquad \text{along } c_0$$

$$\theta = \alpha_i \qquad \text{unknown constant along } c_i \tag{3.155}$$

The minimization of (3.153) produces the Euler equation

$$\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} + 2 = 0 \tag{3.156}$$

and the natural conditions stemming from the variation of the unknown constants α_i

$$\int_{c_i} \frac{\partial}{\partial n} \left(\theta + \frac{y^2 + z^2}{2} \right) ds = 0 \tag{3.157}$$

Altogether (3.156), (3.155) and (3.157) determine uniquely the Prandtl torsion function. Its relation with the warping function X (y, z) under torsion about the x axis is obtained by considering the displacements

$$u = \theta X \qquad v = - xz\theta \qquad w = + xy\theta \tag{3.158}$$

and computing the stresses

$$\tau_{xy} = G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = G \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Comparison with (3.152) yields

$$- \frac{\partial \theta}{\partial y} = \frac{\partial X}{\partial z} + y \qquad \frac{\partial \theta}{\partial z} = \frac{\partial X}{\partial y} - z \tag{3.159}$$

Those equations are easily put in the form of Cauchy-Riemann

$$\frac{\partial X}{\partial y} = \frac{\partial}{\partial z} \left(\theta + \frac{y^2 + z^2}{2} \right) \qquad \frac{\partial X}{\partial z} = - \frac{\partial}{\partial y} \left(\theta + \frac{y^2 + z^2}{2} \right) \tag{3.159}'$$

and (3.156) follows by elimination of X, while we obtain (3.145) by elimination of θ . Also, multiplying (3.159)' respectively by $\partial y/\partial n$ and $\partial z/\partial n$, adding and using (3.154)

$$\frac{\partial X}{\partial n} = \frac{\partial \theta}{\partial s} + z \frac{\partial y}{\partial n} - y \frac{\partial z}{\partial n} \qquad \text{along } c_0 \text{ and } c_i$$

This reduces to (3.146) since Θ is constant along the contours. Finally multiplying the same equations by $\frac{\partial z}{\partial n}$ and $-\frac{\partial y}{\partial n}$ and adding

$$\frac{\partial}{\partial n}\left(\Theta + \frac{y^2 + z^2}{2}\right) = -\frac{\partial X}{\partial s}$$

Hence conditions (3.157) are transformed into the univalued ness conditions for the axial displacement

$$\int_{c_i} \frac{\partial X}{\partial s}\, ds = 0 \qquad (3.160)$$

If the torsion takes place about the axis $y = y_0$, $z = z_0$, the warping of the cross-section is modified by addition of a rigid body motion

$$X^* = X - z_0\, y + y_0\, z$$

this is easily verified by replacing (3.158) by

$$u = \theta X^* \qquad v = -x(z - z_0)\theta \qquad w = x\,(y - y_0)\theta$$

The actual position (y_0, z_0) of the torsion center under a built in end condition $u \equiv 0$ was defined by Weinstein (3.23) as furnishing a least square property

$$\iint (X - z_0\, y + y_0\, z)^2\, dy\, dz \qquad \text{minimum} \qquad (3.161)$$

With due consideration to the assumptions (3.138) and definitions (3.149) it gives

$$Q = z_0\, I_y \qquad R = -y_0\, I_z \qquad (3.162)$$

Which relate Q and R to the "built in" center of torsion, and also determine implicitly the arbitrary axial translation contained in X. The Weinstein definition may look somewhat arbitrary, it is however backed up by the fact that the same center is obtained as flexure center, that is as the locus where shear forces are to be applied to obtain flexure without torsion (3.21). Our definition of flexure without torsion is not the unhappy one of Timoshenko (3.22), that gave rise to so many misunderstandings, but that of Trefftz (3.24) whose object was to make flexure and torsion energies additive. Thus if $\hat{\tau}_{xy}$ and $\hat{\tau}_{xz}$ denote the shear stresses due to flexure and (3.152) those due to torsion, the

condition of additivity is the vanishing of the interference energy

$$\iint (\hat{\tau}_{xy} \frac{\partial \Theta}{\partial z} - \hat{\tau}_{xz} \frac{\partial \Theta}{\partial y}) \, dy \, dz = 0 \qquad (3.163)$$

Integration by parts

$$\int_{c_0, c_i} (\hat{\tau}_{xy} \frac{\partial z}{\partial n} - \hat{\tau}_{xz} \frac{\partial y}{\partial n}) \Theta \, ds + \iint (\frac{\partial \hat{\tau}_{xz}}{\partial y} - \frac{\partial \hat{\tau}_{xy}}{\partial z}) \, dydz = 0$$

Consideration of (3.154) and of

$$\frac{\partial \hat{\tau}_{xz}}{\partial y} - \frac{\partial \hat{\tau}_{xy}}{\partial z} = G \frac{\partial}{\partial x} (\frac{\partial \hat{w}}{\partial y} - \frac{\partial \hat{v}}{\partial z}) = 2 G \frac{\partial}{\partial x} \hat{\omega}_x$$

modify this to read

$$- \int_{c_0, c_i} \Theta (\hat{\tau}_{xy} \, dy + \hat{\tau}_{xz} \, dz) + 2G \frac{d}{dx} \iint \Theta \, \hat{w}_x \, dy \, dz = 0$$

Now, considering (3.155), the first term generates the quantities

$$\int_{c_i} \hat{\tau}_{xy} \, dy + \hat{\tau}_{xz} \, dz = \iint_{\Omega_i} (\frac{\partial \hat{\tau}_{xz}}{\partial y} - \frac{\partial \hat{\tau}_{xy}}{\partial z}) \, dydz$$

$$= 2 G \frac{d}{dx} \iint_{\Omega_i} \hat{\omega}_x \, dydz$$

where due account was taken of the negative sense of description of the contours c_i around the inner cavities Ω_i, within which the displacement field $\hat{v}$ and $\hat{w}$ is analytically defined. The condition of additivity of the flexure and torsion energies can thus be placed in a kinematical form discovered by Fraeijs de Veubeke (3.21)

$$\iint_{\Omega} \hat{\Theta} \, \hat{\omega}_x \, dydz = 0 \qquad (3.164)$$

the integral being extended to the whole domain Ω limited by c_0 and where

$\hat{\Theta} = \alpha_i$ inside the cavity Ω_i

$\hat{\Theta} = \Theta$ elsewhere.

The Timoshenko condition corresponds to

$\hat{\Theta} = 0$ inside the cavities

$\hat{\Theta} = 1$ elsewhere.

Using (3.159), condition (3.163) is also

$$\iint (y\,\hat{\tau}_{xz} - z\,\hat{\tau}_{xy})\,dy\,dz + \iint (\hat{\tau}_{xy}\frac{\partial X}{\partial y} + \hat{\tau}_{xz}\frac{\partial X}{\partial z})dy\,dz = 0$$

The first term is the moment M of shear stresses with respect to the origin; the second will be modified by integration by parts

$$M = -\int_{c_0,c_i} (\hat{\tau}_{xy}\frac{\partial y}{\partial n} + \hat{\tau}_{xz}\frac{\partial z}{\partial n})\,X\,ds + \iint X\,(\frac{\partial \hat{\tau}_{xy}}{\partial y} + \frac{\partial \hat{\tau}_{xz}}{\partial z})\,dy\,dz$$

The contour integrals vanish by virtue of (3.153), the surface integral can be calculated by reference to the axial equilibrium equation

$$\frac{\partial \hat{\tau}_{xy}}{\partial y} + \frac{\partial \hat{\tau}_{xz}}{\partial z} = -\frac{\partial \hat{\sigma}_x}{\partial x} = -\frac{T_y}{I_y}\,y - \frac{T_z}{I_z}\,z$$

where the variation of the axial stress distribution is expressed in terms of the shear loads T_y and T_z. Thus

$$M = -\frac{T_y}{I_y}\iint X\,y\,dy\,dz - \frac{T_z}{I_z}\iint X\,z\,dy\,dz$$

or, introducing the coordinates of Weinstein's center of torsion (3.162),

$$M = -z_0\,T_y + y_0\,T_z$$

showning that this point is also the center of flexure.

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CHAPTER IV

NON-LINEAR SHELL THEORY

1. Coordinate system.

We begin by considering a shell to derive from a flat plate through a finite geometrical transformation. Let x^i ($i = 1, 2, 3$) denote cartesian coordinates of a point in the flat plate configuration, $x^3 = 0$ being the middle plane. In the shell configuration the same point has cartesian coordinates y^j (x^1, x^2, x^3). We can conceive the x^i as lagrangian or convective curvilinear coordinates for the shell. From the position vector of a point of the shell

$$\vec{R} = y^i \vec{e}_i$$

where $\vec{e}_i$ are the cartesian base vectors, follows the definition of the base vectors associated with the curvilinear coordinates

$$\vec{g}_j = D_j \vec{R} = (D_j y^i)\, \vec{e}_i \qquad D_j = \frac{\partial}{\partial x^j}$$

and the corresponding fundamental metric tensor

$$g_{jm} = D_j y^i\, D_m y^n\, \vec{e}_i . \vec{e}_n = D_j y^i\, D_m y^i$$

A considerable simplification of the analysis will be obtained by requiring that the base vector $\vec{g}_3$ remains everywhere orthogonal to $\vec{g}_1$ and $\vec{g}_2$. This condition, expressed by

$$g_{3\alpha} = D_3 y^i\, D_\alpha y^i = 0 \qquad (\alpha = 1, 2) \tag{1.1}$$

can be satisfied by adopting the following structure for the geometrical transformation of coordinates

$$y^i = r^i(x^1, x^2) + x^3\, b^i(x^1, x^2) \tag{1.2}$$

and setting

$$b^i\, (D_\alpha r^i + x^3\, D_\alpha b^i) = 0 \qquad (\alpha = 1, 2) \tag{1.3}$$

Since this should hold for all x^3 values within the thickness boundaries of the shell, we must have separately

$$b^i \, D_\alpha r^i = 0 \qquad (\alpha = 1, 2) \qquad (1.4)$$

$$b^i \, D_\alpha b^i = 0 \qquad (\alpha = 1, 2) \qquad (1.5)$$

The general solution of (1.4), considered as a linear system in the b^i, is

$$b^i = (e_{ijk} \, D_1 r^j \, D_2 r^k) \, c \, (x^1, x^2) \qquad (1.6)$$

where e_{ijk} is the permutation symbol and $c \, (x^1, x^2)$ an arbitrary function. Simultaneously (1.5) requires

$$g_{33} = b^i \, b^i = \text{constant} \qquad (1.7)$$

Without sacrificing generality, the constant can be set equal to unity. Then, substitution of (1.6) into (1.7) determines the function c through the equation

$$1 = c^2 \, e_{ijk} \, e_{ipq} \, D_1 r^j \, D_2 r^k \, D_1 r^p \, D_2 r^q$$

$$= c^2 \, (D_1 r^p \, D_1 r^p \, D_2 r^q \, D_2 r^q - D_1 r^p \, D_1 r^q \, D_2 r^p \, D_2 r^q)$$

In conclusion the functions b^i are completely determined by the geometrical transformation $y^i = r^i \, (x^1, x^2)$ of the middle plane of the plate into the middle surface of the shell. This derivation justifies the classical choice of curvilinear coordinates for shells :

$$\vec{R} = \vec{r} \, (x^1, x^2) + x^3 \, \vec{a}_3 \, (x^1, x^2) \qquad (1.8)$$

where $\vec{r}$ is the position vector of a middle surface point, and where $\vec{a}_3$ can be chosen such that

$$\vec{a}_3 \cdot D_\alpha \vec{r} = 0 \qquad (\alpha = 1, 2) \qquad (1.9)$$

$$\vec{a}_3 \cdot \vec{a}_3 = 1 \qquad (1.10)$$

2. Base vectors for surface and space tensors.

In what follows the greek letter indices will always refer to the surface curvilinear coordinates x^1 and x^2, the latin indices to the complete set of space curvilinear coordinates x^1, x^2 and x^3.

The base vectors, tangent to the middle plane $x^3 = 0$, are given by

$$\vec{a}_\alpha = D_\alpha \vec{r} \tag{2.1}$$

They generate the surface metric tensor

$$a_{\alpha\beta} = \vec{a}_\alpha . \vec{a}_\beta = a_{\beta\alpha} \tag{2.2}$$

and are the coefficients of the first fundamental form

$$d\vec{r} . d\vec{r} = a_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2.3}$$

The conjugate base vectors $\vec{a}^\alpha$ are related by

$$\vec{a}_\beta = a_{\alpha\beta}\vec{a}^\alpha \quad \text{or} \quad \vec{a}^\alpha . \vec{a}_\gamma = \delta^\alpha_\gamma \text{ (Kronecker symbol)} \tag{2.4}$$

and generate the conjugate surface metric tensor

$$a^{\alpha\beta} = \vec{a}^\alpha . \vec{a}^\beta \tag{2.5}$$

The surface metric tensors are used to raise or lower indices of surface tensors.
The determinant of the surface metric will be denoted

$$a = \left| a_{\alpha\beta} \right| \qquad \frac{1}{a} = \left| a^{\alpha\beta} \right| \tag{2.6}$$

The first fundamental form being positive definite a is a positive quantity.
Let $\varepsilon_{\alpha\beta}$ denote a twice covariant antisymmetric surface tensor, it has only one strict component $\varepsilon_{12} = \varepsilon$ ($\varepsilon_{21} = -\varepsilon$). Hence if $e_{\alpha\beta} = e^{\alpha\beta}$ denotes the corresponding permutation symbol ($e_{12} = 1$, $e_{21} = -1$, $e_{11} = e_{22} = 0$) we have

$$\varepsilon_{\alpha\beta} = \varepsilon\, e_{\alpha\beta}$$

The twice contravariant components of this tensor are related by

$$\varepsilon_{\alpha\beta} = a_{\alpha\gamma}\, a_{\beta\mu}\, \varepsilon^{\gamma\mu} = (a_{\alpha\gamma}\, a_{\beta\mu}\, e^{\gamma\mu})\sigma$$

where σ is the strict component of $\varepsilon^{\gamma\mu}$ ($\sigma = \varepsilon^{12}$). The quantity between brackets is $e_{\alpha\beta}a$. Hence, comparing the two expressions of $\varepsilon_{\alpha\beta}$

$$\varepsilon = \sigma\, a \tag{2.7}$$

Further we have

$$\varepsilon^{\alpha\beta}\, \varepsilon_{\alpha\beta} = \sigma\varepsilon\, e^{\alpha\beta}\, e_{\alpha\beta} = 2\sigma\varepsilon$$

and since this is obviously invariant, it can be equated to a fixed quantity. From the invariant $\sigma\varepsilon = \kappa$ and (2.7) we conclude that all antisymmetric second order surface tensors are of the form

$$\varepsilon_{\alpha\beta} = \sqrt{\kappa a}\, e_{\alpha\beta} \qquad \varepsilon^{\alpha\beta} = \sqrt{\frac{\kappa}{a}}\, e^{\alpha\beta}$$

where a is the determinant of the surface metric tensor. The usual ε-tensor for the surface corresponds to the choice $\kappa = 1$:

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \qquad \varepsilon_{12} = \sqrt{a} \tag{2.8}$$

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \qquad \varepsilon^{12} = 1/\sqrt{a}$$

Like the space metric tensor it enjoys the property that its twice covariant and twice contravariant representations are reciprocal : $\varepsilon^{\alpha\beta}\,\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. The unit vector $\vec{a}_3 = \vec{a}^3$ being normal to the surface

$$\vec{a}_3 \cdot \vec{a}^{\alpha} = 0 \qquad \vec{a}_3 \cdot \vec{a}_{\alpha} = 0 \qquad \vec{a}_3 \cdot \vec{a}_3 = 1 \tag{2.9}$$

It can be defined by

$$\vec{a}_3 = \varepsilon^{\alpha\beta}\, \vec{a}_{\alpha} \times \vec{a}_{\beta} \qquad \vec{a}_{\alpha} \times \vec{a}_{\beta} = \varepsilon_{\alpha\beta}\, \vec{a}_3 \tag{2.10}$$

as is easily verified by scalar multiplication by $\vec{a}_{\gamma}$ and $\vec{a}_3$.

Turning now to the base vectors for space tensors, they are defined by

$$\vec{g}_i = D_i \vec{R} \tag{2.11}$$

In view of (1.8) they are related to the surface base vectors through the following equations

$$\vec{g}_{\alpha} = \vec{a}_{\alpha} + x^3\, D_{\alpha} \vec{a}_3 \tag{2.12}$$

$$\vec{g}_3 = \vec{a}_3$$

and generate the fundamental metric tensor

$$\vec{g}_\alpha \cdot \vec{g}_\beta = g_{\alpha\beta} = a_{\alpha\beta} - 2x^3 b_{\alpha\beta} + (x^3)^2 c_{\alpha\beta} \tag{2.13}$$

$$\vec{g}_\alpha \cdot \vec{g}_3 = g_{\alpha 3} = 0 \tag{2.14}$$

$$\vec{g}_3 \cdot \vec{g}_3 = g_{33} = 1 \tag{2.15}$$

(2.14) and (2.15) follow directly from (2.9). (2.13) contains the definitions of the following, obviously symmetric, surface tensors

$$-2b_{\alpha\beta} = \vec{a}_\alpha \cdot D_\beta \vec{a}_3 + \vec{a}_\beta \cdot D_\alpha \vec{a}_3 \tag{2.16}$$

$$c_{\alpha\beta} = D_\alpha \vec{a}_3 \cdot D_\beta \vec{a}_3 \tag{2.17}$$

Simpler formulas for the first are obtained by transformations such as

$$\vec{a}_\alpha \cdot D_\beta \vec{a}_3 = D_\beta (\vec{a}_\alpha \cdot \vec{a}_3) - \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = - \vec{a}_3 \cdot D_\beta \vec{a}_\alpha$$

justified by (2.9) and also

$$D_\beta \vec{a}_\alpha = D_\beta D_\alpha \vec{r} = D_\alpha \vec{a}_\beta \tag{2.18}$$

there comes

$$b_{\alpha\beta} = \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = - \vec{a}_\alpha \cdot D_\beta \vec{a}_3 \tag{2.19}$$

The elements of this tensor are the coefficients of the second fundamental form

$$- d\vec{r} \cdot d\vec{a}_3 = b_{\alpha\beta} \, dx^\alpha \, dx^\beta$$

They appear also when expressing the surface derivative of the unit normal vector to the surface

$$D_\beta \vec{a}_3 = - b_{\alpha\beta} \, \vec{a}^\alpha \tag{2.20}$$

This formula is immediatly justified by (2.4), (2.9) and(2.19) when taking scalar multiplication with the surface base vectors. Substituting (2.20) into (2.17) it is found that the coefficients of the third fundamental form

$$d\vec{a}_3 \cdot d\vec{a}_3 = c_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2.21}$$

are expressible in terms of those of the second

$$c_{\alpha\beta} = b_{\gamma\alpha}\, b_{\varepsilon\beta}\, \vec{a}^\gamma \cdot \vec{a}^\varepsilon = b_{\gamma\alpha}\, b_{\varepsilon\beta}\, a^{\gamma\varepsilon} = b^\varepsilon_\alpha\, b_{\varepsilon\beta} \tag{2.22}$$

It is obvious from the geometrical signifiance of (2.20) or from its equivalent formulation in mixed components,known as the differential equation of Weingarten,

$$D_\beta \vec{a}_3 = - b^\alpha_\beta\, \vec{a}_\alpha \tag{2.23}$$

that the surface tensor generated by the second fundamental form characterizes completely the curvature of the middle surface. With respect to a change of curvilinear surface coordinates, the fundamental invariants of this tensor are

$$H = \frac{1}{2}\, b^\alpha_\alpha \tag{2.24}$$

the so-called mean curvature, and

$$K = \frac{1}{2}\varepsilon^{\alpha\beta}\, \varepsilon_{\lambda\mu}\, b^\lambda_\alpha\, b^\mu_\beta = b^1_1\, b^2_2 - b^2_1\, b^1_2 \tag{2.25}$$

the Gaussian curvature.

3. Covariant surface derivatives.

The surface Christoffel symbols are defined by the following decomposition of the derivatives with respect to the surface curvilinear coordinates of the surface base vectors themselves :

$$D_\beta \vec{a}_\alpha = \overset{\circ}{\Gamma}{}^\gamma_{\beta\alpha}\, \vec{a}_\gamma + \overset{\circ}{\Gamma}{}^3_{\beta\alpha}\, \vec{a}_3 \tag{3.1}$$

In view of (2.18) the Christoffel symbols are symmetrical with respect to the lower subscripts. Scalar multiplication of (3.1) by $\vec{a}_3$ and consideration of the orthogonality of $\vec{a}_3$ with respect to the $\vec{a}_\gamma$ and of (1.10) yields

$$\overset{\circ}{\Gamma}{}^3_{\beta\alpha} = \vec{a}_3 \cdot D_\beta \vec{a}_\alpha = b_{\alpha\beta} \tag{3.2}$$

as shown by (2.19). Hence (3.1) will be replaced by

$$D_\beta \vec{a}_\alpha = \overset{\circ}{\Gamma}{}^\gamma_{\beta\alpha}\, \vec{a}_\gamma + b_{\alpha\beta}\, \vec{a}_3 \tag{3.3}$$

In this form it is known as the differential equation of Gauss. The remaining Christoffel symbols can be calculated from the derivations of the fundamental metric surface tensor. Indeed, scalar multiplication of (3.1) by $\vec{a}_\varepsilon$ produces

$$\vec{a}_\varepsilon \cdot D_\beta \vec{a}_\alpha = \overset{\circ}{\Gamma}_{\beta\ \alpha}^{\ \gamma}\, a_{\gamma\varepsilon} = \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha}$$

However the left hand side is also

$$D_\beta(\vec{a}_\varepsilon \cdot \vec{a}_\alpha) - \vec{a}_\alpha \cdot D_\beta \vec{a}_\varepsilon = D_\beta a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon}$$

Hence by comparison,

$$D_\beta a_{\varepsilon\alpha} = \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon} + \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} \tag{3.4}$$

Similarly, by changing the order of the subscripts,

$$D_\varepsilon a_{\alpha\beta} = \overset{\circ}{\Gamma}_{\varepsilon\beta\alpha} + \overset{\circ}{\Gamma}_{\varepsilon\alpha\beta}$$

$$D_\alpha a_{\beta\varepsilon} = \overset{\circ}{\Gamma}_{\alpha\varepsilon\beta} + \overset{\circ}{\Gamma}_{\alpha\beta\varepsilon}$$

Combining those relations and using the symmetry properties of the Christoffel symbols

$$D_\beta a_{\varepsilon\alpha} - D_\varepsilon a_{\alpha\beta} + D_\alpha a_{\beta\varepsilon} = 2\,\overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} \tag{3.5}$$

In this formula, the central subscript can further be raised by multiplication by the reciprocal surface metric tensor $a^{\varepsilon\gamma}$, producing

$$2\,\overset{\circ}{\Gamma}_{\beta\ \alpha}^{\ \gamma} = a^{\varepsilon\gamma}\,(D_\beta a_{\varepsilon\alpha} - D_\varepsilon a_{\alpha\beta} + D_\alpha a_{\beta\varepsilon}) \tag{3.6}$$

Consider now a vector defined on the middle surface

$$\vec{u} = u^\alpha(x^1, x^2)\,\vec{a}_\alpha + u^3(x^1, x^2)\,\vec{a}_3$$

and calculate its surface derivative

$$D_\beta \vec{u} = (D_\beta u^\alpha)\,\vec{a}_\alpha + (D_\beta u^3)\,\vec{a}_3 + u^\alpha D_\beta \vec{a}_\alpha + u^3 D_\beta \vec{a}_3$$

Substituting (2.23) and (3.3) and reorganizing the terms

$$D_\beta \vec{u} = (u^\alpha||_\beta - b_\beta^\alpha\, u3)\,\vec{a}_\alpha + (D_\beta u3 + b_{\alpha\beta} u^\alpha)\,\vec{a}_3 \tag{3.7}$$

where

$$u^{\alpha}||_{\beta} = D_{\beta}u^{\alpha} + u^{\gamma}\,\overset{\circ}{\Gamma}{}^{\alpha}_{\beta\,\gamma} \tag{3.8}$$

is, by definition, the covariant surface derivative of u^{α}.
In a change of curvilinear coordinates on the middle surface, $\vec{a}_3$ and u^3 are invariant, $D_{\beta}\vec{u}$, $b^{\alpha}_{\beta}\vec{a}_{\alpha}$, $D_{\beta}u^3$ and $b_{\alpha\beta}u^{\alpha}$ are obviously once covariant and the same must then be true for $u^{\alpha}||_{\beta}\,\vec{a}_{\alpha}$. Consequently $u^{\alpha}||_{\beta}$ is a mixed surface tensor.
Let us now look after a formula analogous to (3.1)for the surface derivatives of the dual surface base vectors. Scalar multiplication of (3.1) by $\vec{a}^{\varepsilon}$ yields in view of (2.4) and (2.9)

$$\overset{\circ}{\Gamma}{}^{\varepsilon}_{\beta\,\alpha} = \vec{a}^{\varepsilon} \cdot D_{\beta}\vec{a}_{\alpha} = D_{\beta}\delta^{\varepsilon}_{\alpha} - \vec{a}_{\alpha} \cdot D_{\beta}\vec{a}^{\varepsilon} = \vec{a}_{\alpha} \cdot D_{\beta}\vec{a}^{\varepsilon} \tag{3.9}$$

while $\vec{a}_3 \cdot D_{\beta}\vec{a}^{\varepsilon} = -\vec{a}^{\varepsilon} \cdot D_{\beta}\vec{a}_3$

or, considering (2.23)

$$\vec{a}_3 \cdot D_{\beta}\vec{a}^{\varepsilon} = b^{\alpha}_{\beta}\,\vec{a}_{\alpha} \cdot \vec{a}^{\varepsilon} = b^{\alpha}_{\beta}\delta^{\varepsilon}_{\alpha} = b^{\varepsilon}_{\beta} \tag{3.10}$$

Finally (3.9) and (3.10) are equivalent to the required formula

$$D_{\beta}\vec{a}^{\varepsilon} = -\overset{\circ}{\Gamma}{}^{\varepsilon}_{\beta\,\alpha}\vec{a}^{\alpha} + b^{\varepsilon}_{\beta}\vec{a}^{3} \tag{3.11}$$

If we now express the surface vector $\vec{u}$ by its covariant components

$$\vec{u} = u_{\varepsilon}\vec{a}^{\varepsilon} + u_3\vec{a}^{3}$$

and use the results (3.11) and (2.20) ($\vec{a}^{3} = \vec{a}_3$) there comes

$$D_{\beta}\vec{u} = (u_{\alpha}||_{\beta} - u_3 b_{\alpha\beta})\,\vec{a}^{\alpha} + (D_{\beta}u_3 + b^{\alpha}_{\beta}u_{\alpha})\,\vec{a}^{3} \tag{3.12}$$

where $$u_{\alpha}||_{\beta} = D_{\beta}u_{\alpha} - \overset{\circ}{\Gamma}{}^{\gamma}_{\beta\alpha}u_{\gamma} \tag{3.13}$$

is a twice covariant surface tensor : the covariant surface derivative of u_{α}. Equations (3.8) and (3.13) can be generalized to express the covariant surface derivative of a surface tensor of any order. Take the particular case of the twice covariant fundamental metric tensor $a_{\varepsilon\alpha}$:

$$a_{\varepsilon\alpha}||_{\beta} = D_{\beta}a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}{}^{\gamma}_{\beta\ \varepsilon}a_{\gamma\alpha} - \overset{\circ}{\Gamma}{}^{\gamma}_{\beta\ \alpha}a_{\varepsilon\gamma}$$

This is equivalent to

$$a_{\varepsilon\alpha}||_{\beta} = D_{\beta}a_{\varepsilon\alpha} - \overset{\circ}{\Gamma}_{\beta\alpha\varepsilon} - \overset{\circ}{\Gamma}_{\beta\varepsilon\alpha} = 0 \tag{3.14}$$

by virtue of (3.4). This result is known as Ricci's Lemma. The same result is easily established by direct calculation for $a^{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$. It shows that those tensors behave as constants when calculating surface covariant derivatives.

Another important result concerns the covariant surface differentiation of the $b_{\alpha\beta}$ tensor. From direct application of (3.13)

$$b_{\alpha\beta}||_{\gamma} = D_{\gamma}b_{\alpha\beta} - \overset{\circ}{\Gamma}{}^{\varepsilon}_{\gamma\ \alpha}b_{\varepsilon\beta} - \overset{\circ}{\Gamma}{}^{\varepsilon}_{\gamma\ \beta}b_{\alpha\varepsilon} \tag{3.15}$$

If we now differentiate (2.20) and use (3.11)

$$D_{\gamma}D_{\beta}\vec{a}_3 = - D_{\gamma}b_{\alpha\beta}\vec{a}^{\alpha} - b_{\alpha\beta}\left(- \overset{\circ}{\Gamma}{}^{\alpha}_{\gamma\varepsilon}\vec{a}^{\varepsilon} + b^{\alpha}_{\gamma}\vec{a}_3\right)$$

the result can be put in the form

$$D_{\gamma}D_{\beta}\vec{a}_3 = - b_{\alpha\beta}||_{\gamma}\vec{a}^{\alpha} - \overset{\circ}{\Gamma}{}^{\varepsilon}_{\gamma\ \beta}b_{\alpha\varepsilon}\vec{a}^{\alpha} - c_{\gamma\beta}\vec{a}^{3}$$

The left-hand side and the two last terms of the right-hand side are symmetrical with respect to the subscripts β and γ. The same must then be true of the remaining term. This establishes the Mainardi-Codazzi equation

$$b_{\alpha\beta}||_{\gamma} = b_{\alpha\gamma}||_{\beta} \tag{3.16}$$

Should we have started from (2.23) instead of (2.20) we would have obtained

$$b^{\varepsilon}_{\beta}||_{\gamma} = b^{\varepsilon}_{\gamma}||_{\beta} \tag{3.17}$$

That this is an equivalent formulation of the Mainardi-Codazzi equation is obvious when (3.16) is multiplied by $a^{\alpha\varepsilon}$ and use made of Ricci's Lemma.

The Mainardi-Codazzi equation shows that the curvature tensor $b_{\alpha\beta}$ of a surface imbedded in euclidean space is not independant of the surface metric. From this viewpoint we should consider the system of differential equations of Weingarten (2.23) and Gauss (3.3) and investigate all the integrability conditions to be satisfied for the existence of a field of surface base vectors. In this light the Mainardi-Codazzi equation appears as the integrability condition for the unit normal vector.

The integrability conditions for the other base vectors is obtained by a similar statement of symmetry of second differentials.

Differentiating (3.3) and repeating the use of (2.23) and (3.3)

$$D_{\gamma}D_{\beta}\vec{a}_{\alpha} = \{ D_{\gamma}\overset{\circ}{\Gamma}{}^{\phi}_{\beta\ \alpha} + \overset{\circ}{\Gamma}{}^{\varepsilon}_{\beta\ \alpha}\overset{\circ}{\Gamma}{}^{\phi}_{\gamma\ \varepsilon} - b_{\alpha\beta}b_{\gamma}{}^{\phi} \}\vec{a}_{\phi}$$

$$+\{ b_{\gamma\varepsilon}\overset{\circ}{\Gamma}{}^{\varepsilon}_{\beta\ \alpha} + D_{\gamma}b_{\alpha\beta}\}\vec{a}_3$$

Considering the symmetry of the left-hand side, the brackets in the right-hand side should also be symmetrical with respect to the interchange of β and α. That this is true for the bracket of $\vec{a}_3$ is immediately seen by substituting its second term from (3.15), producing

$$b_{\alpha\beta||\gamma} + \overset{\circ}{\Gamma}_{\beta\ \alpha}^{\ \varepsilon}\, b_{\gamma\varepsilon} + \overset{\circ}{\Gamma}_{\gamma\ \alpha}^{\ \varepsilon}\, b_{\beta\varepsilon} + \overset{\circ}{\Gamma}_{\gamma\ \beta}^{\ \varepsilon}\, b_{\alpha\varepsilon}$$

While the group of the 3 last terms is clearly symmetrical with respect to the interchange of β and γ, the same is true of the first term by virtue of the Mainardi-Codazzi condition. The symmetry condition applied to the first bracket yields after rearranging terms.

$$b_{\alpha\beta} b_{\gamma}^{\phi} - b_{\alpha\gamma} b_{\beta}^{\phi} = R_{\cdot\alpha\gamma\beta}^{\phi}$$

where
$$R_{\cdot\alpha\gamma\beta}^{\phi} = D_\gamma \overset{\circ}{\Gamma}_{\beta\ \alpha}^{\ \phi} - D_\beta \overset{\circ}{\Gamma}_{\gamma\ \alpha}^{\ \phi} + \overset{\circ}{\Gamma}_{\beta\ \alpha}^{\ \varepsilon} \overset{\circ}{\Gamma}_{\gamma\ \varepsilon}^{\ \phi} - \overset{\circ}{\Gamma}_{\gamma\ \alpha}^{\ \varepsilon} \overset{\circ}{\Gamma}_{\beta\ \varepsilon}^{\ \phi} \tag{3.18}$$

depends only on the surface metric. Premultiplication by $a^{\alpha\psi}$, producing

$$b_{\beta}^{\psi} b_{\gamma}^{\phi} - b_{\gamma}^{\psi} b_{\beta}^{\phi} = R_{\cdot\cdot\gamma\beta}^{\phi\psi} \tag{3.19}$$

brings out clearly the antisymmetry with respect to the pair of upper and to the pair of lower indices. As a consequence there is only one scalar condition

$$K = b_1^1 b_2^2 - b_2^1 b_1^2 = R_{\cdot\cdot 21}^{21} \tag{3.20}$$

This is celebrated equation discovered by Gauss from which it appears that the Gaussian curvature depends on the surface metric only and is consequently accessible to measurements made on the surface only. It appears here as the second integrability condition placed on the curvature tensor $b_{\alpha\beta}$.

4. Covariant space derivatives.

The space Christoffel symbols appear as the covariant or contravariant components of derivatives of the space base vectors

$$D_j \vec{g}_i = \Gamma_{j\ i}^{\ h} \vec{g}_h = \Gamma_{jhi} \vec{g}^h \tag{4.1}$$

Since $D_j \vec{g}_i = D_j D_i \vec{R} = D_i \vec{g}_j$ they are also symmetrical with respect to their first and last subscripts. Furthermore, from $D_j (\vec{g}_i \cdot \vec{g}^k) = D_j \delta_i^{\ k} = 0$, we obtain

$$\vec{g}_i \cdot D_j \vec{g}^k = - \vec{g}^k \cdot D_j \vec{g}_i = - \Gamma_{jhi} \vec{g}^k \cdot \vec{g}^h = - \Gamma_{jhi} g^{kh} = - \Gamma_{j\ i}^{\ k}$$

or equivalently,

$$D_j \vec{g}^k = - \Gamma_{j\ i}^{\ k} \vec{g}^i \tag{4.2}$$

Then

$$D_j g_{ih} = D_j (\vec{g}_i \cdot \vec{g}_h) = \vec{g}_i \cdot D_j \vec{g}_h + \vec{g}_h \cdot D_j \vec{g}_i = \Gamma_{jih} + \Gamma_{jhi} \tag{4.3}$$

from which follows by the same type of linear combinations as used in the surface case:

$$2\Gamma_{jih} = D_j g_{ih} - D_i g_{jh} + D_h g_{ji} \tag{4.4}$$

$$\Gamma_{j\ h}^{\ k} = g^{ik} \Gamma_{jih}$$

Consider now a space vector defined by its contravariant components

$$\vec{u} = u^i \vec{g}_i$$

Its derivative formulated by means of (4.1) is

$$D_j \vec{u} = u^i|_j \vec{g}_i \tag{4.5}$$

where $$u^i|_j = D_j u^i + \Gamma_{j\,k}^{\ i}\, u^k \tag{4.6}$$

is defined as the covariant space derivative of u^i, a mixed space tensor. The same vector defined by its covariant components $\vec{u} = u_k \vec{g}^k$ yields for the same derivative, using (4.2)

$$D_j \vec{u} = u_k|_j \vec{g}^k \tag{4.7}$$

with $$u_k|_j = D_j u_k - \Gamma_{j\,k}^{\ m}\, u_m \tag{4.8}$$

The extension of formulas (4.6) and (4.8) to form covariant derivatives of tensors of higher order is immediate.

Clearly, since (4.5) and (4.7) represent the same quantity, expanded first in the system of covariant base vectors and later in the dual system, we must have

$$u_k|_j = g_{ki}\, u^i|_j$$

However the left-hand side is also

$$(g_{ki}\, u^i)\,|_j = g_{ki}|_j\, u^i + g_{ki} u^i|_j$$

From the comparison and the arbitrariness of the vector $\vec{u}$ it can be concluded that

$$g_{ki}|_j = 0 \tag{4.9}$$

This is Ricci's Lemma for the space case. Its direct computational proof relies on (4.3)

$$g_{ih}|_j = D_j g_{ih} - \Gamma_{j\,i}^{\ m}\, g_{mh} - \Gamma_{j\,h}^{\ m}\, g_{im}$$

$$= D_j g_{ih} - \Gamma_{jhi} - \Gamma_{jih} = 0$$

Let $e_{ijk} = e^{ijk}$ denote the permutation symbol, equal to unity if ijk is some even permutation of the sequence 123, to minus unity if the permutation is odd , to zero for all other cases. Any completely antisymmetric tensor is of the form

$$\varepsilon_{ijk} = \varepsilon\, e_{ijk}$$

where $\varepsilon = \varepsilon_{123}$ is the only strict component of the tensor. Similarly the contravariant form of this tensor will have a single strict component σ

$$\varepsilon^{mnp} = \sigma e^{mnp} \qquad \sigma = \varepsilon^{123}$$

The relation between the covariant and contravariant forms

$$\varepsilon_{ijk} = \varepsilon^{mnp}\, g_{mi} g_{nj} g_{pk} = \sigma (e^{mnp} g_{mi} g_{nj} g_{pk})$$

becomes equivalent to $\varepsilon = \sigma g$, when the quantity between brackets is recognized as the expansion of the determinant g of the fundamental metric tensor

$$e^{mnp}\, g_{mi}\, g_{nj}\, g_{pk} = e_{ijk}\, g$$

Furthermore the product of the strict components is easily recognized to be an invariant

$$\sigma\varepsilon = \frac{1}{6}\, \varepsilon^{ijk}\, \varepsilon_{ijk}$$

All completely antisymmetric tensors of this type differ only by the value of this invariant. For the choice $\sigma\varepsilon = 1$, corresponding to

$$\varepsilon = \sqrt{g} \qquad\qquad \sigma = 1/\sqrt{g}$$

we have the usual definition of the space ε- tensor :

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk} \qquad\qquad \varepsilon^{mnp} = \frac{1}{\sqrt{g}}\, e^{mnp} \tag{4.10}$$

An equivalent definition is

$$\varepsilon_{ijk} = (\vec{g}_i \times \vec{g}_j).\vec{g}_k \qquad\qquad \varepsilon^{mnp} = (\vec{g}^m \times \vec{g}^n).\vec{g}^p \tag{4.11}$$

justified by the complete antisymmetry of the mixed products and the fact that when the base vectors are cartesian, $g = 1$ and (4.11) reduces effectively to the permutation symbol.
In conjunction with (4.1) or (4.2) the expressions (4.11) are convenient to compute derivatives of the ε tensor.

$$\begin{aligned} D_h\, \varepsilon_{ijk} &= (D_h\vec{g}_i \times \vec{g}_j) \cdot \vec{g}_k + (\vec{g}_i \times D_h\vec{g}_j).\vec{g}_k + (\vec{g}_i \times \vec{g}_j).\, D_h\vec{g}_k \\ &= (\vec{g}_j \times \vec{g}_k).D_h\vec{g}_i + (\vec{g}_k \times \vec{g}_i)\, .D_h\vec{g}_j + (\vec{g}_i \times \vec{g}_j).\, D_h\vec{g}_k \\ &= \varepsilon_{jkm}\, \Gamma_h{}^m{}_i + \varepsilon_{kim}\, \Gamma_h{}^m{}_j + \varepsilon_{ijm}\, \Gamma_h{}^m{}_k \end{aligned} \tag{4.12}$$

This result is equivalent to stating that the ε-tensor also obeys Ricci's Lemma

$$\varepsilon_{ijk|h} = 0 \tag{4.13}$$

In fact we need only consider the case $i = 1$, $j = 2$, $k = 3$, whence, retaining only the non zero terms on the right hand side

$$D_h\varepsilon_{123} = \varepsilon_{231}\, \Gamma_h{}^1{}_1 + \varepsilon_{312}\, \Gamma_h{}^2{}_2 + \varepsilon_{123}\, \Gamma_h{}^3{}_3 = \varepsilon_{123}\Gamma_h{}^m{}_m \tag{4.14}$$

a result that will be useful later.

5. The structure of space metric in terms of surface metric for a shell.

Between space and surface base vectors exist the following relations deduced from (2.12) and (2.23)

$$\vec{g}_\alpha = \mu_\alpha^\beta\, \vec{a}_\beta \tag{5.1}$$

$$\vec{g}_3 = \vec{a}_3 \tag{5.2}$$

where although dependant on x^3,

$$\mu_\alpha^\beta = \delta_\alpha^\beta - x^3 b_\alpha^\beta \tag{5.3}$$

is a mixed surface tensor, i.e. exhibiting the covariant and contravariant behavior associated with a change of curvilinear coordinates on the surface. The correspondant relations between metric tensors are

$$g_{\alpha\beta} = \mu_\alpha^\beta \mu_\beta^\lambda a_{\gamma\lambda} \tag{5.4}$$

equivalent to (2.13).
Taking the determinant on both sides and noting that

$$g = |g_{ij}| = |g_{\alpha\beta}| \text{ in view of (2.14) and (2.15)}$$

$$g = \mu^2 a \tag{5.5}$$

where $\mu = |\mu_\alpha^\gamma| = 1 - 2 H x^3 + (x^3)^2 K$ (5.6)

as is easily verified by direct expansion of this determinant from (5.3) and consideration of the definitions (2.24) and (2.25). We also need the tensor reciprocal to μ_α^γ and denote it, following Naghdi, by

$$(\mu^{-1})_\alpha^\beta = \frac{1}{\mu} e_{\alpha\gamma} e^{\beta\varepsilon} \mu_\varepsilon^\gamma = \frac{1}{\mu} \{ \delta_\alpha^\beta + x^3 (b_\alpha^\beta - 2 H \delta_\alpha^\beta) \} \tag{5.7}$$

Then from (5.1) in succession

$$(\mu^{-1})_\gamma^\alpha \vec{g}_\alpha = \delta_\gamma^\beta \vec{a}_\beta = \vec{a}_\gamma$$

$$(\mu^{-1})_\gamma^\alpha \vec{g}_\alpha \cdot \vec{g}^\varepsilon = (\mu^{-1})_\gamma^\alpha \delta_\alpha^\varepsilon = (\mu^{-1})_\gamma^\varepsilon = \vec{a}_\gamma \cdot \vec{g}^\varepsilon \quad \text{and finally}$$

$$\vec{g}^\varepsilon = (\mu^{-1})_\gamma^\varepsilon \vec{a}^\gamma \tag{5.8}$$

while

$$\vec{g}^3 = \vec{a}^3 \ (= \vec{a}_3 = \vec{g}_3) \tag{5.9}$$

The reciprocal space metric tensor becomes

$$g^{\varepsilon\phi} = (\mu^{-1})_\gamma^\varepsilon (\mu^{-1})_\lambda^\phi a^{\gamma\lambda} \tag{5.10}$$

$$g^{\varepsilon 3} = 0 \qquad g^{33} = 1 \tag{5.11}$$

We now proceed to calculate the space Christoffel symbols :

$$\Gamma_{\alpha\beta}^\gamma = \vec{g}^\gamma \cdot D_\alpha \vec{g}_\beta = (\mu^{-1})_\lambda^\gamma \vec{a}^\lambda \cdot D_\alpha (\mu_\beta^\varepsilon \vec{a}_\varepsilon)$$

$$= (\mu^{-1})_\lambda^\gamma \{ D_\alpha \mu_\beta^\lambda + \mu_\beta^\varepsilon \overset{\circ}{\Gamma}_{\alpha\varepsilon}^\lambda \}$$

Use was here made in succession of (5.1), (5.8) and (3.1) and (2.9).
The expression in brackets is modified to introduce the surface covariant derivative

$$\mu_{\beta||\alpha}^\lambda = D_\alpha \mu_\beta^\lambda + \mu_\beta^\varepsilon \overset{\circ}{\Gamma}_{\alpha\varepsilon}^\lambda - \mu_\varepsilon^\lambda \overset{\circ}{\Gamma}_{\alpha\beta}^\varepsilon$$

This produces

$$\Gamma_{\alpha\beta}^\gamma = (\mu^{-1})_\lambda^\gamma \{ \mu_{\beta||\alpha}^\lambda + \mu_\varepsilon^\lambda \overset{\circ}{\Gamma}_{\alpha\beta}^\varepsilon \} = \overset{\circ}{\Gamma}_{\alpha\beta}^\gamma + (\mu^{-1})_\lambda^\gamma \mu_{\beta||\alpha}^\lambda \tag{5.12}$$

Finally, in view of (5.7) and (5.3) this can be placed in the form

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \overset{\circ}{\Gamma}_{\alpha\ \beta}^{\ \gamma} - \frac{x^3}{\mu}(1 - 2\,H\,x^3)\ b_{\beta||\alpha}^{\gamma} - \frac{(x^3)^2}{\mu}\ b_{\lambda}^{\gamma}\ b_{\beta||\alpha}^{\lambda} \qquad (5.13)$$

where the dependence on x^3 is more apparent. In particular, as the notation suggested

$$\overset{\circ}{\Gamma}_{\alpha\ \beta}^{\ \gamma} = \left(\Gamma_{\alpha\ \beta}^{\ \gamma}\right)_{x^3 = 0}$$

Moreover the symmetry in the subscripts α and β is guaranteed by the Mainardi-Codazzi condition (3.17).
Next calculate

$$\Gamma_{\alpha\ \beta}^{\ 3} = \vec{g}^{3}.D_{\alpha}\vec{g}_{\beta} = \vec{a}^{3}.D_{\alpha}\left(\mu_{\beta}^{\varepsilon}\,\vec{a}_{\varepsilon}\right)$$

Expanding the derivative and using the orthogonality between $\vec{a}_{\varepsilon}$ and $\vec{a}^{3}$,

$$\Gamma_{\alpha\ \beta}^{\ 3} = \mu_{\beta}^{\varepsilon}\vec{a}^{3}.\left(\overset{\circ}{\Gamma}_{\alpha\ \varepsilon}^{\ \lambda}\vec{a}_{\lambda} + b_{\alpha\varepsilon}\,\vec{a}_{3}\right) = \mu_{\beta}^{\varepsilon}\,b_{\alpha\varepsilon} \qquad (5.14)$$

or, as explicit function of x^3,

$$\Gamma_{\alpha\ \beta}^{\ 3} = b_{\alpha\varepsilon}\left(\delta_{\beta}^{\varepsilon} - x^3\,b_{\beta}^{\varepsilon}\right) = b_{\alpha\beta} - x^3\,c_{\alpha\beta} \qquad (5.15)$$

For $\Gamma_{\beta\ 3}^{\ \alpha} = \Gamma_{3\ \beta}^{\ \alpha}$ we have

$$\Gamma_{\beta\ 3}^{\ \alpha} = \vec{g}^{\alpha}.D_{\beta}\vec{a}_{3} = (\mu^{-1})_{\lambda}^{\alpha}\vec{a}^{\lambda}.(-b_{\beta}^{\gamma}\,\vec{a}_{\gamma}) = -\,b_{\beta}^{\lambda}\,(\mu^{-1})_{\lambda}^{\alpha} \qquad (5.16)$$

or again as an explicit function of x^3

$$\Gamma_{\beta\ 3}^{\ \alpha} = -\,\frac{1 - 2\,H\,x^3}{\mu}\,b_{\beta}^{\alpha} - \frac{x^3}{\mu}\,b_{\beta}^{\lambda}\,b_{\lambda}^{\alpha} \qquad (5.17)$$

Finally it is easily verified that the remaining space Christoffel symbols vanish

$$\Gamma_{3\ 3}^{\ \alpha} = 0 \qquad \Gamma_{3\ \beta}^{\ 3} = 0 \qquad \Gamma_{3\ 3}^{\ 3} = 0 \qquad (5.18)$$

6. The divergence operator and the divergence theorems.

The divergence theorem

$$\int_{S} (\vec{n}.\vec{a})\ ds = \int_{V} \operatorname{div}\vec{a}\ dV \qquad (6.1)$$

applied to a vector field $\vec{a}$ within a volume bounded by a simple closed surface S, provides an intrinsic definition of the divergence operator

$$\operatorname{div}\vec{a} = \lim_{dV \to 0} \frac{\int_{S} (\vec{n}.\vec{a})\,dS}{dV} \qquad (6.2)$$

that can be used to obtain its formulation in general curvilinear coordinates. As elementary volume we take the parallelopiped built on the three infinitesimal vectors $\vec{g}_1\,dx^1$, $\vec{g}_2\,dx^2$, $\vec{g}_3\,dx^3$, issued from the point P where the divergence of the field is to be calculated.

The surface vector $(\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2$ proportional to the aera of one of the facets of the parallelopiped through P, is oriented as the inward normal to this facet. Hence the entering flux of $\vec{a}$ through this facet is

$$(\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2 \,.\, (a^i \vec{g}_i) = a^3\, dx^1\, dx^2\, (\vec{g}_1 \times \vec{g}_2).\vec{g}_3$$

$$= dx^1\, dx^2\, a^3\, \varepsilon_{123}$$

The outgoing flux through the opposite facet is

$$dx^1\, dx^2\, a^3\, \varepsilon_{123} + dx^1\, dx^2\, dx^3\, D_3\, (a^3\, \varepsilon_{123})$$

and the net contribution to the outcoming flux is

$$dx^1\, dx^2\, dx^3\, D_3\, (a^3\, \varepsilon_{123})$$

Taking account of the analogous contributions from the two other pairs of facets and dividing by the volume of the parallelopiped

$$dV = dx^1\, dx^2\, dx^3\, (\vec{g}_1 \times \vec{g}_2).\vec{g}_3 = dx^1\, dx^2\, dx^3\, \varepsilon_{123}$$

the application of definition (6.2) yields

$$\operatorname{div} \vec{a} = \frac{1}{\varepsilon_{123}}\, D_i\, (a^i\, \varepsilon_{123})$$

Expanding the derivative and using our previous result (4.14)

$$\operatorname{div} \vec{a} = D_i\, a^i + a^i\, \Gamma_{i\,m}^{\ m} = a^i|_i \tag{6.3}$$

This allows to write the divergence theorem in the form

$$\int_S (n_i a^i)\, dS = \int_V a^i|_i\, dV \tag{6.4}$$

where n_i are the covariant components of the outward unit normal to the surface. A similar result will prove to be useful for the flux of surface vectors. Let an elementary parallelogram be drawn from the vectors $\vec{a}_1\, dx^1$ and $\vec{a}_2\, dx^2$ in a tangent plane to the middle surface of the shell, and let $\vec{h} = h^\alpha \vec{a}_\alpha$ be a vector field defined on this middle surface. We calculate the outgoing flux $\Sigma\, \vec{\nu}.\, \vec{h}\, ds$, where $\vec{\nu}$ is the outward unit normal to the contour of the parallelogram. Along the side $\vec{a}_1\, dx^1$ the contribution is $(\vec{a}_1 \times \vec{a}_3)\, dx^1.\, h^\alpha \vec{a}_\alpha = dx^1\, h^2\, (\vec{a}_1 \times \vec{a}_3).\vec{a}_2 = -\, dx^1\, h^2\, \varepsilon_{12}$

where use was made of

$$\varepsilon_{12} = (\vec{a}_1 \times \vec{a}_2).\vec{a}_3 \tag{6.5}$$

which is a direct consequence of (2.10). Adding the contribution from the opposite side, the net result is

$$dx^1\, dx^2\, D_2\, (h^2\, \varepsilon_{12}).$$

Finally, adding the analogous contribution from the other pair of sides :

$$dx^1\, dx^2\, D_\alpha\, (h^\alpha\, \varepsilon_{12}) = \Sigma\, \vec{\nu}.\vec{h}\, ds \tag{6.6}$$

The next result is a direct consequence of (2.23), (3.3) and (2.9) :

$$D_\alpha\, \varepsilon_{12} = D_\alpha\, \{(\vec{a}_1 \times \vec{a}_2).\vec{a}_3\} = \varepsilon_{12}\, \overset{\circ}{\Gamma}{}^{\beta}_{\alpha\,\beta} \tag{6.7}$$

It constitutes also the verification that $\varepsilon_{\alpha\beta}$ obeys Ricci's Lemma for the surface derivative. Substituted into (6.6) it gives

$$\begin{aligned} \Sigma\, \vec{\nu}.\vec{h}\, ds &= (D_\alpha\, h^\alpha + \overset{\circ}{\Gamma}{}^{\beta}_{\alpha\,\beta}\, h^\alpha)\, \varepsilon_{12}\, dx^1\, dx^2 \\ &= h^\alpha||_\alpha\, \varepsilon_{12}\, dx^1\, dx^2 \end{aligned} \tag{6.8}$$

Then, by the usual argument of decomposition into elementary cells, we obtain the surface divergence theorem

$$\oint_c \nu_\alpha h^\alpha\, ds = \int_A h^\alpha||_\alpha\, dA \tag{5.9}$$

relating a contour integral to an integral extended over an area A of the middle surface. The surface element is indeed easily observed to be

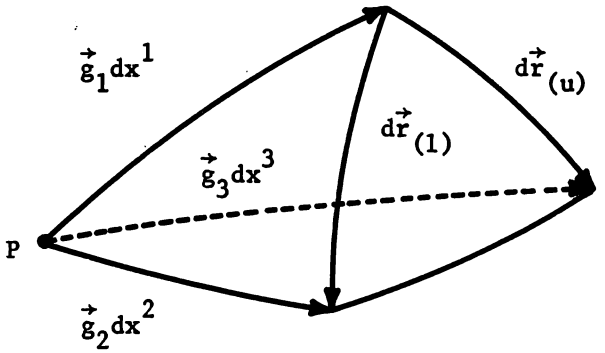
$$dA = \varepsilon_{12}\, dx^1\, dx^2 \tag{5.10}$$

7. Three-dimensional elasticity in curvilinear coordinates.

The approach towards a shell theory will consist in reducing the general three-dimensional elasticity equations to two-dimensional ones by suitable assumptions concerning the structure of the problem with respect to the x^3 coordinate, normal to the curved middle surface. Consequently the three-dimensional elasticity equations in curvilinear coordinates will play an important role and will be briefly recapitulated hereunder.
The basic stress definition will be lagrangian in nature. This implies that infinitesimal forces, that have to be defined as acting on infinitesimal areas of the strained elastic body, are kept invariant but, for the purpose of defining stresses, are manipulated in the metric of a "reference configuration". In the reference configuration, also called initial configuration, the body is by definition unstrained. The geometrical considerations to follow will then apply essentially to the reference configuration.
An oriented surface vector is defined in general as the cross product

$$d\vec{r}_{(1)} \times d\vec{r}_{(2)} = (\vec{g}_i\, dx^i_{(1)}) \times (\vec{g}_j\, dx^j_{(2)})$$

between two infinitesimal vectors. In modulus it is proportional to the area of the parallelogram constructed on these vectors; in orientation it is normal to the plane of the parallelogram.
Consider now an elementary tetrahedron, three lateral facets of which are generated by infinitesimal vectors taken along the local base vectors and issued from a point P.



The fourth facet, called the "inclined" facet, is half the parallelogram constructed on the vectors

$$d\vec{r}_{(1)} = \vec{g}_2\, dx^2 - \vec{g}_1\, dx^1 \qquad d\vec{r}_{(2)} = \vec{g}_3\, dx^3 - \vec{g}_1\, dx^1$$

and has an oriented surface vector that, considering the antisymmetric properties of the cross product, can be written in cyclic form as

$$d\vec{r}_{(1)} \times d\vec{r}_{(2)} = (\vec{g}_1 \times \vec{g}_2)\, dx^1\, dx^2 + (\vec{g}_2 \times \vec{g}_3)\, dx^2\, dx^3 + (\vec{g}_3 \times \vec{g}_1)\, dx^3\, dx^1 \qquad (7.1)$$

Each term on the right represents the surface vector of one of the lateral facets; moreover, if the orientation of the surface vector of the inclined facet is outwards, those of the lateral facets are inwards. Hence (7.1) expresses that the sum of all outward surface vectors of the tetrahedron vanishes.

In (7.1) the left hand side can be replaced by $2\,\vec{n}\,dS$, where $\vec{n}$ is the unit outward normal and dS the area of the inclined facet, while the first term in the right-hand side can be replaced by

$$\vec{g}_1 \times \vec{g}_2\, dx^1\, dx^2 = \varepsilon_{123}\, dx^1\, dx^2\, \vec{g}^3 = 2\,\frac{\vec{g}^3}{\sqrt{g^{33}}}\, dS_3$$

since $\vec{g}^3 / \sqrt{g^{33}}$ is obviously the (inward) unit normal, dS_3 denoting the area, of the lateral facet generated by $\vec{g}_1\, dx^1$ and $\vec{g}_2\, dx^2$. Thus (7.1) becomes

$$\vec{n}\, dS = \frac{\vec{g}^1}{\sqrt{g^{11}}}\, dS_1 + \frac{\vec{g}^2}{\sqrt{g^{22}}}\, dS_2 + \frac{\vec{g}^3}{\sqrt{g^{33}}}\, dS_3$$

or, introducing the covariant components n_i of the unit normal $\vec{n}$

$$\vec{n} = n_i\, \vec{g}^i \qquad (7.2)$$

and identifying

$$dS_i = \sqrt{g^{ii}}\, n_i\, dS \qquad \text{(without summation)} \tag{7.3}$$

Now let $\vec{t}$ and $\vec{t}^{\,i}$ denote respectively the stress vectors acting on the inclined facet and on the lateral facets. Imagining the tetrahedron to be convected in some strained state of the body, those stresses are defined as the ratio of the force acting in the stained state on each facet to the area of the unstrained facet. Hence the translational equilibrium condition of the tetrahedron is, the resultant of mass forces being negligibly small,

$$\vec{t}\, dS = \vec{t}^{\,i}\, dS_i \tag{7.4}$$

The substitution of (7.3) into (7.4) suggests that a lagrangian stress tensor t^{ij} be defined according to equation

$$\sqrt{g^{ii}}\, \vec{t}^{\,i} = t^{ij}\, \vec{g}_j \tag{7.5}$$

so that the equilibrium condition is now expressed by

$$\vec{t} = n_i t^{ij}\, \vec{g}_j \tag{7.6}$$

It follows that the t^{ij} are not physical stress components. The physical components derived from definition (7.5) are given by

$$\sqrt{\frac{g_{jj}}{g^{ii}}}\, t^{ij} \qquad \text{(no summation)} \tag{7.7}$$

If the points of the strained body are submitted to an additional small displacement field $\delta\vec{u}\,(x^1, x^2, x^3)$, the statement of energy conservation is

$$\int_S \vec{t}.\delta\vec{u}\, dS + \int_V \vec{X}.\delta\vec{u}\, dV = \int_V \delta W\, dV \tag{7.8}$$

where surface and volume integrals are taken in the initial configuration. W is the elastic strain energy and $\vec{X}$ the body force field per unit volume in this initial configuration. In curvilinear coordinates this becomes

$$\int_S (n_i t^{ij}\, \delta u_j)\, dS + \int_V X^j\, \delta u_j\, dV = \int_V \delta W\, dV \tag{7.9}$$

When the surface integral is transformed in a volume integral by the divergence theorem (6.4), the local statement of conservation of energy (per unit volume) is

$$(t^{ij}\, \delta u_j)_{|i} + X^j\, \delta u_j = \delta W \tag{7.10}$$

If the additional displacement field is of a rigid body type, there is no energy increase; this can be used to extract from (7.10) the translational and rotational equilibrium equations. In the translational case

$$D_i\, (\delta u_j\, \vec{g}^{\,j}) = \delta u_{j|i}\, \vec{g}^{\,j} = 0 \quad \text{or} \quad \delta u_{j|i} = 0 \tag{7.11}$$

so that (7.10) with $\delta W = 0$ reduces to

$$(t^{ij}{}_{|i} + X^j)\, \delta u_j = 0$$

and, on account of the arbitrariness of the translation vector

$$t^{ij}{}_{|i} + X^j = 0 \qquad (j = 1, 2, 3) \tag{7.12}$$

Those equilibrium equations satisfied by the lagrangian stresses are linear; they allow a first simplification of the energy equation (7.10)

$$t^{ij}\ \delta u_{j|i} = \delta W \tag{7.13}$$

For the rotational case we can write the additional displacement field as

$$\delta\vec{u} = d\vec{\omega} \times (\vec{R} + \vec{u}) \tag{7.14}$$

where $d\vec{\omega}$ is a constant infinitesimal rotation vector and $\vec{u}$ the already prevailing displacement field from the initial configuration. Then

$$D_m\, \delta\vec{u} = d\vec{\omega} \times (\vec{g}_m + D_m\vec{u}) \qquad \text{or}$$

$$\delta u_{i|m}\, \vec{g}^{\,i} = (\delta^p_m + u^p{}_{|m})\, d\vec{\omega} \times \vec{g}_p$$

Scalar multiplication by $\vec{g}_j$ and transformation of

$$(d\vec{\omega} \times \vec{g}_p).\vec{g}_j = d\omega^r\ (\vec{g}_r \times \vec{g}_p).\vec{g}_j = \varepsilon_{rpj}\ d\omega^r$$

furnishes

$$\delta u_{j|m} = \varepsilon_{rpj}\ (\delta^p_m + u^p{}_{|m})\ d\omega^r \tag{7.15}$$

For this value of the covariant derivative of δu_j, the energy increase vanishes again in (7.13) and, since the $d\omega^r$ are arbitrary

$$\varepsilon_{rpj}\ (t^{pj} + t^{ij}\ u^p{}_{|i}) = 0 \qquad \text{all r and j}$$

Considering the complete antisymmetry of ε_{rpj} this rotational equilibrium condition is satisfied if and only if the tensor between bracket is symmetrical in the superscripts p and j. This produces the final rotational equilibrium equations connecting the lagrangian stress tensor components

$$t^{pj} + t^{ij}\ u^p{}_{|i} = t^{ip}\ u^j{}_{|i} + t^{jp} \tag{7.16}$$

There are only three distinct ones, since they are identically satisfied for $p = j$. They reduce to a statement of symmetry of the lagrangian stress tensor only in the case where the covariant derivatives of the contravariant displacement tensor are negligibly small compared to unity (geometrical linearity). A method for automatically satisfying (7.16) consists in writing

$$t^{ij} = (\delta^j_q + u^j{}_{|q})\ s^{qi} \quad \text{with } s^{qi} = s^{iq} \tag{7.17}$$

The new symmetrical stress tensor s^{qi} is that of Kirchhoff and Trefftz, it transforms the energy equation (7.13) into

$$s^{qi}\ (\delta u_{q|i} + u^j{}_{|q}\ \delta u_{j|i}) = \delta W$$

or, making use of the symmetry of the new stress tensor

$$\frac{1}{2}\, s^{qi}\ (\delta u_{q|i} + \delta u_{i|q} + u^j{}_{|q}\ \delta u_{j|i} + u^j{}_{|i}\ \delta u_{j|q}) = \delta W \tag{7.18}$$

Because the formulation is lagrangian, the order sequence of the δ operator and the derivatives with respect to the coordinates x^i may be interchanged. It is then recognized that the tensor between brackets is a perfect differential, provided one also recognizes, e.g. by applying Ricci's Lemma, the equivalence

$$u^j|_i \, \delta u_{j|q} = u_{j|i} \, \delta u^j|_q$$

In fact, introducing the symmetrical tensor

$$\gamma_{qi} = \frac{1}{2} \left(u_{q|i} + u_{i|q} + u^j|_q \, u_{j|i} \right) \tag{7.19}$$

the energy statement (7.18) reduces to

$$\delta W = s^{qi} \, \delta \gamma_{qi} \tag{7.20}$$

It shows that the definition (7.17) is not altogether artificial since it now appears that the strain energy density is a function of the displacement gradients through six distinct combinations of them only, the components of a symmetrical tensor γ_{qi}, and that the Kirchhoff-Trefftz stresses are the corresponding partial derivatives. Obviously the γ_{qi} must represent strains, they are in fact the most commonly used lagrangian measures of strain and are usually derived from a statement of conservation of distance between neighboring points in a rigid body displacement. It is easily verified that

$$d(\vec{R} + \vec{u}).d(\vec{R} + \vec{u}) - d\vec{R}.d\vec{R} = 2\gamma_{qi} \, dx^q \, dx^i \tag{7.21}$$

While the lagrangian viewpoint offers many advantages, the hybrid nature of the stress tensor definitions it requires is well illustrated by the Kirchhoff-Trefftz tensor. They can be interpreted as the same lagrangian stress vectors but decomposed in the metric of the strained body. The base vectors of this metric are naturally defined by

$$\vec{G}_q = D_q(\vec{R} + \vec{u}) = \vec{g}_q + u^j|_q \, \vec{g}_j = (\delta_q^{\ j} + u^j|_q) \, \vec{g}_j \tag{7.22}$$

Then, starting from definition (7.6) of the lagrangian stress tensor and substituting (7.17)

$$\vec{t} = n_i \, s^{qi} \, (\delta_q^j + u^j|_q) \, \vec{g}_j = n_i \, s^{qi} \, \vec{G}_q \tag{7.23}$$

which proves the assertion. This geometrically hybrid character is reflected in the nature of the translational equilibrium equations they satisfy, the Signorini equations, which depend on the displacement field :

$$s^{ij}|_i + (s^{qi} \, u^j|_q)|_i + X^j = 0 \tag{7.24}$$

Due to the symmetry of the Kirchhoff-Trefftz stresses, (7.20) shows that the energy density is in fact a function of the six strain measures.

$$\gamma_{11}, \gamma_{22}, \gamma_{33}, \quad \text{and} \quad \eta_{12} = \gamma_{12} + \gamma_{21}$$

$$\eta_{23} = \gamma_{23} + \gamma_{32}$$

$$\eta_{31} = \gamma_{31} + \gamma_{13} \tag{7.25}$$

with the following general form of the stress-strain relations

$$\sigma^{ii} = \frac{\partial W}{\partial \gamma_{ii}} \qquad \sigma^{ij} = \frac{\partial W}{\partial \eta_{ij}} \quad (i \neq j) \tag{7.26}$$

If, however, it is agreed to distinuish γ_{ij} from γ_{ji} in (7.25), the general formula

$$\sigma^{ij} = \frac{\partial W}{\partial \gamma_{ij}} \tag{7.27}$$

holds true by the chain rule of differentiation.
In writing down an explicit form of the energy density we shall limit ourselves to the generalized Hooke situation (physical linearity).
The energy is then a homogeneous quadratic form in the components of the strain tensor :

$$W = \frac{1}{2} E^{ijkl} \gamma_{ij} \gamma_{kl} \tag{7.28}$$

Because the tensor of elastic moduli is symmetrical with respect to the pair of indices ij and kl and also with respect to the interchange of those pairs, its total number of independant components is not larger than 21.

In the isotropic case, the energy is explicitly

$$W = G \frac{1 - \nu}{1 - 2\nu} I_1^2 - 2GI_2 \tag{7.29}$$

with G the shear modulus and ν Poisson's ratio. The invariants I_1 and I_2 are obtained as coefficients of the powers of γ in the expansion of the determinant.

$$\begin{vmatrix} \gamma_1^1 - \gamma & \gamma_1^2 & \gamma_1^3 \\ \gamma_2^1 & \gamma_2^2 - \gamma & \gamma_2^3 \\ \gamma_3^1 & \gamma_3^2 & \gamma_3^3 - \gamma \end{vmatrix} = -\gamma^3 + I_1 \gamma^2 - I_2 \gamma + I_3$$

$$I_1 = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = \gamma_i^i$$

$$I_2 = \gamma_1^1\gamma_2^2 + \gamma_2^2\gamma_3^3 + \gamma_3^3\gamma_1^1 - \gamma_1^2\gamma_2^1 - \gamma_2^3\gamma_3^2 - \gamma_3^1\gamma_1^3 = \frac{1}{2}(\gamma_i^i\gamma_j^j - \gamma_i^j\gamma_j^i)$$

Going back to the doubly covariant components of strain

$$I_1 = g^{ij} \gamma_{ij} \qquad I_2 = \frac{1}{2} g^{mi} g^{nj} (\gamma_{mi}\gamma_{nj} - \gamma_{mj}\gamma_{ni}) \tag{7.30}$$

and applying formally (7.27)

$$\sigma^{pq} = E^{pq\,ij} \gamma_{ij} \qquad \text{with}$$

$$E^{pqij} = G(g^{pi} g^{qj} + g^{pj} g^{qi} + \frac{2\nu}{1 - 2\nu} g^{pq} g^{ij}) \tag{7.31}$$

8. Stress assumption for shell theory.

The basic stress assumption will be that the direct Kirchhoff-Trefftz σ^{33} normal to the middle surface is zero; it generalizes the assumption common to plate theories. Then, in cartesian local axes, the third axis being oriented like $\vec{a}_3$, the isotropic stress-strain relations reduce to

$$\sigma_1^1 = \frac{E}{1-\nu^2}(\gamma_1^1 + \nu\gamma_2^2) \quad , \quad \sigma_2^2 = \frac{E}{1-\nu^2}(\gamma_2^2 + \nu\gamma_1^1) \quad , \quad \sigma_3^3 = 0$$

$$\sigma_i^j = \frac{E}{1-\nu^2}(1-\nu)\gamma_i^j \qquad (i \neq j)$$

where E is Young's modulus. Building up the corresponding energy density by means of Clapeyron's theorem

$$W = \frac{1}{2}\sigma_i^j \gamma_j^i$$

the resulting expression is easily put in the form

$$W = \frac{E}{2(1-\nu^2)}\left\{(\gamma_\alpha^\alpha)^2 + 2(1-\nu)\gamma_\alpha^3\gamma_3^\alpha - (1-\nu)(\gamma_\alpha^\alpha\gamma_\beta^\beta - \gamma_\alpha^\beta\gamma_\beta^\alpha)\right\}$$

It is clearly valid also in the curvilinear system of coordinates of the shell in view of the invariance of the different terms with respect to any change in metric not involving the third axis.
Turning then to the covariant strains measures

$$W = \frac{E}{2(1-\nu^2)}\left\{(g^{\alpha\beta}\gamma_{\alpha\beta})^2 + 2(1-\nu)\, g^{\alpha\beta}\gamma_{\alpha 3}\gamma_{3\beta} - (1-\nu)g^{\alpha\lambda}g^{\beta\mu}(\gamma_{\alpha\lambda}\gamma_{\beta\mu} - \gamma_{\alpha\mu}\gamma_{\beta\lambda})\right\} \quad (8.1)$$

By taking formal derivatives it is found that

$$\left.\begin{aligned} E^{\lambda\mu\alpha\beta} &= \frac{E}{2(1-\nu^2)}\left\{2\nu g^{\lambda\mu}g^{\alpha\beta} + (1-\nu)(g^{\lambda\alpha}g^{\mu\beta} + g^{\lambda\beta}g^{\mu\alpha})\right\} \\ E^{\lambda 3\alpha 3} &= 2\,G\,g^{\lambda\alpha} \\ E^{\lambda\mu\alpha 3} &= 0 \qquad E^{\lambda\mu 33} = 0 \qquad E^{\lambda 3\,33} = 0 \qquad E^{33\,33} = 0 \end{aligned}\right\} \quad (8.2)$$

9. Displacement assumptions for a shell theory.

Following the basic stress assumption $\sigma^{33} = 0$, there are, like in plate theory, different ways open for reaching the goal of reducing a shell theory to essentially two dimensions. Further assumptions can be made on stress distributions, generalizing E. Reissner's approach to the plate bending formulation. Displacement assumptions can be used instead, as in Hencky's plate bending theory, or both systems can be mixed to take advantage of two-field variational principles.
We shall here discuss a displacement formulation, first because it is more widely in use, secondly because it can be specialized to include a discussion of the Kirchhoff-Love hypothesis. The simplest assumption is that of linearity across the shell thickness of the displacement vector :

$$\vec{u} = \vec{u}_{(0)} + x^3\,\vec{u}_{(1)} \quad (9.1)$$

where the two component vectors depend only on the middle surface coordinates. Covariant components of surface displacement tensors are consequently introduced by the definitions

$$\vec{u}_{(0)} = v_\alpha \vec{a}^\alpha + w \vec{a}^3 \tag{9.2}$$

$$\vec{u}_{(1)} = w_\alpha \vec{a}^\alpha + p \vec{a}^3 \tag{9.3}$$

The v_α are middle surface displacements, w is a transversal displacement normal to the middle surface, the w_α are sectional rotations and finally p is a transverse "pinch". The pinch component is usually omitted; it is however essential to preserve the ability of (9.1) to represent a finite rigid body rotation and moreover it lends more symmetry to the subsequent derivations. We next introduce the surface tensors generated by taking surface derivatives of the component vectors

$$D_\beta \vec{u}_{(0)} = \lambda_{\gamma\beta} \vec{a}^\gamma + \phi_\beta \vec{a}^3 = \lambda^\gamma_{.\beta} \vec{a}_\gamma + \phi_\beta \vec{a}_3 \tag{9.4}$$

$$D_\beta \vec{u}_{(1)} = \rho_{\alpha\beta} \vec{a}^\gamma + \psi_\beta \vec{a}^3 = \rho^\gamma_{.\beta} \vec{a}_\gamma + \psi_\beta \vec{a}_3 \tag{9.5}$$

where, from application of formula (3.12), we find

$$\lambda_{\gamma\beta} = v_{\gamma||\beta} - b_{\gamma\beta} w \tag{9.6}$$

$$\rho_{\gamma\beta} = w_{\gamma||\beta} - b_{\gamma\beta} p \tag{9.7}$$

$$\phi_\beta = D_\beta w + b^\varepsilon_\beta v_\varepsilon \tag{9.8}$$

$$\psi_\beta = D_\beta p + b^\varepsilon_\beta w_\varepsilon \tag{9.9}$$

To prepare the computation of the Green strain tensor, we calculate the various derivatives $u_{i|q}$.

$$\left.\begin{aligned}
u_{\alpha|\beta} &= \vec{g}_\alpha \cdot D_\beta \vec{u} = \mu^\varepsilon_\alpha \vec{a}_\varepsilon \cdot \{ (\lambda_{\gamma\beta} + x^3 \rho_{\gamma\beta}) \vec{a}^\gamma + (\phi_\beta + x^3 \psi_\beta) \vec{a}^3 \} \\
&= \mu^\gamma_\alpha (\lambda_{\gamma\beta} + x^3 \rho_{\gamma\beta}) \\
u_{\alpha|3} &= \vec{g}_\alpha \cdot D_3 \vec{u} = \mu^\varepsilon_\alpha \vec{a}_\varepsilon \cdot \vec{u}_{(1)} = \mu^\varepsilon_\alpha w_\varepsilon \\
u_{3|\beta} &= \vec{g}_3 \cdot D_\beta \vec{u} = \vec{a}_3 \cdot D_\beta \vec{u} = \phi_\beta + x^3 \psi_\beta \\
u_{3|3} &= \vec{g}_3 \cdot D_3 \vec{u} = \vec{a}_3 \cdot \vec{u}_{(1)} = p
\end{aligned}\right\} \tag{9.10}$$

Similar calculations, using (5.8), produce

$$\left.\begin{aligned}
u^\alpha|_\beta &= (u^{-1})^\alpha_\gamma (\lambda^\gamma_{.\beta} + x^3 \rho^\gamma_{.\beta}) \\
u^\alpha|_3 &= (u^{-1})^\alpha_\varepsilon w^\varepsilon \\
u^3|_\beta &= u_{3|\beta} \\
u^3|_3 &= u_{3|3}
\end{aligned}\right\} \tag{9.11}$$

Substitution into the definitions (7.19) gives then

$$(9.12)\quad \left[\begin{aligned} 2\gamma_{\alpha\beta} &= \lambda_{\alpha\beta} + \lambda_{\beta\alpha} + \lambda_{\varepsilon\alpha}\lambda^{\varepsilon}_{\cdot\beta} + \phi_\alpha\phi_\beta \\ &\quad + x^3(\rho_{\alpha\beta} + \rho_{\beta\alpha} - b^{\varepsilon}_{\alpha}\lambda_{\varepsilon\beta} - b^{\varepsilon}_{\beta}\lambda_{\varepsilon\alpha} + \rho_{\varepsilon\alpha}\lambda^{\varepsilon}_{\cdot\beta} + \rho_{\varepsilon\beta}\lambda^{\varepsilon}_{\cdot\alpha} + \phi_\alpha\psi_\beta + \phi_\beta\psi_\alpha) \\ &\quad + (x^3)^2(-b^{\varepsilon}_{\alpha}\rho_{\varepsilon\beta} - b^{\varepsilon}_{\beta}\rho_{\varepsilon\alpha} + \rho_{\varepsilon\alpha}\rho^{\varepsilon}_{\cdot\beta} + \psi_\alpha\psi_\beta) \\ 2\gamma_{\alpha 3} &= w_\alpha + \phi_\alpha + w_\varepsilon\lambda^{\varepsilon}_{\cdot\alpha} + p\phi_\alpha + x^3(\psi_\alpha - b^{\varepsilon}_{\alpha}w_\varepsilon + w_\varepsilon\rho^{\varepsilon}_{\cdot\alpha} + p\psi_\alpha) \\ 2\gamma_{33} &= 2p + p^2 + w^{\varepsilon}w_\varepsilon \end{aligned}\right.$$

Some additional physical interpretation of the surface tensors in equations (9.4) and (9.5) can be derived in the case of small material rotations. The incremental displacement field in some infinitesimal neighborhood of a point can then be analyzed, following Helmholtz, as a superposition of a pure strain and a small rotation :

$$d\vec{u} = \vec{\omega} \times d\vec{R} + \gamma_{iq}\, dx^q\, \vec{g}^{i} \qquad \text{with } \gamma_{qi} = \gamma_{iq} \tag{9.13}$$

The rotational component can be explicited, either by introducing an antisymmetrical tensor ω_{iq}

$$\vec{\omega} \times d\vec{R} = \omega_{iq}\, dx^q\, \vec{g}^{i} \tag{9.14}$$

$$\omega_{qi} = -\omega_{iq} \tag{9.15}$$

or a pseudo-tensor

$$\vec{\omega} = \omega^m \vec{g}_m \tag{9.16}$$

Then

$$\vec{\omega} \times d\vec{R} = \omega^m\, dx^q\, \vec{g}_m \times \vec{g}_q = \varepsilon_{mqr}\, \omega^m dx^q\, \vec{g}^{r}$$

Hence the relationship

$$\omega_{rq} = \varepsilon_{mqr}\, \omega^m \tag{9.17}$$

The antisymmetrical tensor is more convenient because its dependance on x^3 is simpler. Inserting (9.15) into (9.14), replacing

$$d\vec{u} = D_q\vec{u}\, dx^q$$

and comparing coefficients of dx^q, there comes

$$D_\beta\vec{u} = (\omega_{i\beta} + \gamma_{i\beta})\, \vec{g}^{i}$$

$$D_3\vec{u} = (\omega_{i3} + \gamma_{i3})\, \vec{g}^{i}$$

Scalar multiplication of those equations by $\vec{g}_j$ and evaluation of the left-hand sides as in the previous computation of the strain components, yields

$$\omega_{\alpha\beta} + \gamma_{\alpha\beta} = \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta})$$

$$\omega_{3\beta} + \gamma_{3\beta} = \phi_{\beta} + x^3 \psi_{\beta}$$

$$\omega_{\beta 3} + \gamma_{\beta 3} = \mu^{\varepsilon}_{\beta} w_{\varepsilon}$$

$$\gamma_{33} = p$$

From here, playing on the symmetry characteristics,

$$\left.\begin{aligned} 2\gamma_{\alpha\beta} &= \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta}) + \mu^{\varepsilon}_{\beta} (\lambda_{\varepsilon\alpha} + x^3 \rho_{\varepsilon\alpha}) \\ 2\gamma_{\beta 3} &= \phi_{\beta} + x^3 \psi_{\beta} + \mu^{\varepsilon}_{\beta} w_{\varepsilon} \\ 2\gamma_{33} &= 2p \end{aligned}\right\} \qquad (9.18)$$

and

$$\left.\begin{aligned} 2\omega_{\alpha\beta} &= \mu^{\varepsilon}_{\alpha} (\lambda_{\varepsilon\beta} + x^3 \rho_{\varepsilon\beta}) - \mu^{\varepsilon}_{\beta} (\lambda_{\varepsilon\alpha} + x^3 \rho_{\varepsilon\alpha}) \\ 2\omega_{3\beta} &= \phi_{\beta} + x^3 \psi_{\beta} - \mu^{\varepsilon}_{\beta} w_{\varepsilon} \end{aligned}\right\} \qquad (9.19)$$

Expressions (9.18) can be recognized as the linearized versions of (9.11), (9.12) and (9.13); they are formed from all the linear terms of the general strain components. The surface tensor of the "rotation of the normal" is, by definition

$$(\omega_{3\beta})_{x^3=0} = \frac{1}{2} (\phi_{\beta} - w_{\beta}) \qquad (9.20)$$

while the "rotation about the normal" is

$$(\omega_{12})_{x^3=0} = \frac{1}{2} (\lambda_{12} - \lambda_{21}) = \frac{1}{2} (v_{1||2} - v_{2||1}) \qquad (9.21)$$

The rate of change in the direction of the normal of the rotation about the normal, or "normal twist", plays an important role in plate theory where it can be shown to be responsible for boundary layer effects, while it vanishes under a Kirchhoff-Love hypothesis. Its expression in shell theory can be defined from

$$D_3 (\omega^3 \vec{g}_3) = (D_3 \omega^3) \vec{g}_3$$

as measured by $D_3 \omega^3$. Since according to (9.17)

$$\omega_{21} = \varepsilon_{312} \omega^3 = \sqrt{g}\, \omega^3 \qquad (9.22)$$

and in view of (5.5), (5.6) and (2.24) and (9.19)

$$2\sqrt{a}(1-2Hx^3 + (x^3)^2 K)\omega^3 = \lambda_{21} - \lambda_{12} + x^3(\rho_{21} - \rho_{12} - b^{\varepsilon}_{2}\lambda_{\varepsilon 1} + b^{\varepsilon}_{1}\lambda_{\varepsilon 2})$$

$$- (x^3)^2 (b_2 \rho_{\varepsilon 1} - b^{\varepsilon}_{1} \rho_{\varepsilon 2})$$

$$\omega^3 = \omega^3_{(0)} + x^3 \omega^3_{(1)} + (x^3)^2 \omega^3_{(2)} + \cdots$$

where $2\sqrt{a}\,\omega^3_{(0)} = \lambda_{21} - \lambda_{12}$

$$2\sqrt{a}\,(-2H\omega^3_{(0)} + \omega^3_{(1)}) = \rho_{21} - \rho_{12} - b^\varepsilon_2\lambda_{\varepsilon 1} + b^\varepsilon_1\lambda_{\varepsilon 2}$$

...

Hence for the normal twist on the middle surface

$$\omega^3_{(1)} = \frac{1}{2\sqrt{a}}\{b^\varepsilon_\varepsilon(\lambda_{21} - \lambda_{12}) + \rho_{21} - \rho_{12} - b^\varepsilon_2\lambda_{\varepsilon 1} + b^\varepsilon_1\lambda_{\varepsilon 2}\} \tag{9.23}$$

$$= \frac{1}{2\sqrt{a}}\{\rho_{21} - \rho_{12} + b^\varepsilon_1\lambda_{2\varepsilon} - b^\varepsilon_2\lambda_{1\varepsilon}\} = \frac{1}{2}\varepsilon^{\alpha\beta}(\rho_{\beta\alpha} + b^\varepsilon_\alpha\lambda_{\beta\varepsilon})$$

The Kirchhoff-Love hypothesis requires $\gamma_{\beta 3} = 0$. It can be satisfied in the linearized case by setting

$$w_\beta = -\phi_\beta = -D_\beta w - b^\varepsilon_\beta v_\varepsilon \tag{9.24}$$

thus relating the sectional rotations to the middle surface and transverse displacement. Also we should have

$$\psi_\beta - b^\varepsilon_\beta w_\varepsilon = D_\beta p = 0 \tag{9.25}$$

a condition identically satisfied by suppressing the pinch component in the original displacement assumptions. Under those conditions

$$\rho_{21} = w_{2||1} = -D_1D_2w - (b^\varepsilon_2 v_\varepsilon)_{||1}$$

and

$$\rho_{21} - \rho_{12} = b^\varepsilon_1 v_{\varepsilon||2} - b^\varepsilon_2 v_{\varepsilon||2} + v_\varepsilon\,(b^\varepsilon_{1||2} - b^\varepsilon_{2||1})$$

The last term vanishes by virtue of the Mainardi-Codazzi condition, hence, using (9.6)

$$\rho_{21} - \rho_{12} = b^\varepsilon_1\lambda_{\varepsilon 2} - b^\varepsilon_2\lambda_{\varepsilon 1}$$

$$\omega^3_{(1)} = \frac{1}{2\sqrt{a}}\{b^\varepsilon_1(\lambda_{\varepsilon 2} + \lambda_{2\varepsilon}) - b^\varepsilon_2(\lambda_{\varepsilon 1} + \lambda_{1\varepsilon})\}$$

$$= \frac{1}{2}\varepsilon^{\alpha\beta}b^\varepsilon_\alpha(\lambda_{\varepsilon\beta} + \lambda_{\beta\varepsilon}) \tag{9.26}$$

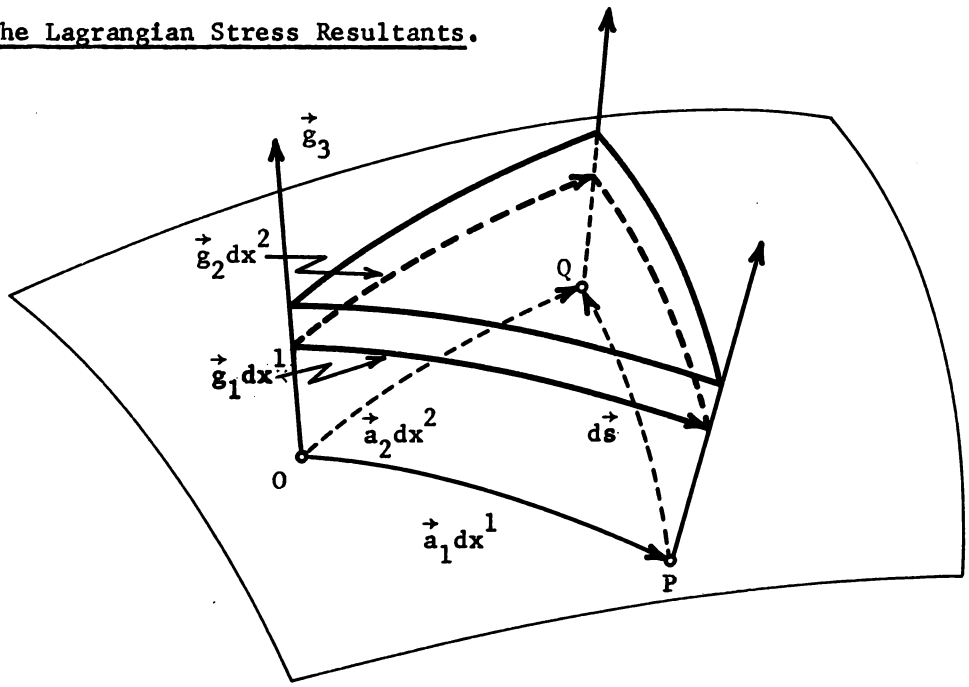
Hence, under a Kirchhoff-Love hypothesis, there remains a normal twist due to the curvature of the middle surface. Substracted from (9.23) it furnishes the "additional normal twist"

$$\Delta\omega^3_{(1)} = \frac{1}{2}\varepsilon^{\alpha\beta}(\rho_{\beta\alpha} - b^\varepsilon_\alpha\lambda_{\varepsilon\beta}) \tag{9.27}$$

Again, in view of the antisymmetry of $\varepsilon^{\alpha\beta}$ and the Mainardi-Codazzi condition, this can be put in the form

$$\Delta\omega^{3}_{(1)} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\,(w_{\beta} - b^{\varepsilon}_{\beta}\,v_{\varepsilon})_{||\alpha} \tag{9.28}$$

10 The Lagrangian Stress Resultants.



Let O, P and Q be three points in the middle surface, $\vec{OP}$ is taken along the coordinate curve x^2 = constant, $\vec{OQ}$ along the coordinate curve x^1 = constant. The "inclined" side of the triangle, $\vec{PQ}$ is an element $d\vec{s}$ of a contour drawn on the middle surface. Erecting normals to the middle surface in those three points, and measuring equal distances x^3 and dx^3 on each, we define a triangular slab of which the translational equilibrium will be expressed. Obviously at the limit when both P and Q tend to coincide with O, the only forces to take into account will be due to the stresses acting on the lateral sides of the slab. The first lateral side has a surface vector. $\vec{g}_1 \times \vec{g}_3\, dx^1\, dx^3 = -\sqrt{g}\ \vec{g}^2\, dx^1\, dx^3$ oriented outwards. The stress vector on this is $\vec{t}^2$ and the resultant force applied to it

$$-\sqrt{g}\ \sqrt{g^{22}}\ \vec{t}^2\, dx^1\, dx^3 = -\sqrt{g}\ t^{2j}\ \vec{g}_j\, dx^1\, dx^3$$

Similarly for the second lateral side with outwardly oriented surface vector

$$-\vec{g}_2 \times \vec{g}_3\, dx^2\, dx^3 = -\sqrt{g}\ \vec{g}^1\, dx^2\, dx^3$$

on which the resultant force is

$$-\sqrt{g}\ \sqrt{g^{11}}\ \vec{t}^1\, dx^2\, dx^3 = -\sqrt{g}\ t^{1j}\ \vec{g}_j\, dx^2\, dx^3$$

Then, if $d\vec{f}$ denotes the force acting on the third (inclined) lateral side, equilibrium demands

$$d\vec{f} = \sqrt{g}\ (t^{2j}\ \vec{g}_j\ dx^1 + t^{1j}\ \vec{g}_j\ dx^2)\ dx^3 \tag{10.1}$$

Let, as in section 5, $\vec{\nu}$ denote the outward normal to a closed contour described in the positive sense on the middle surface, then

$$\vec{\nu}\ ds = d\vec{s} \times \vec{a}_3 = (-\vec{a}_1\ dx^1 + \vec{a}_2\ dx^2) \times \vec{a}_3$$

$$= \sqrt{a}\ (\vec{a}^2\ dx^1 + \vec{a}^1\ dx^2)$$

or, introducing the covariant components of $\vec{\nu}$ by

$$\vec{\nu} = \nu_\alpha\ \vec{a}^\alpha$$

and comparing

$$\nu_1\ ds = \sqrt{a}\ dx^2 \qquad\qquad \nu_2\ ds = \sqrt{a}\ dx^1 \tag{10.2}$$

This result can be substituted into (10.1) after using (5.5)

$$d\vec{f} = \mu(\nu_2 t^{2j} + \nu_1 t^{1j})\ \vec{g}_j\ dsdx^3 = \mu\nu_\alpha t^{\alpha j}\vec{g}_j dsdx^3$$

or, in view of (5.1) and (5.2),

$$d\vec{f} = \mu(\nu_\alpha t^{\alpha\beta}\mu_\beta^\gamma\ \vec{a}_\gamma + \nu_\alpha t^{\alpha 3}\ \vec{a}_3)\ ds\ dx^3 \tag{10.3}$$

In order to see what are the stress resultants to introduce, we form the scalar product of this elementary force with the incremental displacement vector corresponding to the assumptions (9.1), (9.2) and (9.3)

$$\delta\vec{u} = (\delta v_\varepsilon + x^3 \delta w_\varepsilon)\ \vec{a}^\varepsilon + (\delta w + x^3 \delta p)\ \vec{a}^3$$

then integrate over the thickness of the shell :

$$\int_{-h}^{h} d\vec{f}.\delta\vec{u}\ dx^3 = (\nu_\alpha N^{\alpha\varepsilon}\delta v_\varepsilon + \nu_\alpha M^{\alpha\varepsilon}\delta w_\varepsilon + \nu_\alpha Q^\alpha dw + \nu_\alpha P^\alpha \delta p)\ ds \tag{10.4}$$

with the membrane resultants

$$N^{\alpha\varepsilon} = \int_{-h}^{h} \mu\mu_\beta^\varepsilon t^{\alpha\beta} dx^3 \tag{10.5}$$

the bending and twisting moments

$$M^{\alpha\varepsilon} = \int_{-h}^{h} \mu\mu_\beta^\varepsilon t^{\alpha\beta} x^3 dx^3 \tag{10.6}$$

the shear loads

$$Q^\alpha = \int_{-h}^{h} \mu t^{\alpha 3}\ dx^3 \tag{10.7}$$

and pinching loads

$$P^{\alpha} = \int_{-h}^{h} \mu t^{\alpha 3} x^3 \, dx^3 \tag{10.8}$$

If (10.4) is integrated over a closed contour (c) on the middle surface and transformed into an integral over the enclosed area A by means of theorem (5.9) the contribution of the stresses to the energy increase per unit initial area of the middle surface is obtained in the form

$$\delta F_1 = (N^{\alpha\varepsilon} \delta v_{\varepsilon} + M^{\alpha\varepsilon} \delta w_{\varepsilon} + Q^{\alpha} \delta w + P^{\alpha} \delta p)_{||\alpha} \tag{10.9}$$

There is also a contribution from the externally applied loads, either through body forces, or through surface tractions on the faces $x^3 = \pm h$, of the form

$$\delta F_2 = N^{\varepsilon} \delta v_{\varepsilon} + M^{\varepsilon} \delta w_{\varepsilon} + Q \delta w + P \delta p \tag{10.10}$$

From the energy equation $\delta F = \delta F_1 + \delta F_2$ the equilibium equations and the constitutive equations can be deduced by a procedure analogous to that of the general three dimensional equations in section 7.

a. Translational equilibrium equations for a piece of shell.

An additional displacement field is of translational type if

$D_3 \, \delta \vec{u} = 0$, which implies $\delta w_{\beta} = 0$ and $\delta p = o$ (10.11)

$D_{\beta} \, \delta \vec{u} = 0$, which implies $\delta \lambda_{\alpha\beta} = \delta v_{\alpha||\beta} - b_{\alpha\beta} \delta w = 0$ (10.12)

and $\delta \phi_{\beta} = D_{\beta} \delta w + b_{\beta}^{\varepsilon} \delta v_{\varepsilon} = 0$ (10.13)

The energy equation can be reduced to the form

$$\delta F = N^{\alpha\varepsilon}{}_{||\alpha} \delta v_{\varepsilon} + b_{\varepsilon\alpha} N^{\alpha\varepsilon} \delta w + Q^{\alpha}{}_{||\alpha} \delta w - Q^{\alpha} b_{\alpha}^{\varepsilon} \delta v_{\varepsilon} + N^{\varepsilon} \delta v_{\varepsilon} + Q \delta w$$

and, since $\delta F = 0$, equating separately to zero the coefficients of δv_{ε} and δw

$$N^{\alpha\varepsilon}{}_{||\alpha} - Q^{\alpha} b_{\alpha}^{\varepsilon} + N^{\varepsilon} = 0 \quad (\varepsilon = 1, 2) \tag{10.14}$$

$$Q^{\alpha}{}_{||\alpha} + b_{\varepsilon\alpha} N^{\alpha\varepsilon} + Q = 0 \tag{10.15}$$

b. Rotational equilibrium equations.

In an expanded form of (10.9), the covariant derivatives of the membrane forces and transverse shear forces can now be substituted from the equilibrium equations (10.14) and (10.15). If, likewise, the covariant derivatives of the incremental displacements are taken from the incremental form of definitions (9.6), (9.7), (9.8) and (9.9)

$$\begin{aligned} \delta F = {} & N^{\alpha\varepsilon} \delta \lambda_{\varepsilon\alpha} + M^{\alpha\varepsilon} \delta \rho_{\varepsilon\alpha} + (M^{\alpha\varepsilon}{}_{||\alpha} - b_{\alpha}^{\varepsilon} P^{\alpha} + M^{\varepsilon}) \delta w_{\varepsilon} \\ & + Q^{\alpha} \delta \phi_{\alpha} + P^{\alpha} \delta \psi_{\alpha} + (P^{\alpha}{}_{||\alpha} + b_{\varepsilon\alpha} M^{\alpha\varepsilon} + P) \delta p \end{aligned} \tag{10.16}$$

It is now necessary to derive the particular values taken by the incremental displacement parameters in a small rigid body rotation superinposed on the strained state of the elastic body. Starting from (7.14) and separating the contributions with respect to the x^3 distribution

$$\delta \vec{u}_{(o)} = d\vec{w} \times (\vec{r} + \vec{u}_{(o)}) \tag{10.17}$$

$$\delta \vec{u}_{(1)} = d\vec{w} \times (\vec{a}_3 + \vec{u}_{(1)}) \tag{10.18}$$

In these equations we substitute (9.2) and (9.3), together with

$$\vec{r} = r^{\alpha}\vec{a}_{\alpha} + r^{3}\,\vec{a}_{3}$$

$$d\vec{w} = dw^{\lambda}\,\vec{a}_{\lambda} + dw^{3}\,\vec{a}_{3}$$

and make use of

$$\vec{a}_{\lambda} \times \vec{a}_{\alpha} = \varepsilon_{\lambda\alpha}\,\vec{a}^{3} \qquad\qquad \vec{a}_{\lambda} \times \vec{a}_{3} = \varepsilon_{\alpha\lambda}\,\vec{a}^{\alpha}$$

Then, comparing covariant components

$$\left.\begin{aligned} \delta v_{\beta} &= \varepsilon_{\beta\lambda}\,(r^{3} + w)\,dw^{\lambda} + \varepsilon_{\alpha\beta}\,(r^{\alpha} + v^{\alpha})\,d\omega^{3} \\ \delta w &= \varepsilon_{\lambda\alpha}\,(r^{\alpha} + v^{\alpha})\,d\omega^{\lambda} \\ \delta w_{\beta} &= \varepsilon_{\beta\lambda}\,(1 + p)\,d\omega^{\lambda} + \varepsilon_{\alpha\beta}w^{\alpha}d\omega^{3} \\ \delta p &= \varepsilon_{\lambda\alpha}w^{\alpha}d\omega^{\lambda} \end{aligned}\right\} \qquad (10.19)$$

Next, taking partial derivatives with respect to x^{β} on both sides of (10.17) and (10.18) and remembering (9.4) and (9,5)

$$\delta\lambda_{\gamma\beta}\vec{a}^{\gamma} + \delta\phi_{\beta}\vec{a}^{3} = \Lambda^{\alpha}_{\cdot\beta}\,d\vec{\omega} \times \vec{a}_{\alpha} + \phi_{\beta}d\vec{\omega} \times \vec{a}_{3}$$

$$\delta\rho_{\gamma\beta}\vec{a}^{\gamma} + \delta\psi_{\beta}\vec{a}^{3} = R^{\alpha}_{\cdot\beta}\,d\vec{\omega} \times \vec{a}_{\alpha} + \psi_{\beta}d\vec{\omega} \times \vec{a}_{3}$$

where, for concision, the following notations were introduced

$$\Lambda^{\alpha}_{\cdot\beta} = \delta^{\alpha}_{\beta} + \lambda^{\alpha}_{\cdot\beta} \qquad\qquad R^{\alpha}_{\cdot\beta} = \rho^{\alpha}_{\cdot\beta} - b^{\alpha}_{\beta} \qquad (10.20)$$

From those equations we deduce

$$\left.\begin{aligned} \delta\lambda_{\gamma\beta} &= \varepsilon_{\alpha\gamma}\Lambda^{\alpha}_{\cdot\beta}\,d\omega^{3} + \varepsilon_{\gamma\lambda}\phi_{\beta}\,d\omega^{\lambda} \\ \delta\phi_{\beta} &= \varepsilon_{\lambda\alpha}\Lambda^{\alpha}_{\cdot\beta}\,d\omega^{\lambda} \\ \delta\rho_{\gamma\beta} &= \varepsilon_{\alpha\gamma}R^{\alpha}_{\cdot\beta}\,d\omega^{3} + \varepsilon_{\gamma\lambda}\psi_{\beta}\,d\omega^{\lambda} \\ \delta\psi_{\beta} &= \varepsilon_{\lambda\alpha}R^{\alpha}_{\cdot\beta}\,d\omega^{\lambda} \end{aligned}\right\} \qquad (10.21)$$

The results (10.19) and (10.21) are to be inserted into (10.16) and since $\delta F = 0$ again, while $d\vec{w}$ is an arbitrary constant vector, the following general conditions for rotational equilibrium are found :

b.1. equilibrium about the normal to the middle surface, obtained by equating to zero the coefficient of $d\omega^{3}$

$$\varepsilon_{\gamma\beta}\left[N^{\alpha\beta}\Lambda^{\gamma}_{\cdot\alpha} + M^{\alpha\beta}R^{\gamma}_{\cdot\alpha} + S^{\beta}w^{\gamma}\right] = 0 \qquad (10.22)$$

with the definition

$$S^{\beta} = M^{\alpha\beta}{}_{||\alpha} - b^{\beta}_{\alpha}P^{\alpha} + M^{\beta} \qquad (10.23)$$

Considering the antisymmetry of $\varepsilon_{\gamma\beta}$, this is equivalent to require the symmetry of the bracket in (10.22) with respect to β and γ.

b.2. equilibrium about axes tangent to the middle surface, expressed by equating to zero the coefficient of $\varepsilon_{\gamma\lambda}\, d\omega^{\lambda}$

$$N^{\alpha\gamma}\phi_{\alpha} + M^{\alpha\gamma}\psi_{\alpha} + S^{\gamma}(1+p) - Q^{\alpha}\Lambda^{\gamma}_{\cdot\alpha} - P^{\alpha}R^{\gamma}_{\cdot\alpha} - Tw^{\gamma} = 0 \qquad (10.24)$$

$$(\gamma = 1, 2)$$

with the definition

$$T = P^{\alpha}{}_{||\alpha} + b_{\varepsilon\alpha}M^{\alpha\varepsilon} + P$$

As in the case of the general three-dimensional equilibrium equations satisfied by lagrangian stresses, the lagrangian resultants obey linear translational equilibrium conditiond (10.14) and (10.15), but their rotational equilibrium conditions (10.22) and (10.24) also involve deformations.

An attempt to calculate δF directly from the general result (7.13) and the relation

$$\delta W dV = \delta W\sqrt{g}\, dx^1dx^2dx^3 = \delta W\mu\sqrt{a}\, dx^1dx^2dx^3 = \delta W\mu dx^3dS$$

or

$$\delta F = \int_{-h}^{h} \mu\delta W\, dx^3$$

gives, in view of definitions (10.5) to (10.8) and results (9.10)

$$\delta F = N^{\beta\gamma}\delta\lambda_{\gamma\beta} + M^{\beta\gamma}\delta\rho_{\gamma\beta} + Q^{\beta}\delta\phi_{\beta} + P^{\beta}\delta\psi_{\beta} + S^{\beta}\delta w_{\beta} + T\delta p \qquad (10.26)$$

which is similar to (10.16) and shows that the tensors defined by (10.23) and (10.25) are also resultants

$$S^{\beta} = \int_{-h}^{h} \mu\mu^{\beta}_{\alpha}\, t^{3\alpha}\, dx^3 \qquad T = \int_{-h}^{h} \mu t^{33}\, dx^3 \qquad (10.27)$$

They do not, however, constitute internal loads, since they involve lagrangian stresses which are defined at interfaces between shell pieces.
As a matter of fact, the rotational equilibrium condition about the displaced normal to the middle surface does not contain S^{β} nor T.
To show it we calculate the base vectors in the deformed metric from (7.22) and (9.11) obtaining

$$\vec{G}_{\gamma} = (\Lambda^{\varepsilon}_{\cdot\gamma} + x^3R^{\varepsilon}_{\cdot\gamma})\,\vec{a}_{\varepsilon} + (\phi_{\gamma} + x^3\psi_{\gamma})\,\vec{a}_3 \qquad (10.28)$$

$$\vec{G}_3 = w^{\varepsilon}\vec{a}_{\varepsilon} + (1+p)\,\vec{a}_3 \qquad (10.29)$$

The displaced base vectors on the middle surface are then given by

$$\vec{A}_{\gamma} = \Lambda^{\varepsilon}_{\cdot\gamma}\vec{a}_{\varepsilon} + \phi_{\gamma}\vec{a}_3 \qquad (10.30)$$

$$\vec{A}_3 = w^{\lambda}\vec{a}_{\lambda} + (1+p)\,\vec{a}_3 = \vec{G}_3 \qquad (10.31)$$

Thus, if we denote by $R^{\gamma\beta}$ the bracket in (10.22) and by R^{γ} the left-hand side of (10.24)

$$\varepsilon_{\gamma\beta} R^{\gamma\beta} d\omega^3 + \varepsilon_{\gamma\lambda} R^{\gamma} d\omega^{\lambda} = 0$$

is the general rotational equilibrium condition, while for a rotation $d\Omega^3$ about the displaced normal $\vec{A}^3$

$$d\omega^{\lambda} = w^{\lambda} d\Omega^3 \qquad d\omega^3 = (1 + p)\, d\Omega^3$$

and the corresponding condition is

$$(1 + p)\, \varepsilon_{\gamma\beta} R^{\gamma\beta} + w^{\lambda} \varepsilon_{\gamma\lambda} R^{\gamma} = 0$$

or symmetry of

$$(1 + p)\, R^{\gamma\beta} + w^{\beta} R^{\gamma}$$

In this linear combination S^{β} and T do precisely appear in symmetrical groupings

$$(1 + p)\, (w^{\gamma} S^{\beta} + w^{\beta} S^{\gamma}) - T w^{\gamma} w^{\beta}$$

and may consequently be dropped leaving the condition

$$(1 + p)\, N^{\alpha\beta} \Lambda^{\gamma}_{\cdot\alpha} + w^{\beta} \phi_{\alpha} N^{\alpha\gamma} + (1 + p)\, M^{\alpha\beta} R^{\gamma}_{\cdot\alpha} + w^{\beta} \psi_{\alpha} M^{\alpha\gamma} - Q^{\alpha} w^{\beta} \Lambda^{\gamma}_{\cdot\alpha} - P^{\alpha} w^{\beta} R^{\gamma}_{\cdot\alpha} \quad \text{symmetrical in } (\beta, \gamma) \tag{10.32}$$

The equilibrium conditions (10.24) can now be incorporated into the energy equation (10.26) by using them to eliminate S^{β}. This gives

$$\begin{aligned} \delta F = {} & N^{\beta\gamma} \left(\delta\lambda_{\gamma\beta} - \frac{1}{1 + p} \phi_{\beta} \delta w_{\gamma} \right) + M^{\beta\gamma} \left(\delta\rho_{\gamma\beta} - \frac{1}{1 + p} \psi_{\beta}\, \delta w_{\gamma} \right) \\ & + Q^{\beta} \left(\delta\phi_{\beta} + \frac{1}{1 + p} \Lambda^{\gamma}_{\cdot\beta} \delta w_{\gamma} \right) + P^{\beta} \left(\delta\psi_{\beta} + \frac{1}{1 + p} R^{\gamma}_{\cdot\beta} \delta w_{\gamma} \right) \\ & + T \left(\delta p + \frac{1}{1 + p} w^{\gamma} \delta w_{\gamma} \right) \end{aligned} \tag{10.33}$$

This new form of the energy equation cannot be expected to turn into a canonical form similar to (7.20), from which constitutive equations can be rationally derived, untill the last rotational equilibrium equation (10.32) can be solved and the constraint it constitutes between the lagrangian stress resultants can be liberated.

11. Kirchhoff-Trefftz stress parameters.

Since the Kirchhoff-Trefftz stress tensor contains implicitly the rotational equilibrium conditions in its property of symmetry, an immediate solution to our problem consists in transforming the definitions (10.5) to (10.8) by means of the relations (7.17). In our case, considering equations (9.11), we can substitute

$$\begin{aligned} t^{\alpha\beta} &= s^{\alpha\beta} + u^{\beta}|_{\varepsilon} s^{\varepsilon\alpha} + u^{\beta}|_{3} s^{3\alpha} \\ &= s^{\alpha\beta} + (\mu^{-1})^{\beta}_{\gamma} (\lambda^{\gamma}_{\cdot\varepsilon} + x^3 \rho^{\gamma}_{\cdot\varepsilon})\, s^{\varepsilon\alpha} + (\mu^{-1})^{\beta}_{\gamma} w^{\gamma} s^{3\alpha} \\ t^{\alpha 3} &= s^{\alpha 3} + u^{3}|_{\beta} s^{\beta\alpha} + u^{3}|_{3} s^{3\alpha} \\ &= s^{\alpha 3} + (\phi_{\beta} + x^3 \psi_{\beta}) s^{\beta\alpha} + p s^{3\alpha} \\ t^{3\alpha} &= s^{3\alpha} + u^{\alpha}|_{\beta} s^{\beta 3} + u^{\alpha}|_{3} s^{33} \\ &= s^{3\alpha} + (\mu^{-1})^{\alpha}_{\gamma} (\lambda^{\gamma}_{\cdot\beta} + x^3 \rho^{\gamma}_{\cdot\beta}) s^{\beta 3} + (\mu^{-1})^{\alpha}_{\gamma} w^{\gamma} s^{33} \\ t^{33} &= s^{33} + u^{3}|_{\beta} s^{\beta 3} + u^{3}|_{3} s^{33} \\ &= s^{33} + (\phi_{\beta} + x^3 \psi_{\beta})\, s^{\beta 3} + p\, s^{33} \end{aligned} \tag{11.1}$$

The lagrangian stress resultants are then expressed in terms of the following Kirchhoff-Trefftz stress parameters

$$
\begin{aligned}
n^{\alpha\beta} &= \int_{-h}^{h} \mu s^{\alpha\beta} dx^3 = n^{\beta\alpha} \\
m^{\alpha\beta} &= \int_{-h}^{h} \mu s^{\alpha\beta} x^3 dx^3 = m^{\beta\alpha} \\
1^{\alpha\beta} &= \int_{-h}^{h} \mu s^{\alpha\beta} (x^3)^2 \, dx^3 = 1^{\beta\alpha} \\
q^{\beta} &= \int_{-h}^{h} \mu s^{\beta 3} \, dx^3 \\
p^{\beta} &= \int_{-h}^{h} s^{\beta 3} \, x^3 \, dx^3 \\
n &= \int_{-h}^{h} \mu s^{33} \, dx^3
\end{aligned}
\tag{11.2}
$$

There comes

$$
\begin{aligned}
N^{\alpha\varepsilon} &= \Lambda^{\varepsilon}_{\cdot\gamma} \, n^{\gamma\alpha} + R^{\varepsilon}_{\cdot\gamma} \, m^{\gamma\alpha} + w^{\varepsilon} q^{\alpha} \\
M^{\alpha\varepsilon} &= \Lambda^{\varepsilon}_{\cdot\gamma} \, m^{\gamma\alpha} + R^{\varepsilon}_{\cdot\gamma} \, 1^{\gamma\alpha} + w^{\varepsilon} p^{\alpha} \\
Q^{\alpha} &= (1 + p) \, q^{\alpha} + \phi_{\gamma} \, n^{\gamma\alpha} + \psi_{\gamma} \, m^{\gamma\alpha} \\
P^{\alpha} &= (1 + p) \, p^{\alpha} + \phi_{\gamma} \, m^{\gamma\alpha} + \psi_{\gamma} \, 1^{\gamma\alpha}
\end{aligned}
\tag{11.3}
$$

and also

$$
\begin{aligned}
S^{\beta} &= \Lambda^{\beta}_{\cdot\alpha} q^{\alpha} + R^{\beta}_{\cdot\alpha} p^{\alpha} + w^{\beta} n \\
T &= (1 + p) \, n + \phi_{\alpha} q^{\alpha} + \psi_{\alpha} p^{\alpha}
\end{aligned}
\tag{11.4}
$$

It is easily verified that all the rotational equilibrium conditions (10.22) and (10.24) or (10.32) are thereby satisfied. Furthermore the energy equation (10.26) takes the canonical form

$$
\begin{aligned}
\delta F = {} & n^{\gamma\alpha} \overset{\circ}{\delta\gamma}_{\gamma\alpha} + m^{\gamma\alpha} \overset{1}{\delta\gamma}_{\gamma\alpha} + 1^{\gamma\alpha} \overset{2}{\delta\gamma}_{\gamma\alpha} \\
& + 2q^{\alpha} \overset{\circ}{\delta\gamma}_{\alpha 3} + 2p^{\alpha} \overset{1}{\delta\gamma}_{\alpha 3} + n \delta\gamma_{33}
\end{aligned}
\tag{11.5}
$$

where the strains are the coefficients of the powers in x^3 of the Green strain tensor (9.12)

As a matter of fact, this result is equivalent to a straight forward application of a Rayleigh-Ritz process based on the displacement assumptions (9.1) to (9.3) to the expression

$$\delta F = \int_{-h}^{h} \mu \delta W \, dx^3$$

where δW is given by (7.20). Applied to the form (8.1) of the energy density it introduces the following rigidity coefficients for the shell :

$$D^{\alpha\beta}_{(m)}{}^{\gamma\delta} = \frac{E}{1-\nu^2} \int_{-h}^{h} \mu g^{\alpha\beta} g^{\gamma\delta} (x^3)^m \, dx^3 \qquad (m = 0,1,\ldots 4) \qquad (11.6)$$

$$C^{\alpha\beta}_{(m)} = \frac{E}{1-\nu^2} \int_{-h}^{h} \mu g^{\alpha\beta} (x^3)^m \, dx^3 \qquad (m = 0,1,2) \qquad (11.7)$$

Taking due account of the symmetries, the total number of rigidity coefficients of type (11.6) is $6 \times 5 = 30$, of type (11.7) $3 \times 3 = 9$.
The energy per mit area is then expressible as

$$\begin{aligned}
2F = {} & D^{\alpha\lambda}_{(0)}{}^{\beta\mu} \{ \nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{\circ}{\gamma}_{\beta\mu} + (1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{\circ}{\gamma}_{\beta\lambda} \} \\
& + 2 D^{\alpha\lambda}_{(1)}{}^{\beta\mu} \{ \nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{1}{\gamma}_{\beta\mu} + (1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{1}{\gamma}_{\beta\lambda} \} \\
& + D^{\alpha\lambda}_{(2)}{}^{\beta\mu} \{ 2\nu \overset{\circ}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + \nu \overset{1}{\gamma}_{\alpha\lambda} \overset{1}{\gamma}_{\beta\mu} + 2(1-\nu) \overset{\circ}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} + (1-\nu) \overset{1}{\gamma}_{\alpha\mu} \overset{1}{\gamma}_{\beta\lambda} \} \\
& + 2 D^{\alpha\lambda}_{(3)}{}^{\beta\mu} \{ \nu \overset{1}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + (1-\nu) \overset{1}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} \} \\
& + D^{\alpha\lambda}_{(4)}{}^{\beta\mu} \{ \nu \overset{2}{\gamma}_{\alpha\lambda} \overset{2}{\gamma}_{\beta\mu} + (1-\nu) \overset{2}{\gamma}_{\alpha\mu} \overset{2}{\gamma}_{\beta\lambda} \} \\
& + 2(1-\nu) \{ C^{\alpha\beta}_{(0)} \overset{\circ}{\gamma}_{\alpha 3} \overset{\circ}{\gamma}_{\beta 3} + 2 C^{\alpha\beta}_{(1)} \overset{\circ}{\gamma}_{\alpha 3} \overset{1}{\gamma}_{\beta 3} + C^{\alpha\beta}_{(2)} \overset{1}{\gamma}_{\alpha 3} \overset{1}{\gamma}_{\beta 3} \}
\end{aligned} \qquad (11.8)$$

The constitutive equations of the shell follow by taking the partial derivatives of (11.8) as indicated in the total differential (11.5).
In particular, since (11.8) is independant of γ_{33}, we find $n = 0$ in accordance with the assumption $\sigma^{33} = 0$.
The integrals in (11.6) and (11.7) are not easily carried out in closed form because of the complicated dependance of the reciprocal metric tensor (5.10) on the coordinate x^3. They can be evaluated by numerical quadrature methods or expanded in powers of a (small) parameter.

It follows from (11.5) that the Kirchhoff-Trefftz stress parameters are internal conjugate variables to the arguments of the strain energy per unit surface :

$$n^{\gamma\alpha} = \frac{\partial F}{\partial \overset{\circ}{\gamma}_{\gamma\alpha}} \qquad m^{\gamma\alpha} = \frac{\partial F}{\partial \overset{1}{\gamma}_{\gamma\alpha}} \qquad 1^{\gamma\alpha} = \frac{\partial F}{\partial \overset{2}{\gamma}_{\gamma\alpha}}$$

$$2q^{\alpha} = \frac{\partial F}{\partial \overset{\circ}{\gamma}_{\alpha 3}} \qquad 2p^{\alpha} = \frac{\partial F}{\partial \overset{1}{\gamma}_{\alpha 3}} \qquad n = \frac{\partial F}{\partial \gamma_{33}} \tag{11.9}$$

If those arguments as given by (9.12) are expressed through definitions (9.6) to (9.9) in terms of v_α, w_α, w, p and their derivatives, it is easily found by the chain rule of differentiation and by the results (11.3) and (11.4) that

$$\frac{\partial F}{\partial v_{\alpha||\beta}} = N^{\beta\alpha} \qquad \frac{\partial F}{\partial w_{\alpha||\beta}} = M^{\beta\alpha}$$

$$\frac{\partial F}{\partial w_{||\alpha}} = Q^{\alpha} \qquad \frac{\partial F}{\partial p_{||\alpha}} = P^{\alpha} \tag{11.10}$$

$$\frac{\partial F}{\partial p} = T - b_{\alpha\beta} M^{\alpha\beta} \qquad \frac{\partial F}{\partial w_{\alpha}} = b^{\alpha}_{\varepsilon} P^{\varepsilon} + S^{\alpha}$$

$$\frac{\partial F}{\partial w} = - b_{\alpha\beta} N^{\alpha\beta} \qquad \frac{\partial F}{\partial v_{\alpha}} = b^{\alpha}_{\varepsilon} Q^{\varepsilon} \tag{11.11}$$

Equations (11.10) can then be considered as the constitutive equations of the shell in terms of Lagrangian resultants. Since they involve (11.3) and (11.4) they automatically satisfy the rotational equilibrium equations. They also allow to apply in a simple fashion the variational principle

$$\delta \{ \int_S F dS - \int_S (N^{\alpha} v_{\alpha} + M^{\alpha} w_{\alpha} + Qw + Pp) dS + F_3 \} = 0 \tag{11.12}$$

where F_3 denotes the potential energy of external loads applied at the boundary. From (11.10) and (11.11) there comes

$$\int_S \{ N^{\beta\alpha} \delta v_{\alpha||\beta} + M^{\beta\alpha} \delta w_{\alpha||\beta} + Q^{\alpha} \delta w_{||\alpha} + P^{\alpha} \delta p_{||\alpha}$$

$$+ (b^{\alpha}_{\varepsilon}\, Q^{\varepsilon} - N^{\alpha})\ \delta v_{\alpha} + (S^{\alpha} + b^{\alpha}_{\varepsilon}\, P^{\varepsilon} - M^{\alpha})\ \delta w_{\alpha}$$

$$- (Q + b_{\alpha\beta}\, N^{\alpha\beta})\ \delta w + (T - b_{\alpha\beta}\, M^{\alpha\beta} - P)\ \delta p\ \}\ dS + \delta F_3 = 0$$

After integration by parts, the following Euler-Lagrange equilibrium equations are obtained

for δv_{α} $$- N^{\beta\alpha}{}_{||\beta} + b^{\alpha}_{\varepsilon}\, Q^{\varepsilon} - N^{\alpha} = 0 \qquad (11.13)$$

δw_{α} $$- M^{\beta\alpha}{}_{||\beta} + S^{\alpha} + b^{\alpha}_{\varepsilon}\, P^{\varepsilon} - M^{\alpha} = 0 \qquad (11.14)$$

δw $$- Q^{\alpha}{}_{||\alpha} - Q - b_{\alpha\beta}\, N^{\alpha\beta} = 0 \qquad (11.15)$$

δp $$- P^{\alpha}{}_{||\alpha} + T - b_{\alpha\beta}\, M^{\alpha\beta} - P = 0 \qquad (11.16)$$

Associated with boundary conditions resulting from the variational equation

$$\delta F_3 + \phi\nu_{\beta}\,(N^{\beta\alpha}\delta v_{\alpha} + M^{\beta\alpha}\delta w_{\alpha} + Q^{\alpha}\delta w + P^{\alpha}\delta p)\ ds = 0 \qquad (11.17)$$

In equations (11.13) and (11.15) we recognize the translational equilibrium equations already obtained in (10.14) and (10.15), while (11.14) and (11.16) are respectively equivalent to the definitions (10.23) and (10.25) of S^{α} and T.
The field equations of the two-dimensional non linear shell problem consist of the 6 linear equilibrium equations (11.13) to (11.16) together with the 15 resultants-displacement derivatives equations (11.10). Those are to be worked out by using in succession (11.8), (9.12) and (9.6) to (9.9). The total number of unknowns is also 21 : the 6 displacements (v_{α}, w_{α}, w, p) and the 15 components of resultants in

$N^{\alpha\beta}$, $M^{\alpha\beta}$, Q^{α}, P^{α}, S^{α}, T because of the identities

$$\frac{\partial F}{\partial v_{\alpha}} = b^{\alpha}_{\varepsilon}\,\frac{\partial F}{\partial \phi_{\varepsilon}} = b^{\alpha}_{\varepsilon}\,\frac{\partial F}{\partial w_{||\varepsilon}}$$

and

$$\frac{\partial F}{\partial w} = - b_{\alpha\beta}\,\frac{\partial F}{\partial \lambda_{\alpha\beta}} = - b_{\alpha\beta}\,\frac{\partial F}{\partial v_{\alpha||\beta}}$$

stemming from (9.9) and (9.6), equations (11.11) must not be counted in the constitutive equations.

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A P P E N D I X

The ASEF Computer Program - Version 20

by P. BECKERS, M. WEELEN, G. SANDER
University of Liège, Belgium

SUMMARY

A1 General description

ASEF stands for "Analyse des Structures par Eléments finis". The version 20 is the result of the evolution over 6 years of a computer program for static, linear dual analysis of structures. The main function of the ASEF program is to provide a research tool for investigations of new finite elements and new methods of solution. The second function is to provide the capability of solving occasionally large, practical problems and evaluate on these examples the impact of the advantages brought by new theories or new elements. Compared with other existing programs, ASEF presents the following peculiarities or advantages :

- It is written in FORTRAN IV G for a relatively small computer (IBM 360-44). The programming methods are completely machine-independant. The minimum configuration to run the program requires 200 K bytes or 25 K words (double precision on IBM 360) of core storage, 10 sequential files on tape, disk or drum and an operating system similar to the IBM-OS-360. The overlay feature is obviously mandatory. The version of the program accompanying this report is specifically adapted to an IBM-360-65 using a 256 K bytes memory, 5 disk drives 2314, 8 disk drives 2311. It was tested for the O.S. Release 17 working in MFT mode.

- Variable dimensions are used for the core storage allocations. Hence it is possible to take advantage of the multiprocessing capabilities of modern computers (by using small partitions to solve small problems). This is controlled by changing two statements only.

- New elements of any type : conforming, equilibrium, hybrid, membrane, plate, shell, three dimensional etc, are very easy to introduce.

- The existing library contains various finite elements for conforming, equilibrium and hybrid analysis of membrane and flat plate structures. The degree of these elements range from the simplest to the most sophisticated.

- It is possible to idealize a structure using conforming elements of variable degree (see section 2.1)

- Although the program works basically according to the displacement method, additional linear constraints in any number,of any kind,can be processed efficiently (like in the combined method).

- The modular presentation allows easy modifications to suit special problems.

- It is possible to solve problems of up to 5000 degrees of freedom on a 256 K bytes machine, 8 to 10 000 degrees of freedom on a 512 K bytes machine.

- The input data are easy to prepare if due consideration is given to the large number of finite element models proposed.

- Comprehensive checks for input data errors are incorporated.

- The possibility is provided to subdivide a solution in steps allowing partial checking, recovery in case of error or modification to the problem during the solution.

- A direct method of solution (Substructures method) is proposed as well as an iterative method based on conjugate gradients. Very good accuracy is obtained with both methods due to the systematic use of double precision (on IBM 360).

- Running cost of the program is relatively cheap

- An interface program is available for a graphic computer program using a BENSON-FRANCE tracing table.

- Another interface program is available for a dynamic analysis program DYNAM-5 (Ref A1)

A2. Organisation of the program

A general flow diagram is presented on figure A1. Figure A2 displays the detailed organisation of the overlay structure. The names of the 8 branching points are indicated after each of the "overlay" statements, while the names of the 225 subroutines or common areas appear after the "insert" statements of the different sections.

The execution of the program is subdivided into 6 main phases indicated by the circled numbers in the general flow chart. The names of the main subroutines controlling each phase is also indicated on figure A1. A normal execution starts by the phases 1, 2, 3, then either 4A or 4B, and ends by phases 5 and 6. These phases are grouped in two sets : the phases 1, 2, 6 which share the characteristic of using long and numerous subroutines but requiring a relatively small working space in the core storage, and the phases 3, 4, 5 which use smaller subroutines but require the largest possible core storage working space for the solution of the system of linear equations. The common areas named ASEFCO and BSEFCO are the corresponding core storage areas used as working space in these two main sections. The various arrays are dynamically allocated in them by the program itself.

The function of the different phases can be described as follows :

PHASE 1 : INPUT DATA PREPARATION

The input data given in the simple form detailed in the next section are first listed as card images. Comprehensive checking follows for their consistency. Various additional data are next deduced as described in the section dealing with the details of each phase. It is also checked that the core storage allocation is sufficient for the solution of the problem. The output of this module in another set of card images (much larger than the primary input) stored on the unit of symbolic name U5. In a normal execution, this data set is transmitted as input for the next phases. An optional exit exists for partial checking of these "prepared data" or their "manual" modification.

PHASE 2 : ELEMENTS GENERATION

From phase 2 to 5 the input data are read sequentially on unit U5 which simulates a card reader. They have been sorted in the required order by phase 1. In phase 2, for each element of the structure, the subroutine corresponding to the type of element specified is selected in the elements library. The stiffness matrix in global axes is stored with its localization vector on unit 2. All the informations necessary to compute the

stresses are stored on unit 4 for use in phase 6. The consistent loading corresponding to body forces, temperature distribution, transverse loading on plates or condensation of internal degrees of freedom is computed and stored on unit 13. When necessary the particular solution for the stresses and the corresponding potential energy term due to those body forces is also computed.

PHASE 3 : APPLICATION OF THE BOUNDARY CONDITIONS

The load vector (s) is (are) assembled by combining the external loading with the consistent loads eventually computed in phase 2, due to body forces. The degrees of freedom fixed are read from unit 5. Then the diagonal terms of the global stiffness matrix are checked to determine their minimum and maximum values which are printed for checking purposes. If zero diagoanl terms are found, they are automatically fixed, but a warning message is printed.

PHASE 4 : DEPENDING OF THE OPTION SELECTED, THE SOLUTION IS EITHER BY AN ITERATIVE METHOD (PHASE 4A) OR BY A DIRECT METHOD (PHASE 4B).

Phase 4A : Iterative method

The conjugate gradients method is used which is at the same time an iterative method and an exact method, if n steps are achieved,n being the order of the system (Ref. A2, 23). The algorithm used takes advantage of the possibility not to assemble physically the stiffness matrix when iterative method are used (Ref. A4). As it is only necessary to compute products of the stiffness matrix by a vector, the stiffness matrix of each element in turn is multiplied by the selected terms of the vector and the results addressed correspondingly.

The element stiffness matrices are first read from unit 2, scaled in such a way that the diagonal terms of the global stiffness matrix are all equal to unity, and stored on unit 3 in a compacted form in which all fixations have already been applied. The iteration algorithm works only from this compact set of matrices. The precision is controlled by the magnitude of the square of the residual vector and the iterations stopped when a given level of precision or when a given maximum number of iterations is reached. As starting vector, the standard option is zero, but if a better approximation is known, it can be put in optionally from unit 1.

Phase 4B : Direct solution

The classical substructure method is used (Ref. A5). The algorithm used for the inversion of the sub-matrices is due to Crout-Banachiewicz-Cholesky (Ref. A6). The solution of the final system of equations uses a Gauss

algorithm. As the Cholesky method may not be used if zero diagonal terms are present, a situation which can occur when additional constraints equations are considered, special care has to be taken in introducing the constraining elements to avoid it. This is treated in detail in the section describing the elements of constraint.

The dimensions of the various substructures can be defined in different ways : the user can group the elements into substructures "manually". In this case the program checks that their dimensions do not exceed the available core storage. A second possibility is to order the elements in such a way that the cut in the structure always contains less degrees of freedom than the maximum number allowed by the core storage allocation. In this case the program choses a subdivision in substructures such that their dimensions are as close as possible to the maximum. The third possibility is to input an optimum dimension for a substructure, but less than the maximum dimension. The program tries to satisfy this requirement as much as possible. Subdivisions are in fact decided during Phase 1.

While it is only possible to solve for one load case when the iterative method is used, with the direct method many load cases can be treated simultaneously and economically. This is achieved however at the expense of a reduction in the maximum dimension of the substructures.

PHASE 5 : PRINTING OF THE DISPLACEMENTS

The only displacements printed are those for which a non zero stiffness has been found. The standard output yields at each node and along each interface the values of the various displacement components. The word component is taken here in a generalized sense : it means each generalized displacement attached to a node or to an interface in the conventional sequence indicated for each particular element used. In addition the second member (s) is (are) recomputed by multiplying the original stiffness matrix by the computed solution. This result is printed with the displacement components and allows a very efficient checking of the accuracy of the solution.

PHASE 6 : STRESS COMPUTATION

This phase is quite independant of the preceding ones. It is possible to interrupt the program after phase 5 to achieve partial checking and to restart it for phase 6 afterwards. For each type of element a particular subroutine is chosen which selects the displacements pertaining to the element and, using the informations stored on unit 4, computes the stresses, moments or fluxes and prints them. Local axes are always used. They are

defined conventionally for each type of element depending on the order of numbering of the nodes. The program does not allow to combine the stresses for different loading cases but it is easy to store the output for such an operation.

Partial execution of the program

With the purpose to allow slicing the execution of large problems into steps of reasonable length and to achieve partial checking during the execution, four entry points and five exit points are provided. The program can be started by phases 1, 2, 3 or 6; the exit can be started after phases1, 2, 3 or 6. Eventually the final condensed stiffness matrix of the last substructure can be saved with the corresponding loads to be used either in another program (such as in dynamics) or as an input for another execution.

A3. Input data preparation

This section describes the input data required for a normal execution of the program. They are read during phase 1 only.
The input data are grouped in six categories. To each of them corresponds an input data form represented in figures A3 to A8.
The meaning of the variables indicated on these forms is the following :

FORM 1 : General informations and control variables

The first card for an execution contains the control variables :

ISTART : Defines the entry point of the program and can take the values 1, 2, 3, 4 only. The corresponding entry points are : before phase 1 (normal execution), between phases 1 and 2, between phases 2 and 3, between phases 5 and 6.

ISTOP : Defines the exit point and can take the values 1, 2, 3, 4, 5 only. The corresponding exit points are : after phase 1, after phase 2, after phase 5, after phase 6 and eventually treatment of the next problem (normal execution), after phase 6 without considering another problem that might be stacked behind.
N.B.: If ISTART equals 4, ISTOP is set to 5

IU5 : FORTRAN symbolic number of the unit on which the input data are to be stored between phase 1 and 2. This unit contains card images. In case ISTART = 2, the unit IU5 can be the card reader (standard option : IU5 = 5). Otherwise the standard option is IU5 = 10.

These 3 variables are given once only for an execution while the set of all the others can be repeated as many times as there are problems to be executed in the run.

TITLE : Any title to be printed in front of the output and composed of 2 cards of 80 alphanumeric characters.

The next card furnishes general informations for the problem. None of these informations is strictly necessary : a standard option exists for all of them. If none is specified a blank card must however be present.

NDC : Defines the number of displacement components along an interface in global, three dimensional axes. This variable is only used for conforming membrane elements (triangle, quadrilateral, parallelogram or bar). It corresponds to a general indication that the particular data of the elements can override. (This is the case when variable

degree elements are used). NDC can take the values 0, 3, 6, 9, 12 only which correspond to displacement fields in the elements which are linear, quadratic, cubic, quartic or quintic along the interfaces. If NDC is given another value, it is set equal to zero.
For elements other than conforming membranes and bars, this variable is set by the program to the correct value according to the type of elements used, no mather what NDC is on the input data card.

IFO : Must be 0 or 1 (if different, it is set equal to 0). IFO = 0 specifies that no body forces are considered in the problem, overriding eventually the specifications given in the elements characteristics. IFO = 0 specifies that the consistant loading corresponding to the body forces specified in the element characteristics is to be computed.

IGRAD : Must be 0 or 1 (otherwise it is set equal to 0).
IGRAD = 0 specifies that the direct method of solution is to be used.
IGRAD = 1 specifies that the conjugate gradients method is to be used.
If IGRAD = 0 the following three variables are ineffective.

NITER : Maximum number of iterations for the conjugate gradients method. If NITER $\leq$ 0, it is set equal to the largest number given to a degree of freedom in the problem.

LPREC : Defines the level of precision required in the conjugate gradients method. It must be such that $1 \leq$ LPREC ≤ 8 otherwise it is set equal to 5. The iterations are stopped when the square of the scaled residual vector becomes less or equal 10^{-2} exponent LPREC.

IDEP : Must be 0 or 1 (otherwise it is set equal to 0).
If IDEP = 0 the initial vector in the conjugate gradients method is zero.
If IDEP = 1 the initial vector is read on unit 1.
N.B.: IDEP = 1 implies that ISTART = 2

NS : Number of substructures (If not specified it is set equal to 1).
If NS = 1, an automatic subdivision into substructures is achieved if necessary. Their dimensions depend upon the dimension of the common area BSEFCO and upon the variable NCSOPT.

NTO : Number from which the numbering of the degrees of freedom is to be started. This number is arbitrary. However it must be mentionned that some of the working spaces in the common area ASEFCO and BSEFCO are proportional to the largest number given to a degree of freedom. This option should therefore be used only when different structures have to be joined which were computed separately and otherwise NTO is set equal to zero.

NC : Number of boundary conditions on the displacements. If NC = 0, it is set equal to 1. For each type of boundary condition all the fixations and loads have to be redefined.

NCSOPT : Optimum maximum dimension for a substructure. If NCSOPT = 0, the maximum dimension allowed in the common area BSEFCO is taken. NCSOPT has only to be specified if the automatic subdivision in substructures is used and if it is desirable to use substructures smaller than the maximum dimension allowed by the common area BSEFCO. If in a substructure the program finds that it is impossible to keep the dimension smaller than NCSOPT, it uses the smallest dimension between NCSOPT and the maximum dimension. If it is impossible to keep the dimension of a substructure smaller than the maximum dimension, an error message is printed and the program skips to the next problem.

NOPi : Printing option in the various phases of the program.

NOPi = 0 corresponds to printing only the results and some intermediate output allowing to check that the execution is normally echieved

NOPi = 5 corresponds to printing all the intermediate results including many matrices

1 < NOPi < 5 corresponds to printing only a part of the intermediate output

NOP1 applies to the phase 1

NOP2 applies to the phase 2

NOP3 applies to the phase 3

NOP4 applies to the phases 4 and 5

NOP5 applies to the phase 6

For large problems (more than 20 elements or more than 200 degrees of freedom) the maximum printing options are not allowed and set equal to 0 automatically.

E, NU, T, DA, DB (one card, eventually blank)

These are 5 characteristic data applicable to a large number of elements in the problem. In most types of element they represents :

E : Young's modulus

NU : Poisson's ratio

T : thickness or cross section

DA : distributed load per unit area

DB : distributed load per unit of length or temperature

For certain special elements the informations carried by these variables are different (see details in elements description).

When these characteristic data are applicable to an element they need not to be repeated and the program prints a message : "general data are assumed". The data given in the description of the elements always override these general data.

Important notice : NU has an important secondary function. When set to a negative value, it indicates that a solution by the Southwell analogies is required. In this case a warning message is printed. If a solution by the stress function method is desired with the Poisson's ratio equal to zero, NU has nevertheless to be given an arbitrary non zero negative value in the general data and then specified equal to zero in the elements characteristics.

FORM 2 : Nodal points coordinates

All the nodal points defining the geometrical structure of the finite elements have to be given at this step. In addition a certain number of "dunny" nodal points used to define the additional constraints (if any) have to be included. (See details in the section dealing with the constraint elements). The numbering of the nodal points is arbitrary and need not be continuous nor in ascending order. It should be remembered however that the work spaces in ASEFCO are determined by the largest node number used. The end of the nodes set is indicated by a card starting by 99999. This card must always be present.

FORM 3 : Definition of the substructures and elements

A substructure is identified by a card containing the substructure number NOS and the number of elements it groups, NEL.

The different substructures must be given in ascending order and their numbering must be continuous. If the automatic subdivision in substructures is desired, only one substructure has to be defined which contains all the elements sorted in such a way that the cross section in the structure, virtually defined after assembling any of the elements, does not contain more degrees of freedom than the maximum dimension allowed by the space BSEFCO for a substructure. In other words the program is able to define how many elements to use in each substructure but not to reorder the elements. If the order of the elements is such that it implies substructures too large for the allocation in BSEFCO,an error message is printed and the execution stops.

In the case of a solution by the conjugate gradients method, the order of the elements does not matter and only one substructure can be defined. In all cases, the number of elements NEL defined in a substructure has to be exact. In addition the end of the elements definition is indicated by a separation card 99999.

One card is used to define one element. It contains the following two groups of variables. The first group concerns variables which are strictly necessary. These are :

NOEL : Numbering of the element. It is a dummy variable which is used only to identify the element in the output. It can therefore be arbitrary (including blank)

ITYP : Defines the type of the element according to the convention given in the section dealing with the details of the elements.

NNO : Number of nodal points defining the elements. Note that certain elements require more nodal points than those defining the geometrical contour. (This is the case for the beam element where an additional node defines the orientation of the inertia axes).

Ni : Nodal point defining the element, given in the conventional order indicated in the element description.

ISIMPL : If ISIMPL = 0 the element matrices are computed.
If ISIMPL = 1 the element is assumed to be identical to the preceding one (This means it can be superimposed by translation only and its elastic properties are identical). In this case, the localization vector only is computed while the stiffness matrix is that of the preceding element.

ISTRES : If ISTRES = 0 the stresses of the element are computed. Otherwise they are not.

IAD : Number of cards required to describe the particular data of the element. If IAD = 0 the general data are assumed (E, NU, T, DA, DB). This number can be determined from the description of the element. When ISIMPL = 1, IAD must be 0.

The second group of variables on the element definition card is never strictly necessary : standard options are always provided. The variables which can be used in this group are defined in the element description.

FORM 4 : Elastic and geometric data of the elements

These cards contain the particular data of the elements, such as Young's modulus, Poisson's ratio, thickness (es), cross section (s), moments of inertia etc.

They are identified by the element number NOEL which must be the same as on the card defining the element on FORM 3. The number of card for an element has been defined by IAD on FORM 3.

The program checks that the card containing the particular data are given in the same sequence as those defining the elements in each substructure and that their number corresponds to IAD. The subdivision in substructures is not indicated. The order of the elements must be that given by the substructures taken sequentially. An error causes a program interruption.

When IAD = 0 has been specified for an element, no corresponding card exists in the present set, and the general data are assumed (E, NU, T, DA, DB). The meaning of the variables used as particular data is defined in the description of the elements.

The end of the set of the particular data is indicated by 99999.

FORM 5 : Zero prescribed displacements

The zero prescribed displacements are defined by components at a node or along an interface. NODE1 and NODE2 are the 2 nodal points defining an interface. It is necessary that NODE2 > NODE1. If the fixation is to be applied at a vertex, its number has to be entered as NODE1, while NODE2 is set equal to zero. The number of the components fixed along an interface or at a vertex has to be given according to the conventional sequence defined for each particular element.

Example : for a beam element, the sequence of the 6 displacement components at the nodal points is (u, v, w, ϕ_x, ϕ_y, ϕ_z).

If one requires to fix u, v, ϕ_y, ϕ_z at the node number 12, the card can read :

NODE1	NODE2	COMPONENTS			
12	0	1	6	2	5

If all the components of displacement at a node or along an interface have to be fixed, simply enter 99999 as first fixed component.
The order of the fixation is arbitrary and the nodes and interfaces can be mixed. If two different fixations are entered for the same node or interface, the last one overrides the other.
It is important to note that the degrees of freedom for which a zero diagonal term is found in the assembled stiffness matrix are automatically fixed by the program. Therefore when such a situation is known in advance, it is not required to fix explicitly such degrees of freedom. For instance, the out of plane displacement component of a membrane element lying in a plane of coordinates. However the program cannot detect mechanisms in the structure. Therefore the out of plane displacement of a membrane which is not in a plane of coordinates is not fixed by this implicit process.
If non zero prescribed displacements are to be entered, they must come after the zero prescribed, separated of them by a card 88888 and presented according to FORM 6. Anyhow the end of the set of fixation cards (zero prescribed and non zero prescribed) is indicated by a separation card 99999.

FORM 6 : Non zero prescribed displacements
Load components

The identification of a node or an interface component is the same as in FORM 5 but only one component per card is allowed. It is followed by the value of the prescribed displacement or the intensity of the load component.
The ordering of the load or displacement components is arbitrary. The various loading cases are separated by a card starting with 88888, while the end of the loads is indicated by 99999. If there are no loads (like in problems with body forces only) one card with a dummy load component must be present and the displacement component loaded (with zero intensity) must be one for which a non zero stiffness exists.

Input data flow chart

The order of the input data cards is summarized on figure A9. The different problems to be solved in the same run of the program are separated by a card starting by - - - - (4 minus signs). The last problem in a run is followed by a card starting with 8 minus signs (- - - - - - - - -).
When an error is detected by ASEF in a problem, it tries to skip to the next problem. The main control variables ISTART, ISTOP, IU5 cannot be re-defined during an execution.

A4. DESCRIPTION OF THE FINITE ELEMENTS LIBRARY

The following elements have been retained for ASEF-20 and are described in this section :

TYPE 1 : Frame element

TYPE 2 : Bar element with variable degree of displacement field and cross section

TYPE 3 : Conforming triangular element of variable degree

TYPE 4 : Conforming quadrilateral element of variable degree

TYPE 5 : Not used

TYPE 6 : Special bulkhead element for displacement analysis

TYPE 7 : Conforming beam element including shear deformation

TYPE 8 : Conforming plate bending element of moderate thickness (Theory of Hencky)

TYPE 9 : Equilibrium plate bending element of moderate thickness (Reissner theory)

TYPE 10 : Not used

TYPE 11 : Equilibrium bar element with constant axial shear

TYPE 12 : Equilibrium quadrilateral membrane element with constant stresses

TYPE 13 : Equilibrium triangular plate bending element, Kirchhoff Theory

TYPE 14 : Hybrid rectangular membrane element of variable degree

TYPE 15 : Conforming quadrilateral plate bending theory, Kirchhoff Theory

TYPE 16 : Equilibrium quadrilateral membrane element with linear stresses

TYPE 17 : Equilibrium bar element with linear axial shear and linear cross section

TYPE 18 : Not used

TYPE 19 : Not used

TYPE 20 : Conforming parallelogram membrane element with bubble functions and variable degree.

In addition 3 types of linear constraints can be introduced in the form of "constraint" elements which are identified by the following types :

TYPE 51 : Constraint between the displacements at 2 nodes

TYPE 52 : Constraint between the displacement at a node and the foundation

TYPE 53 : Constraint to reduce the degree along an interface.

For all these elements, references are given in which a detailled derivation of the various matrices required can be found and, usually, more details concerning the program. The assumptions are recalled and followed by a short description of the elements. The informations necessary to prepare the element definition card and to introduce the particular data are detailed. Finally the conventions used to compute the stresses are summarized and the names of the FORTRAN subroutines composing the element generation program are given.

- References : (2.1) or (2.13)

- Assumptions : $u = \alpha_1 + \alpha_2 x$

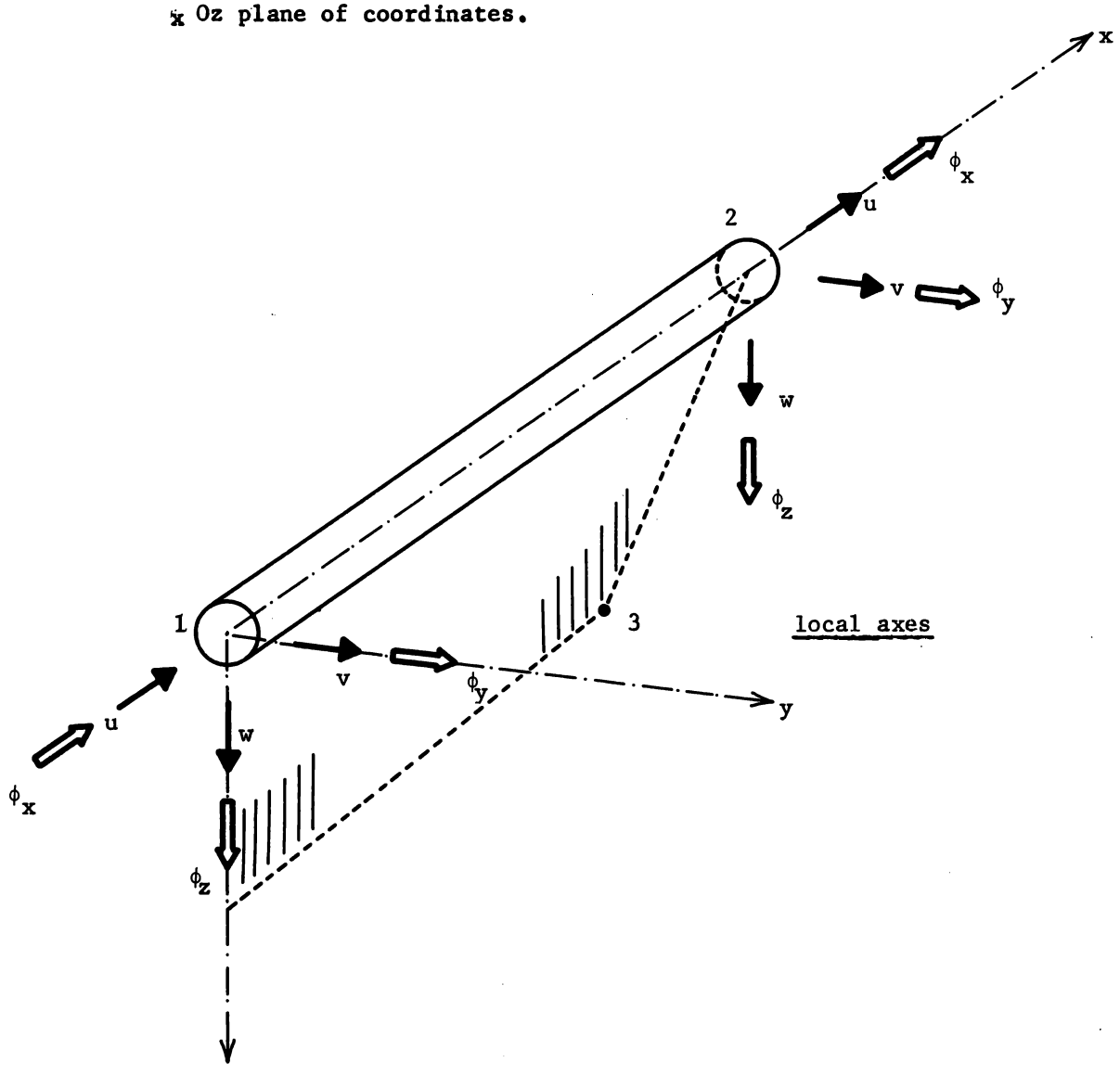
$$v = \alpha'_1 + \alpha'_2 x + \alpha'_3 x^2 + \alpha'_4 x^3$$

$$w = \alpha''_1 + \alpha''_2 x + \alpha''_3 x^2 + \alpha''_4 x^3$$

$$\phi_x = \beta_1 + \beta_2 x$$

- The frame element is a straight prismatic member for which deformation modes in extension, bending in two perpendicular planes and in torsion are assumed. It follows the simplest engineering beam theory.

- The local axes of the element are the inertia axes with the Ox axis oriented from node 1 to node 2 while a third node defines the local xOz plane of coordinates.



- Element definition card content :

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD

All these variables have been defined in the section A3, FORM3.

ITYP = 1 NNO = 3

IAD can only be equal to 1 or 2, otherwise an error message is printed.

The node 3 can be any node of the structure, used to define vertices of elements or not.

- Particular data

If IAD = 1 the following variables are read :

A cross section

RTY moment of inertia with respect to the local Oy axis

RTZ moment of inertia with respect to the local Oz axis

RJ moment of inertia in twist

G shear modulus in twist.

The Young's modulus is assumed to be that of the general data.

If IAD = 2 the Young's modulus is read on the second card.

Any one of these particular data can be zero.

- Degrees of freedom

The presence of an element of TYPE 1 in a structure causes all the nodal points to be considered as having 6 degrees of freedom ordered, in global axes :

$$(u, v, w, \phi_x, \phi_y, \phi_z)$$

If the element is connected with conforming membrane elements of first degree, it is compatible in extension. If elements of higher degree are used as membrane, the cross section of this element should be zero and the extension mode of deformation represented by superimposing a bar of TYPE 2. If a bending stiffness exists in the plane of the membrane, this element cannot be conforming. In that case the beam element of TYPE 7 should be used, which allows conformity of the bending modes with membrane elements of second and third degree.

- Stress output

The stress ouput are the 3 forces F_x, F_y, F_z and the 3 bending and twisting moments M_x, M_y, M_z. They are always given in global axes and in local axes.

- Subroutines used

SP1 CP1 AUXPT TSBEAM (plus matrix package)

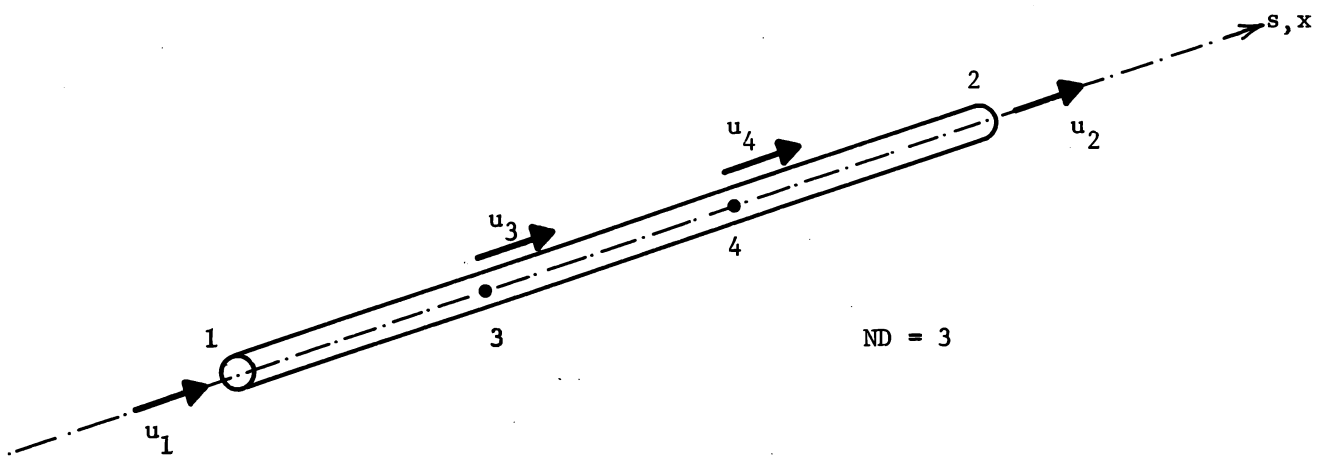
- References : 2.1 or A.9

- Assumption : $u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \ldots + \alpha_{10} x^9$

(complete polynomial truncated at a specified degree)

- Description

This element has only the axial deformation mode. It is intented to reinforce the conforming membrane elements of types 3, 4, 6 or 20 and the hybrid element of type 14. The degrees of the polynomial approximation for u and that used to represent the axial distribution of the cross section can be fixed independantly to any value between 0 and 9. Analytical integration is always used for obtaining the stiffness matrix.



- Local axis is directed from node 1 to node 2.

- Degrees of freedom

The length of the bar is subdivided in ND sections, ND being the degree of the polynomial used to represent the axial displacement u. This defines ND-1 points between the ends 1 and 2 which are numbered 3, 4 ... (ND + 1) from 1 to 2.

The displacement sequence in global axes is :

$$q' = (d_1\ d_2\ d_3\ d_4 \ldots d_{ND+1})$$

where $d_i = (u_i\ v_i\ w_i)$

Eventually, at points 3 to (ND + 1) the local axes can be specified in which case the stiffness corresponding to v_i w_i is obviously zero. This specification is achieved by the phase 1 if the elements of membranes meeting along the interface 1-2 are coplanar and not in a plane of coordinates.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, ISIMPL, ISTRES, IAD, ND, MD

All these variables, but ND and MD, have been defined in section A3, FORM 3.

ITYP = 2 NNO = 2

ND is the degree of the polynomial used for the displacement field. It can take the value 0 to 9. If ND = 0 the degree is fixed by the general data NDC. If $1 \leq ND \leq 9$ its value overrides that given by the general data NDC. Note that ND can be reduced if the bar is connected with another element for which ND is smaller. The general rule during phase 1 is to assign to the interfaces the degree of the simplest adjacent element. This rule overrides any specification of ND.

MD is the degree of the polynomial representing the cross section axial variation. It can take the values 0 to 9.

IAD is the number of additional card containing particular data. It must be consistent with the specification of MD : IAD = $\frac{MD + 6}{5}$ (default quotient).

- Particular data

The cross section area has to be specified at MD + 1 stations and followed by the Young's modulus. If this latter is zero, the general value is assumed.

- Stress output

The axial stress is computed at ND - 1 stations along the length of the element.

- Subroutines used

SP2

SPBAR2 (plus the matrix operation package)

CONNEC

- Reference : A 9

- Assumption : $u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$
$+ \alpha_7 x^3 + \alpha_8 x^2y + \alpha_9 xy^2 + \alpha_{10} y^3$
$+ \ldots + \alpha_{15} y^4 + \ldots \ldots + \alpha_{21} y^5$

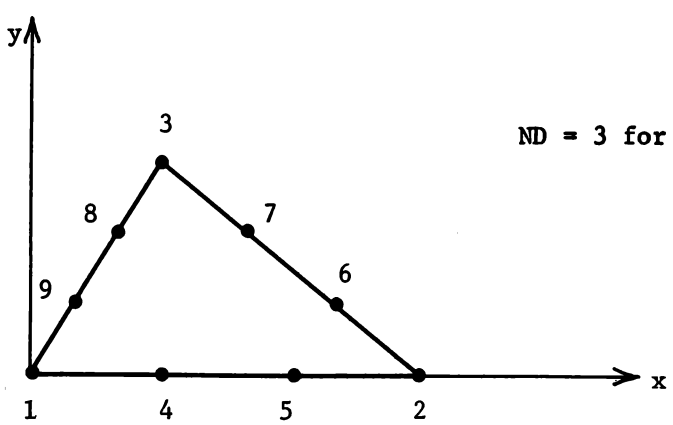
$v = \alpha'_1 + \alpha'_2 x + \alpha'_3 y + \alpha'_4 x^2 + \alpha'_5 xy + \alpha'_6 y^2$
$+ \alpha'_7 x^3 + \alpha'_8 x^2y + \alpha'_9 xy^2 + \alpha'_{10} y^3$
$+ \ldots \alpha'_{15} y^4 + \ldots + \alpha'_{21} y^5$

(complete polynomial in x, y truncated at a specified degree)

- Description

This element generalizes the classical triangular membrane element in this sense that it can be of any degree between 1 and 5 along any of the 3 interfaces and inside, independently. The thickness is assumed to vary linearly between the 3 vertices. The Hooke's law modification necessary to use the element by the Southwell analogies as a plate bending element is also provided (in this case the thickness has however to be constant).

- Local axes and degrees of freedom



ND = 3 for all edges.

The local axes are directed as indicated above with the axis Ox coinciding with the edge 1-2. They influence only the stress ouput.

Each edge is subdivided in ND equal segments, ND being the degree assigned by the phase 1 to each edge. These segments define a certain number of points along the interfaces which are numbered following the same sense of rotation as for the 3 geometrical nodes 1.2.3. The example given above applies for ND = 3 on all edges. The displacements sequence is :

$$q' = (d_1\ d_2\ d_3\ d_4\ d_5\ \ldots\ d_n)$$

where $d_i = (u_i\ v_i\ w_i)$ in global axes. Eventually, if along an interface, only 2 membrane elements are connected (plus eventually a bar of type 2) which are coplanar and not in a plane of coordinates, the phase 1 decides to express the displacements of the interface in special local axes. These axes are defined by the tangent and normal to the interface. Their orientation is given by a reference node printed in the output. If this node is positive the normal is directed in the half plane containing this node. If the reference node is negative the normal is directed in the opposite half plane. The advantage of these special axes is that the out of plane stiffness is automatically zero at the interface points and therefore is automatically fixed by the program. If the 2 membrane elements are "almost" coplanar, the special local axes are selected if the angle is smaller than .01 radiants.

Bubble functions of displacement

If the degree specified for the element is higher than the degree of the interfaces, bubble functions of displacement are automatically included. This is also the case if the degree is higher than 2 in which case internal degrees of freedom have to be defined. They can be interpreted also as bubble functions. The presence of bubble modes can be forced in various ways : for instance by specifying the variable NDB or by specifying a degree higher than that of the surrounding adjacent elements. Eventually bar elements of zero cross section can be used to force the reduction of degree along certain interfaces.

Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD, ND, NDB, NDT, IFLUX, IPERF.

The 9 first variables are defined in section A3, FORM 3.

ND is the degree assigned to the element. It must be smaller than or equal 5. If ND = 0, the degree is that fixed by the general data NDC.

NDB is the degree assigned inside the element. If NDB > ND it forces the inclusion of bubble functions. If NDB = 0 it is assumed equal ND.

NDT is the degree of the stresses in the output. Normally, and in particular if NDT = 0, the degree of the computed stress field is equal ND - 1. However the stresses can be computed at additional points corresponding to a displacement field of higher degree. The additional stresses are obtained by interpolation. This option is useful for graphic representation of the stress output. The maximum value of NDT is 5.

IFLUX is an option for stress computation. If IFLUX = 1 the fluxes are computed instead of the stresses.

IPERF = 1 specifies that the stresses have to be punched or output on unit 7.

- Particular_data

If IAD = 0 the general data are assumed. The thickness is constant and equals T.

If IAD = 1 the first additional card contains the Young's modulus and the thickness at points 1.2.3 sequentially.

If the thickness at points 2 and 3 are zero, the thickness is assumed constant and equals the thickness at point 1.

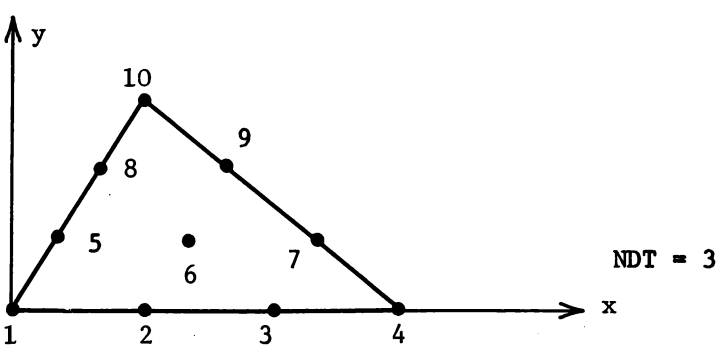
If IAD = 2 the second additional card contains the Poisson's ratio wich otherwise is given the general value.

- Stress_output

Depending of IFLUX, stresses or fluxes (stresses times local thickness) are computed in local axes. They are printed at a certain number of reference points, depending of NDT, which are numbered as indicated below along lines parallel to the local Ox axis.

The figure corresponds to NDT = 3. Note that the number of points decreases by one along succesive lines.

When the element is used in the stress function method, the stresses printed are in fact bending moments.



Numbering of the reference points for stress output.

- Subroutines used

SP3	BS2103	BS2107	BS16
DS21	BS2104	BS2108	BS17
BS2101	BS2105	BS2109	BS18
BS2102	BS2106	BS10	BS19

(plus the matrix operation package)

- Reference : A 9

- Assumption : $u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$

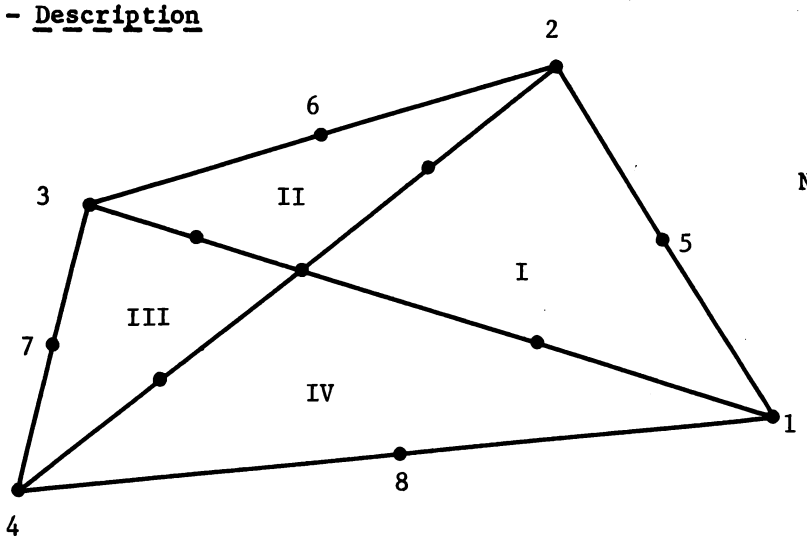
$$+ \alpha_7 x^3 + \alpha_8 x^2y + \alpha_9 xy^2 + \alpha_{10} y^3$$

$$v = \alpha'_1 + \alpha'_2 x + \alpha'_3 y + \alpha'_4 x^2 + \alpha'_5 xy + \alpha'_6 y^2$$

$$+ \alpha'_7 x^3 + \alpha'_8 x^2y + \alpha'_9 xy^2 + \alpha'_{10} y^3$$

(complete polynomial in x, y truncated at a specified degree in each triangle independantly)

- Description



This quadrilateral is a super-element composed of 4 triangular membrane elements of type 3. The internal interfaces are defined by the diagonals of the quadrilateral. The internal degrees of freedom corresponding to the internal interfaces and eventually to internal modes of the triangles are eliminated by condensation. The degrees of the 4 triangles are assumed to be identical and not higher than 3. The reduction of the degree along the external interfaces of the quadrilateral is possible and follows the same rules for the type 3 element. The possibility of using the element as a plate bending element by the Southwell analogies is also provided.

The thickness can vary linearly along each external interface and is defined by the local values at the vertices. The thickness at the diagonal intersection point is interpolated as follows : a linear variation is assumed along the two diagonals. The thickness at the central node is the average of 2 thicknesses so defined at this point. If the 4 vertices of the quadrilateral are not coplanar, a correction of the warping is achieved by defining a mean plane and projecting the 4 nodes in this plane. The importance of the warping in NOT checked.

- Degrees of freedom

The sequence of the generalized displacements is

$$q' = (d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7\ d_8\ \ldots\ )$$

following the same conventions as for the type 3 element.
The selection of the special local axes on the interfaces follows also the same rules.

- Bubble functions of displacement

Such functions can be introduced following the rules defined for the element of type 3.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD, ND, NDB, NDT, IFLUX, IPERF.

The only difference with type 3 is the presence of 4 nodes.

- Particular data

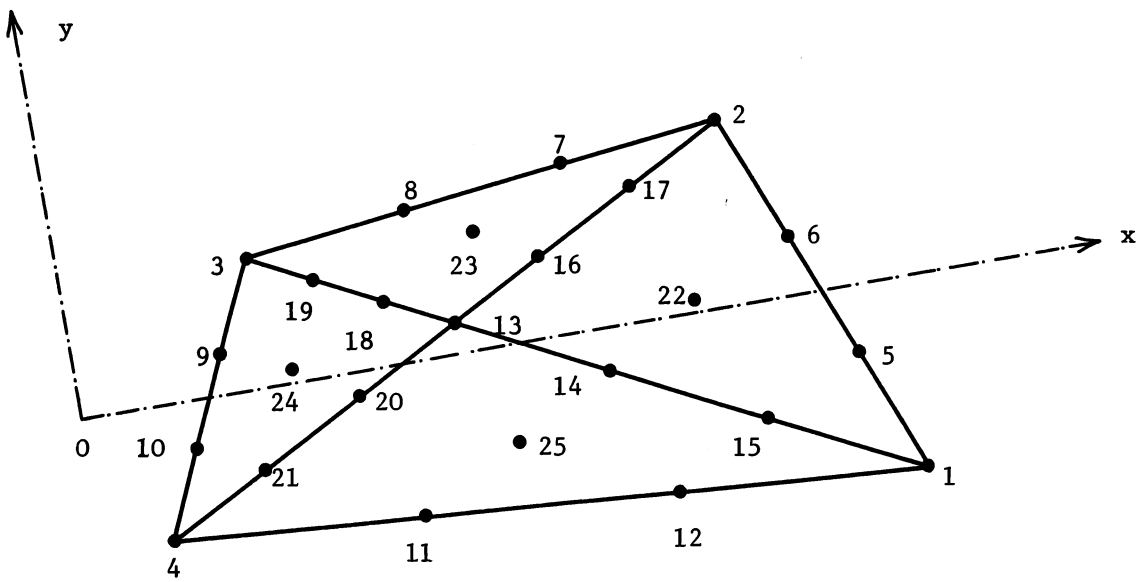
The only difference with type 3 is that, if IAD $\geq$ 1 the first additional card contains the Young's modulus and 4 thicknesses at points 1.2.3.4 sequentially.

- Stress output

The numbering of reference points follows a different sequence which is indicated on the figure below for NDT = 3 which is the maximum stress output. The logic of the numbering is :

- 4 vertices,
- interface points (if any),
- diagonal intersection point (always present)
- internal interfaces points (if any), numbered from center to nodes 1.2.3.4 sequentially,
- points internal to the triangles I, II, III, IV (if any).

The stresses or fluxes are given in local axes defined by the Ox axis joining the mid-edge 4-3 to the mid-edge 1-2 and by the Oy axis directed in the half plane containing the node 3. Along internal interfaces of the element the stresses (or fluxes) printed are the averages of the values computed in each adjacent triangular sub-element.



Local axes for stress output and numbering of the reference points for NDT = 3

- Subroutines used

SP4	BS2103	BS2107	BS16
DS21	BS2104	BS2108	BS17
BS2101	BS2105	BS2109	BS18
BS2102	BS2106	BS10	BS19

(plus the matrix operation package)

- Reference : (A.7)

- Assumptions : $u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + y (\alpha_5 + \alpha_6 x + \alpha_7 x^2 + \alpha_8 x^3)$

$$v = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 + y (\beta_5 + \beta_6 x + \beta_7 x^2 + \beta_8 x^3)$$

(eventually reduced to 2^d degree along the edge 1-2)

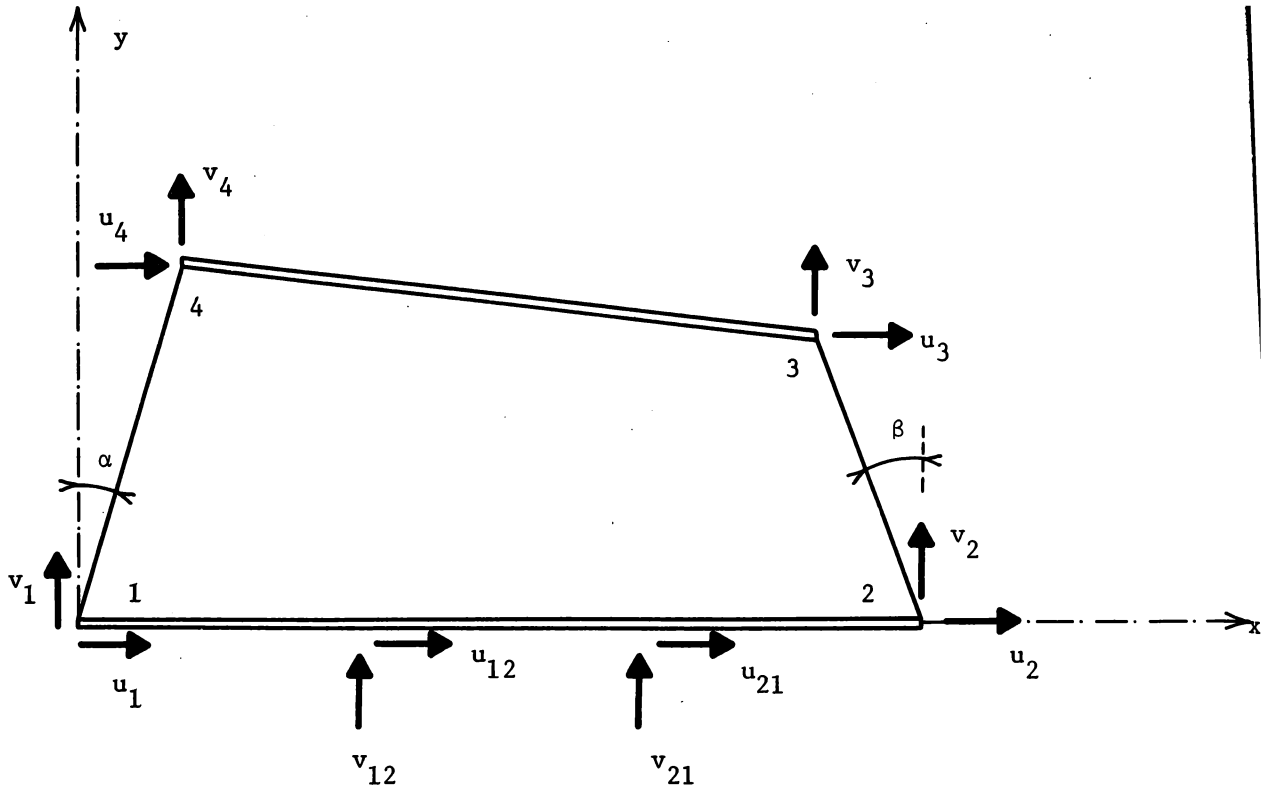
- Description

The special bulkhead element is a kind of frame or spar element designed to reinforce conforming membrane elements for out of plane bending and for which the neutral axis is not in the plane of the membrane.
This situation is often met in the idealization of bulkheads in fuselage analysis. The element is a quadrilateral membrane which can be connected with the membrane elements of the fuselage skin only by the edge 1-2 along which it is conforming. The opposite edge is always supposed to be free. The 2 lateral edges 2-3 and 4-1 can be connected with similar elements or with general membrane elements. This connection is stricly conforming only if the lateral edges are perpendicular to the edge 1-2. Otherwise it introduces a slight lack of conformity which, for bulkhead analysis, is not important.

It has the advantage over standard membrane elements (like TYPE 4) of a better representation of the bending modes which include the effects of shear deformation and shortening of the cross section, but with less degrees of freedom. Note that connection is not allowed along the edge 3-4 due to the condensation of the 4 interface degrees of freedom along that edge.

The high of the element can vary linearly and two different reinforcing flanges of linearly varying cross section are provided. The thickness of the web is constant but can be different in extension and in shear to allow a correct representation of the effect of holes in the web. The loading can be achieved, in addition to tip concentrated forces, by a linearly distributed line load applied on the edge 1-2 in the plane of the web. This allows to input the pressurisation of the fuselage with the consistent loads.

- Local axes are oriented with Ox along 1-2



- The angles α and β should be small.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD, ND

All these variables, but ND, have been defined in the section A3, FORM 3.

ITYP = 6 NNO = 4

IAD can take the values 0, 1, 2, 3 or 4

ND is the degree of the displacement field along the edge 1-2

It can only take the values 2 or 3. If it is equal 2 the two couples of generalized displacements along 1-2 are reduced to one to be conforming with the standard membrane elements of second degree (ITYP = 3 or 4)

- Particular data

If IAD = 0 the general data are assumed for E and NU while the web thickness is taken equal T in shear and in elongation. The flange cross sections are assumed zero

If IAD = 1 the variables T1 (thickness of the web in elongation) and T2 (thickness in shear) are read

If IAD = 2 the second additional card contains :

EA = Young's modulus of the web

ANUA = Poisson's ratio of the web

E1 = Young's modulus of the flange 1-2

E2 = Young's modulus of the flange 3-4

If IAD = 3 the third additional card contains the cross sections of the flanges A12, A21, A34, A43 respectively at points 1, 2, 3, 4.
It is assumed that :
if A21 = 0 then A21 = A12
if A12 = 0 there is no flange 1-2
if A43 = 0 then A43 = A34
if A34 = 0 there is no flange 3-4

If IAD = 4 the fourth additional card contains P1 and P2 which are the local intensities of the distributed line load at points 1 and 2. If P2 equals zero, then P2 = P1 is assumed.

- Degrees of freedom

The sequence of the generalized displacements in global axes is

$$q' = (d_1\ d_2\ d_3\ d_4\ d_{12}\ d_{21})$$

where $d_i = (u_i\ v_i\ w_i)$, if ND = 3.

If ND = 2 the triplet d_{21} is droped and d_{12} is expressed at mid-edge 1-2.
NB : When this element is connected with membrane elements of type 3 or 4 it forces on the corresponding interfaces the choice of global axes instead of local axes (see details in the description of the elements 3 and 4)

- Stress output

The stress output are :

σ_x σ_y τ_{xy}	computed at points 1, 2, 3 and 4
σ_x σ_y τ_{xy}	at the diagonals intersection
$\bar{\tau}_{xy}$	average of the shear stress over the element
N1, N2, N12	normal forces in the flange 1-2 at points 1, 2 and in the mid-edge 1-2
N3, N4, N34	normal forces for flange 3-4

- Subroutines used : (plus the matrix operation package)

SP6	MC6	STS6	INT6
LAND6	STA6	COR6	PINT6
CD6	TA6	MP6	

- Reference : A.8

- Assumptions : $u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$

$$v = \alpha'_1 + \alpha'_2 x + \alpha'_3 x^2 + \alpha'_4 x^3$$

$$w = \alpha''_1 + \alpha''_2 x + \alpha''_3 x^2 + \alpha''_4 x^3$$

$$\phi_x = \beta_1 + \beta_2 x + \beta_3 x^2$$

$$\phi_y = \beta'_1 + \beta'_2 x + \beta'_3 x^2$$

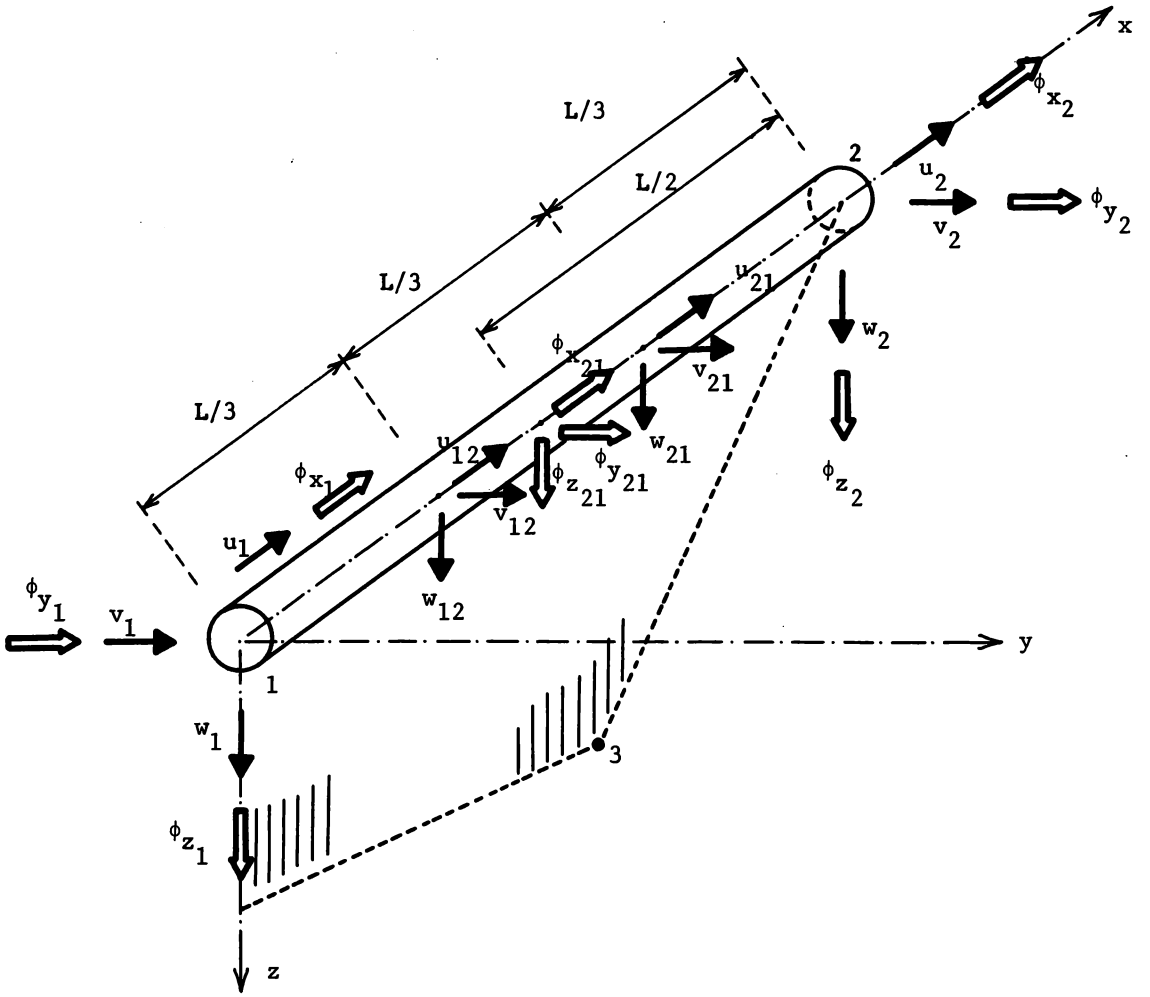
$$\phi_z = \beta''_1 + \beta''_2 x + \beta''_3 x^2$$

(eventually the displacements u, v, w are reduced to 2nd degree)

- Description

This element is a straight prismatic member which deforms in extension, bending and torsion. The local axes are the inertia axes of the cross section. As the shear deformation is included in the theory used for bending, the displacements u, v, w are represented by modes independent of those used to describe the rotations. (ϕ_y is not equal to $\frac{\partial w}{\partial X}$, etc...). The difference between the elements of type 1 and type 7 is the same as between the plate bending elements of type 15 (Kirchhoff) and type 8 (Hencky). This element should be used to represent beams reinforcing membrane elements and which have a bending stiffness in the plane of the membranes. The connection can be conforming with membrane elements of 2nd and 3rd degree (Type 3, 4 or 20).

All the characteristics of the cross section can vary linearly along the element neutral axis.



- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD, ND

All these variables, but ND, have been defined in the section A 3, FORM 3

ITYP = 7 NNO = 3

The third node N3 is used to define, with N1 and N2, the local Oxz plane. This nodal point is not necessarily a node used to define the element vertices.

ND is the degree of displacements u, v, w along Ox. It can be equal to 2 or 3 only. Note that the degree of the polynomials used for the rotations is always 2. ND should match the degree of the membrane elements to which this frame element is connected. IAD can only be equal to 3 or 4.

- Particular data

If IAD = 3 the following data are read on 3 cards, sequentially

A1, A2	cross section area at nodes 1 and 2
ARY1, ARY2	cross section area in shear along the local Oy axis

ARZ1, ARZ2 cross section area in shear along the local Oz axis

RJ1, RJ2 moment of inertia in torsion

RIY1, RIY2 moment of inertia in bending with respect to Oy

RIZ1, RIZ2 moment of inertia in bending with respect to Oz

The Young's modulus is that of the general data and the shear modulus is equal to $\frac{E}{2(1+\nu)}$

Conventionnally, for each couple of characteristics of the cross section, if the value at node 2 is zero, the characteristic is assumed to be constant over the length of the element, with the value given at node 1. If both values are zero, the characteristic is zero. If it is really required to represent a triangular distribution of a characteristic, a very small, but non zero, value should be input at one end.

If IAD = 4 the fourth card contains the Young's modulus and the shear modulus. This latter can be zero.

- Degrees of freedom

The sequence of the generalized displacements in global axes is :

$$q' = (d_1 , \omega_1 , d_2 , \omega_2 , d_{21} , \omega_{12})$$

where $d_i = (u_i \; v_i \; w_i)$

$\omega_i = (\phi_{xi} \; \phi_{yi} \; \phi_{zi})$ if ND = 3.

If ND = 2 the triplet d_{21} is droped and d_{12} is expressed at mid-edge 1-2.

- Stress output

The resultant forces $F_x \; F_y \; F_z$ and bending and twisting moments $M_x \; M_y \; M_z$ in local axes are computed at points 1, 2 and at mid-length 1-2.

- Special choice of local axes for interface displacements

When the membrane elements that join along 1-2 are coplanar and not in a plane of coordinates, the program chooses to use local axes for the interface displacements (not at the nodal points 1 or 2). These are defined by the tangent and normal to the interface. (See details in description of elements 3 or 4).

- Subroutines used

SP7	ST7
LCV7	CD7
MC7	CONNEC

(plus the matrix operation package)

TYPE 8 CONFORMING TRIANGULAR PLATE BENDING ELEMENT OF MODERATE THICKNESS (THEORY OF HENCKY)

- Reference : 2.7

- Assumptions : $u = z\,\phi_x(x, y)$ $\qquad v = z\,\phi_y(x, y)$

$$\phi_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

$$\phi_y = \alpha'_1 + \alpha'_2 x + \alpha'_3 y + \alpha'_4 x^2 + \alpha'_5 xy + \alpha'_6 y^2$$

$$w = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$
$$\beta_7 x^3 + \beta_8 x^2 y + \beta_9 xy^2 + \beta_{10} y^3$$

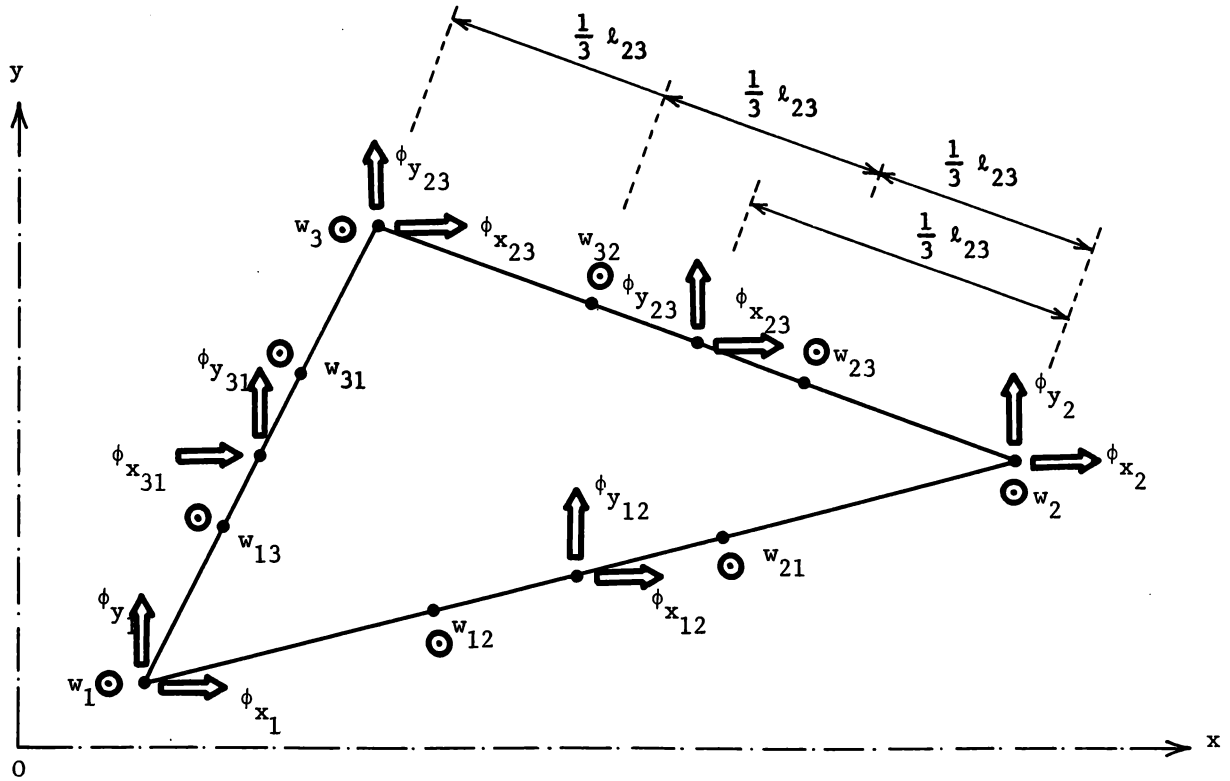
- Description

This plate bending element is derived according to the theory of Hencky for plates of moderate thickness. It differs from the Kirchhoff plate theory by the independence of the rotations ϕ_x, ϕ_y of the cross section and the slopes $\partial w/\partial x$, $\partial w/\partial y$ of the mean plane of the plate. It allows to take into account the shear deformation and the edge effect due to the transverse rotation around the axis normal to the mean plane. The element is derived with fixed degree and constant thickness. The Young's modulus used to compute the bending energy and the shear modulus used to compute the transverse shear energy can be independant. This allows to use the element to model non homogeneous materials like sandwich, multi-layer trusses etc.

- Degrees of freedom

The element can only be used in the X-Y plane. The sequence of the generalized displacements is, in the global XY axes.

$$q' = (w_1\ \phi_{x1}\ \phi_{y1}\quad w_2\ \phi_{x2}\ \phi_{y2}\quad w_3\ \phi_{x3}\ \phi_{y3}$$
$$w_{12}\ \phi_{x12}\ \phi_{y12}\ w_{21}\ w_{23}\ \phi_{x23}\ \phi_{y23}\ w_{32}\ w_{31}\ \phi_{x31}\ \phi_{y31}\ w_{13})$$



The internal degree of freedom w necessary to derive the element is eliminated by condensation.

To allow a correct representation of the boundary conditions, special axes can be defined at the 6 points where rotations are expressed. Only one set of special axes can be defined per element, by giving their direction cosines in the LOCAL axes. This transformation of coordinates does not necessarily affect all the rotations.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD, (ICOR(I), I = 1, 6)

All the variables, but ICOR, are defined in section A3, FORM 3

ITYP = 8 NNO = 3

ICOR(I) indicates at which points the special axes have to be used instead of the global ones. Each component of ICOR corresponds to a point in the sequence : (1, 2, 3, 12, 23, 31)

ICOR(I) = 0 specifies the global axes

ICOR(I) = 1 specifies the special axes. In this case the program checks that IAD = 2 so that the direction cosines are effectively specified.

- Particular_data

If IAD = 0 the 5 general data are assumed to represent :

E : the Young's modulus in bending

NU : the Poisson's ratio

T : the constant thickness

DA : the shear modulus for the transverse shear energy

DB : the transverse distributed load (per unit area)

If IAD = 1 the five variables E, NU, T, DA, DB are read

If IAD = 2 the second additional card contains CLI and CMI, the direction cosines of the special axes in the local axes. If the transverse shear modulus DA is zero, it is assumed to be given by $\frac{E}{2(1+NU)}$

- Stress_output

The bending moments and shear forces are computed at the 3 vertices and mid-edges in local axes oriented so that Ox coincides with the edge 1-2. The antisymmetrical torsion (around an axis normal to the element) is also computed at the 3 vertices. It is independent of the choice of axes.

- Subroutines used

SP8

COND8

SPL8B3

(plus the matrix operation package)

TYPE 9 EQUILIBRIUM QUADRILATERAL PLATE BENDING ELEMENT OF MODERATE THICKNESS (REISSNER THEORY)

- Reference : 2.7

- Assumptions : In each triangular region, the bending moment field in the oblique axes defined by the 2 internal interfaces of length a and b is :

$$M_x = \beta_1 + \beta_2 \frac{x}{a} + \beta_3 \frac{y}{b}$$

$$M_y = \beta_4 + \beta_5 \frac{x}{a} + \beta_6 \frac{y}{b}$$

$$M_{xy} = \beta_7 + \beta_8 \frac{x}{a} + \beta_9 \frac{y}{b}$$

A particular solution for constant distributed load p is superimposed

$$p\sin\alpha = -\frac{6}{ab}\beta_{10}$$ (α is the angle of the oblique axes)

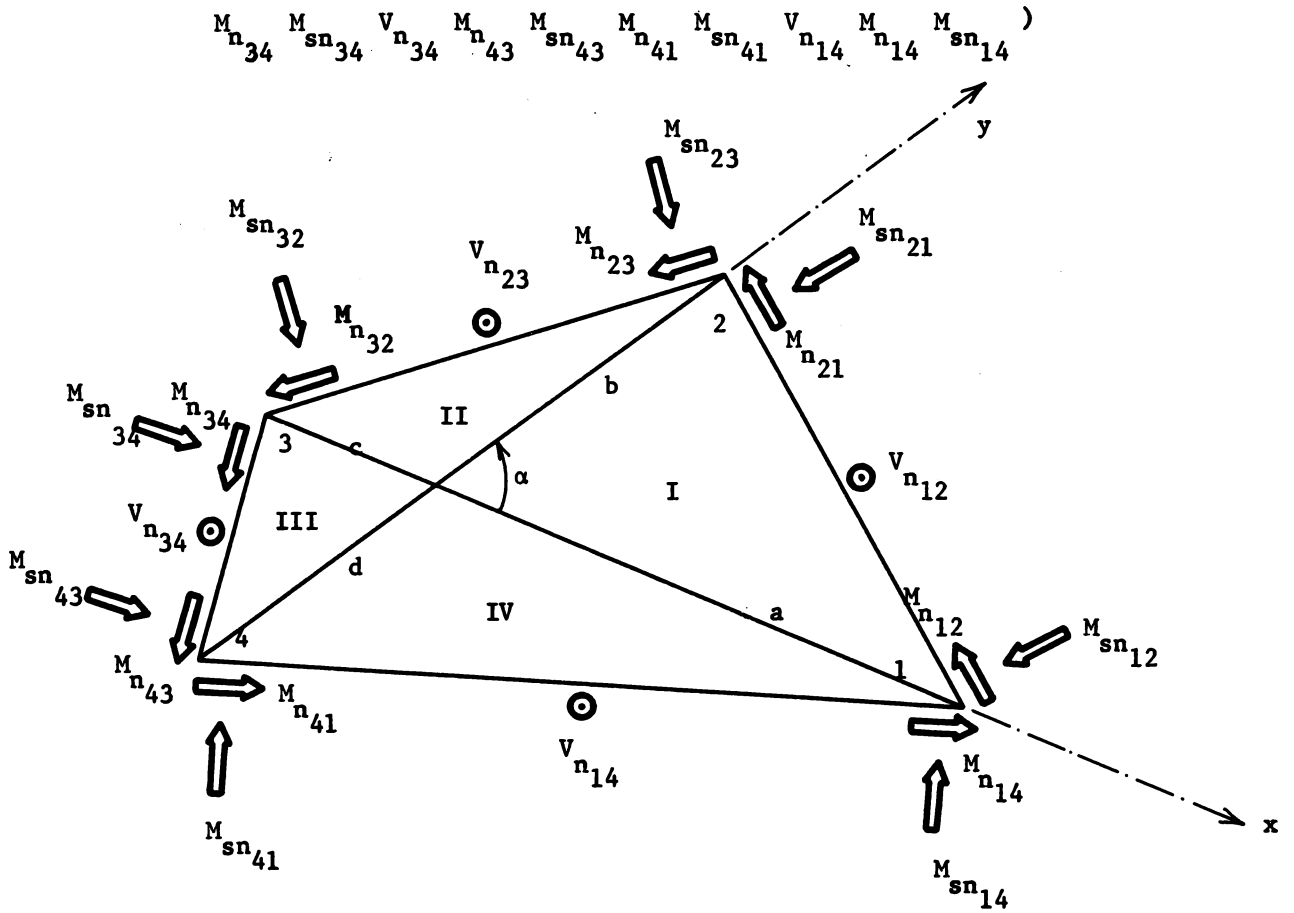
- Description

This plate bending element is derived by the equilibrium theory of Reissner for plates of moderate thickness. It differs from the equilibrium formulation of the Kirchhoff theory by the form of the complementary stress energy which includes the contribution of the transverse shear and by the continuity requirements for the surface tractions. In this element the bending and twisting moments are continuous across an interface as well as the shear forces. There are no corner loads.
Although the form of the bending moment field is the same as in the equilibrium Kirchhoff plate bending element of type 13, it is impossible to derive a triangular element free of additional constraints (or kinematic deformation modes)due to the increased number of interface generalized forces. The solution of building a super-element composed of 4 triangles allows to reduce the additional constraints inside the assemblage.
The element is derived with fixed degree and constant thickness.
The Young's modulus can be independent of the transverse shear modulus.

-Degrees of freedom

The element can only be used in the X-Y plane. The sequence of generalized forces is :

$$g' = (M_{n_{12}} \; M_{sn_{12}} \; V_{12} \; M_{n_{21}} \; M_{sn_{21}} \; M_{n_{23}} \; M_{sn_{23}} \; V_{n_{23}} \; M_{n_{32}} \; M_{sn_{32}}$$



The 4 internal degrees of freedom corresponding to the distributed load in each triangle are eliminated by condensation. All the generalized forces being expressed in local tangent-normal axes on each interface, there is no need of special coordinates transformation in this element.

- Element definition card : it contains simply
  NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD

- Particular data
  If IAD = 0 the 5 general data are assumed to represent :
  E : the Young's modulus in bending
  NU : Poisson's ratio
  T : the constant thickness
  DA : the transverse shear modulus
  DB : the transverse distributed load (per unit area)
  If IAD = 1 these 5 variables are read on one card.
  If the transverse shear modulus is zero, it is assumed to be equal $\frac{E}{2(1+NU)}$

- Stress_output

The bending moments and shear forces are computed at the 4 vertices and at the diagonal intersection point (reference point nbr 0) in the GLOBAL X-Y axes. The values printed are the averages of the stresses in the triangular regions adjacent to the point, in each element.
In addition the generalized forces, bending moments and shear forces, defined along the interfaces are also computed.

- Subroutines used

SP9	STREL9	COND9
SPL9	LOCEF9	
FLEXI9	CONN9	

(plus the matrix operation package)

- Reference : 2.1

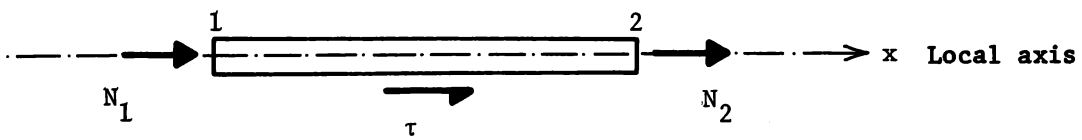
- Assumption : τ = constant

  or $N(x) = \beta_1 + \beta_2 x$

- Description

  This equilibrium bar element is stressed by a constant longitudinal shear and two tip forces. It is intended to reinforce the equilibrium membrane element of type 12. The section is supposed to be constant.

- Degrees of freedom



The local axis is oriented from node 1 to node 2 and the generalized forces are N_1, N_2 and τ. $\ell_{12} = N_{12}$. These 3 local forces are expressed in global axes in the sequence

$$g' = (F_{x_1}\ F_{y_1}\ F_{z_1}\ F_{x_2}\ F_{y_2}\ F_{z_2}\ F_{x_{12}}\ F_{y_{12}}\ F_{z_{12}})$$

Eventually, if only 2 membrane elements meet along the interface 1-2 which are coplanar and not in a plane of coordinates the phase 1 decides to keep the local axis for the interface force. The total number of generalized forces remains the same but $F_{y_{12}}$ is replaced by N_{12} while a zero stiffness is given to 2 other components.

- Element definition card : it contains

  NOEL, ITYP, NNO, N1, N2, ISIMPL, ISTRES, IAD

  All these variables are defined in section A3, FORM 3.

  ITYP = 11 NNO= 2 IAD = 0 or 1

- Particular data

  If IAD = 0 the general data are assumed

  E is the Young's modulus

  T is the cross section area

  If IAD = 1 the varaibles E and the cross section area are read.

- Stress output

  The stresses at both ends are computed in local axes.

- Subroutines used

  SP11

  CP11

  (plus the matrix operation package)

TYPE 12 EQUILIBRIUM QUADRILATERAL MEMBRANE ELEMENT WITH CONSTANT STRESSES

- Reference : 2.1 or 2.11

- Assumptions : In each triangular regions, the stress field in the oblique axes defined by the 2 diagonals is :

$$\sigma_x = \beta_1$$

$$\sigma_y = \beta_2$$

$$\tau_{xy} = \beta_3$$

- Description

This quadrilateral is the simplest equilibrium membrane element. It is subdivided by the diagonals in 4 triangular regions in each of which a constant stress field is assumed. The 12 parameters are reduced to 5 by the constraints of continuity of the normal and tangential surface tractions along the internal interfaces. The element is free of remaining constraints (or kinematical deformation modes). However the connection with other elements is achieved by identifying the simple averages of the displacements along each interface which is equivalent to a pin joint. Therefore special care has to be taken in expressing the boundary conditions to avoid possible mechanism in the structure. Such mechanisms are always avoided if the element is bordered by bar elements of type 11.
If the 4 nodes are not coplanar, a correction of twist is achieved by defining a mean plane in which the 4 nodes are projected. The importance of the warping is NOT checked.

- Degrees of freedom

Along each interface of length ℓ_{ij} the generalized forces are in local cartesian axes normal and tangential to the interface

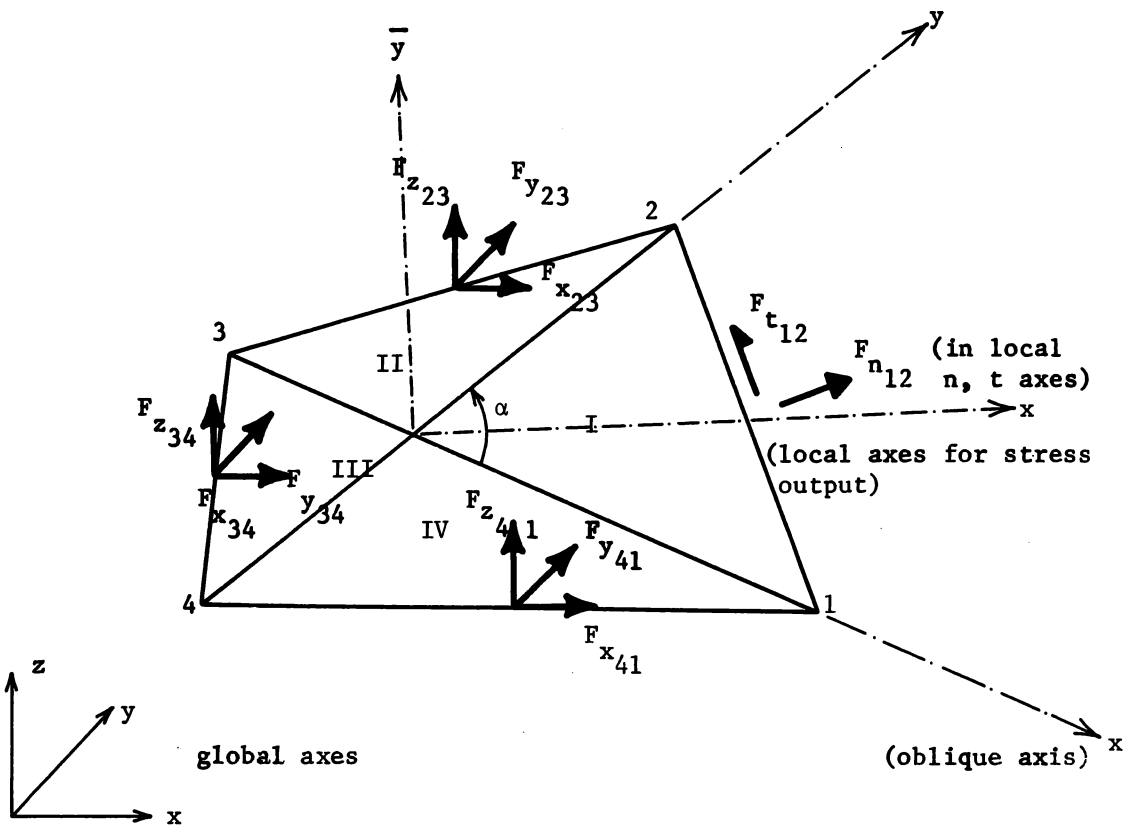
$$F_{n_{ij}} = \sigma_n \cdot \ell_{ij} \qquad \text{and} \qquad F_{t_{ij}} = \tau_{sn} \cdot \ell_{ij}$$

They are expressed in global axes in the sequence

$$g' = (F_{x_{12}}\ F_{y_{12}}\ F_{z_{12}}\ F_{x_{23}}\ F_{y_{23}}\ F_{z_{23}}\ F_{x_{34}}\ F_{y_{34}}\ F_{z_{34}}$$

$$F_{x_{41}}\ F_{y_{41}}\ F_{z_{41}})$$

If along an interface there are only 2 adjacent membrane elements which are coplanar and not in a plane of coordinates, the local cartesian axes are kept for the interface and the third component of force is given a zero value.



- Element definition card : it contains

  NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD

  All these variable are defined in section A3, FORM 3

  ITYP = 12 NNO = 4 IAD = 0 or 1

- Particular data

  If IAD = 0 the general data are assumed

  E is the Young's modulus

  NU is the Poisson's ratio

  T is the constant thickness

  If IAD = 1 the variables E, NU, T are read on one card

- Stress output

  The stresses are computed in local cartesian axes defined by an Ox axis directed from the diagonal intersection point to the middle of the edge 1-2 and the Oy axis directed toward the edge 2-3.

The stresses are computed in each of the 4 triangular regions.

- Subroutines used :

  SP12

  CP12

  BS2112

  (plus the matrix operation package)

- References : 2.18, 2.7

- Assumption : $M_x = \beta_1 + \beta_2 x + \beta_3 y + \beta_{10} (1 - x) x$

$$M_y = \beta_4 + \beta_5 x + \beta_6 y + \beta_{10} (1 - y) y$$

$$M_{xy} = \beta + \beta_8 x + \beta_9 y - \beta_{10} xy$$

The parameter β_{10} controls the particular solution under a uniform distributed load.

- Description

This classical equilibrium plate bending element is derived from a linear bending moments field. Such a field satisfies the homogeneous equilibrium equations. A particular solution is superimposed which is in equilibrium with a constant distributed transverse load.
The thickness is assumed to be constant.

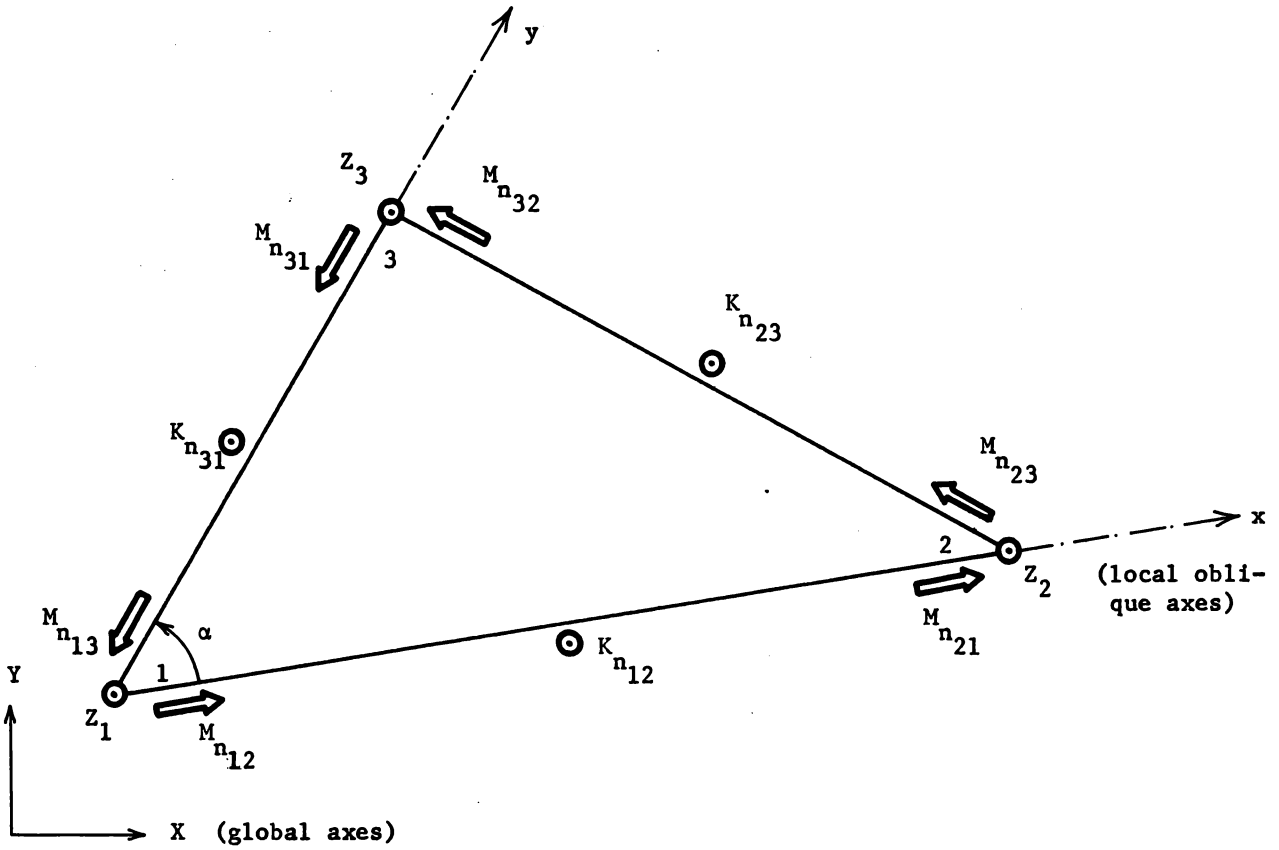
- Degrees of freedom

The element can only be used in the X Y plane. The generalized forces which insure the equilibrium along the interfaces (in the sense of Kirchhoff) are : 2 local values of the normal bending moment M_n, the constant value of the normal Kirchhoff shear force $K_n = V_n + \frac{\partial M_{sn}}{\partial s}$ and the 3 corner loads equal to the jump of twisting moment $Z_i = M_{sn_{i+o}} - M_{sn_{i-o}}$.

They are computed in the sequence :

$$g' = (Z_1\ Z_2\ Z_3\ M_{n_{12}}\ K_{n_{12}}\ M_{n_{21}}\ M_{n_{23}}\ K_{n_{23}}\ M_{n_{32}}\ M_{n_{31}}\ K_{n_{31}}\ M_{n_{13}})$$

The internal degree of freedom corresponding to the distributed load is eliminated by condensation.



- Element definition card : it contains

  NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD

  All these variables are defined in section A3, FORM 3

  ITYP = 13 NNO = 3 IAD = 0 or 1

- Particular data

  If IAD = 0 the general data are assumed to mean :

  E : the Young's modulus

  NU : the Poisson's ratio

  T : the thickness

  DA : the distributed transverse load

  If IAD = 1 these 4 variables are read on one card

- Stress output

  The bending moments are computed at the 3 vertices in local cartesian axes having the Ox axis oriented along the edge 1-2. The constant shear forces V_x and V_y are also computed. In addition the normal Kirchhoff shear forces $K_{n_{ij}}$ are given along the 3 interfaces.

- Subroutines used

  SP13 CP13 COND13 (plus the matrix operation package)

- Reference : A 10, 2.8 and chap. 2 of the present report

- Assumptions : 1) $u = \alpha_1 + \alpha_2 s + \alpha_3 s^2 + \alpha_4 s^3 + \alpha_5 s^4 + \alpha_6 s^5 \ldots$

$$v = \alpha'_1 + \alpha'_2 s + \alpha'_3 s^2 + \alpha'_4 s^3 + \alpha'_5 s^4 + \alpha'_6 s^5 \ldots$$

s being a current coordinate tangent to the edges.

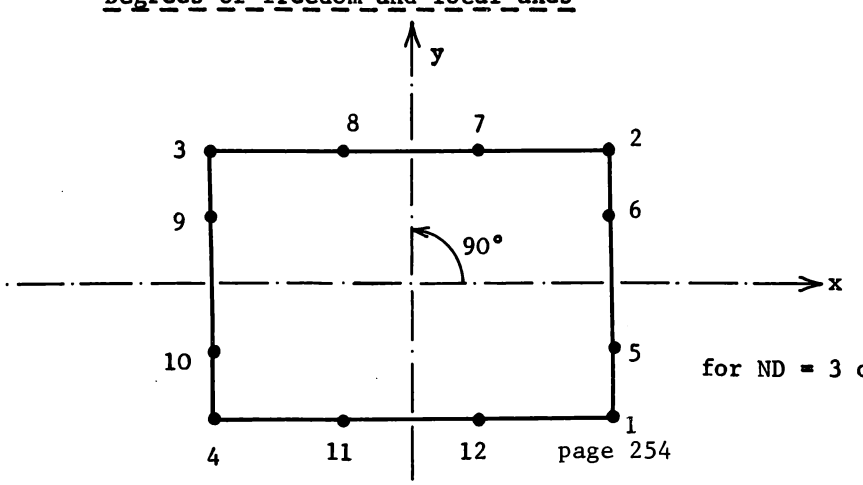
$$2)\ \sigma_x = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

ϕ (x,y) being a complete plynomial in x and y.
The polynomials for u, v, ϕ are truncated at the degrees requested by the user.

- Description

This rectangular membrane element is derived according to the theory of hybrid elements deducible of the Reissner principle and presented in the chapter 2 of the present report. It covers the family of such elements up to the 5th degree for the displacement field assumed along each interface independently and with practically no limitation for the degree of the stress field assumed inside. Numerical difficulties arise however if the degree of the stress field exceeds 8. The nature and the sequence of the generalized displacements defined along the interfaces are such that the element can be joined indifferently to any conforming membrane element of types 3, 4, 6, 20 and to the bar or beam elements of types 2 or 7. It should be noted that no check is incorporated that the element is effectively a plane rectangle. The thickness is supposed constant.

- Degrees of freedom and local axes



for ND = 3 on all interfaces

The sequence of the generalized displacements follows the same logic as for the conforming membrane elements of types 3,4 or 20 : the displacements of the vertices are followed by those of points defined on the interfaces taken sequentially turning anticlockwise.

$$q' = (d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7\ \ldots\ d_{12})$$

with $d_i = (u_i\ v_i\ w_i)$

The displacements of the vertices are always expressed in global axes while on each individual interface the special local axes defined for the elements of types 3 or 4 can be selected by the phase 1.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD, ND, NT, ITENS

Most of the variables have been defined in section A3, FORM 3

ITYP = 14 NNO=4 IAD = 0 or 1

ND is the degree of the displacements along the 4 interfaces. If it is not defined, the value given by NDC in the general data is assumed. Eventually the phase 1 can reduce this degree along interfaces with adjacent elements of smaller degree.

NT is the degree of stress field. It should be remembered that some combinations of ND and NT are restricted by the presence of kinematic deformation modes (See figure 2.13, chapter 2). If NT is not specified, it is assumed equal to ND.

ITENS is composed of 5 digits which mean :

- 1st digit : If equal to 1, the stress output are punched
- 2nd and 3rd digits : Degree of the stress output along the Ox local axis
- 4th and 5th digits : Degree of the stress output along the Oy local axis.

If the 5 digits are zero, the degree of the stress output corresponds to NT.

- Particular data

If IAD = 0 the general are assumed

- E is the Young's modulus
- NU is the Poisson's ratio
- T is the thickness

If IAD = i the variables E, NU, T are read on one card.

- Stress output

The stresses are computed in the local axes in a number of points defined by NT or ITENS and which correspond to a regular subdivision of the element in subrectangles. The sequence of the reference points starts at node 4 and follows lines parallel to the Ox axis in ascending order.

- Subroutines used

SP14	SEXY14	BS1404	TC14
NODE14	CECI14	BS1416	
CONN14	R14	BS1417	

(plus the matrix operation package)

TYPE 15 CONFORMING QUADRILATERAL PLATE BENDING ELEMENT
(KIRCHHOFF THEORY)

- Reference : 2.1, 2.7, 2.10 or 2.26

- Assumptions : In each triangular region

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

$$+ \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3$$

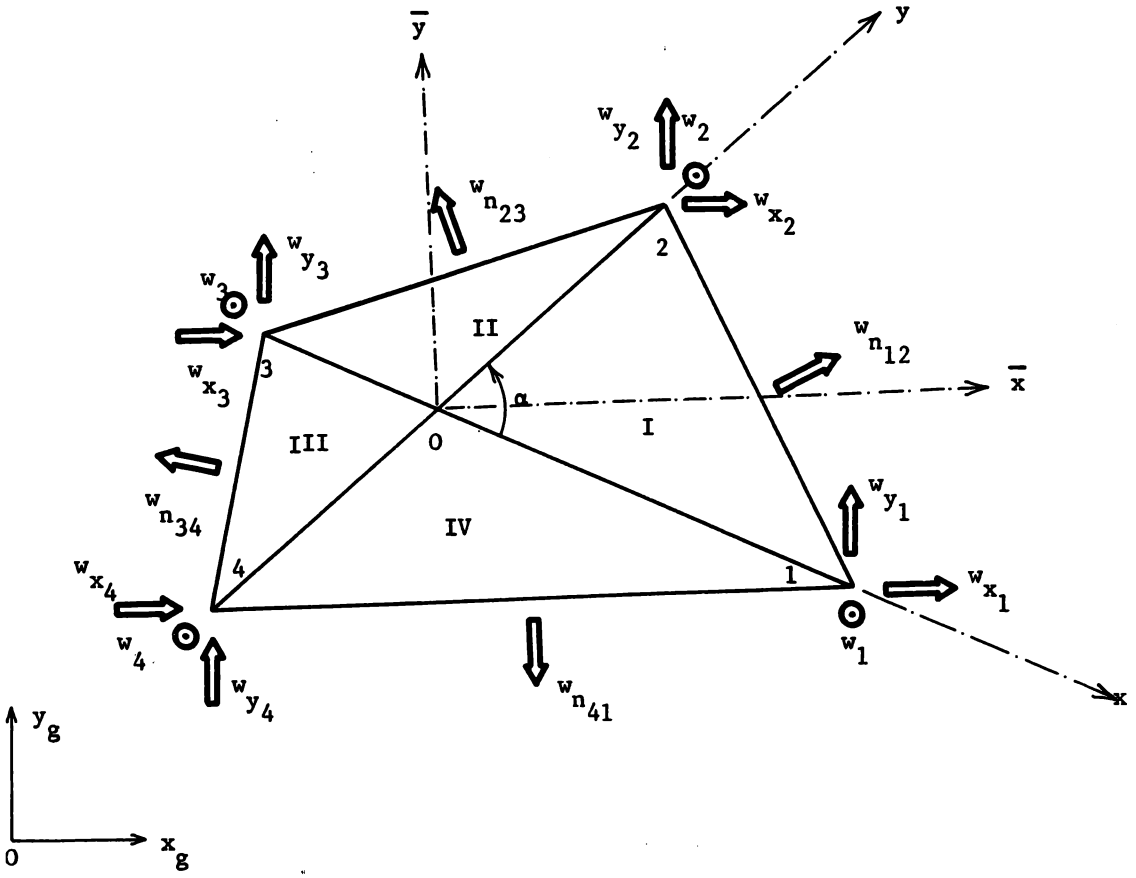
- Description

This plate bending super-element is obtained by assembling the 4 triangular elements defined by the diagonals of the quadrilateral.
In each of these triangles the deflection is represented by a complete cubic. The 40 corresponding parameters are reduced to 16 by expressing the continuity requirements of the deflection and normal slope along the internal interfaces of the quadrilateral. These 16 independent parameters are finally expressed in terms of the 16 generalized displacements (w, $w_{,x}$, $w_{,y}$, at the 3 vertices and $w_{,n}$ along each interface) necessary to insure a strict continuity of the deflection and normal slope along the external interface of the quadrilateral.
The element is programmed with the possibility of using a variable thickness, anisotropic stress-strain relations and various special support options. The element can be used as an equilibrium membrane element by the stress function method. It is the Southwell analog of the element type 16. In this case the thickness has to be constant.

- Degrees of freedom

The element can only be used in the global X-Y plane. The sequence of the 16 generalized displacements is :

$$q' = (w_1 \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial y_2} \quad w_2 \frac{\partial w}{\partial x_2} \frac{\partial w}{\partial y_2} \quad w_3 \frac{\partial w}{\partial x_3} \frac{\partial w}{\partial y_3} \quad w_4 \frac{\partial w}{\partial x_4} \frac{\partial w}{\partial y_4} \frac{\partial w}{\partial n_{12}} \frac{\partial w}{\partial n_{23}} \frac{\partial w}{\partial n_{34}} \frac{\partial w}{\partial n_{41}})$$



Oxy	oblique local axes
$O\bar{x}\,\bar{y}$	cartesian local axes
$Ox_g\ y_g$	global axes

The local axes $O\bar{x}$ is directed from 0 toward the middle of the edge 12.

The normal slopes $w_{n_{ij}}$ are expressed in the middle of each edge.

- Special options

1°/ Variable thickness : the thickness is constant unless the variable ICHOI = 1 In this case the thickness at point 0 is interpolated from the values defined along the 2 diagonals at this point by a linear variation. The average of the 2 values is assumed. When the thickness is variable all the other special options are ineffective.

2°/ Anisotropy : it is controlled by the variable IANISO. IANISO = 0 corresponds to the isotropic case. Four anisotropic layers can be superimposed to the parent plate. Each of these layer can have an independent thickness t_{ani} and the bending rigidity of each layer is $\frac{1}{12} E\, t_{ani}^3$.

The orientation of these layers are independent and defined by 2 nodes which are not necessarily vertices of elements.

3°/ Internal support : when a point support is close to a free edge it is better to replace a generalized displacement of that edge by a local deflection expressed inside the element. This avoids using very small elements in the neighbourhood of the support. This option is controlled by the variable NAPPUI which equals 1 in case of an internal support. Anyone of the variable of an unconnected edge can be changed into a point support.

4°/ Local axes for stress output : their choice is controlled by the variable ISTRAX. If ISTRAX = 3 or 4 the stresses are computed in the global X Y plane. If ISTRAX = 0 or 1 the local axes $O\bar{x}\ \bar{y}$ are used.

5°/ Special coordinate transformation for the slopes : At each vertex a possibility exists to change the cartesian slopes $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ into oblique slopes tangent to the edges $\frac{\partial w}{\partial s_{ij}}\ \frac{\partial w}{\partial s_{jk}}$ where $i < j < k$. This is controlled by the array IAX (4) each component corresponding to a vertex. If IAX (i) = 0 the cartesian slopes are used at the vertex i. IAX (i) = 1 otherwise.

- Element definition card and particular data

The element definition card contains :

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD, ICHOI

All the variables, but ICHOI, are defined in section A3, FORM 3.

ICHOI controls the combinations of the special options allowed.

1. ICHOI = 0 implies ISTRAX = 3 IANISO = 0 NAPPUI = 0
IAX = 0 IAD = 0 or 1

If IAD = 0 the general data are assumed

E is the Young's modulus

NU is the Poisson's ratio

T is the thickness

DA is the constant distributed load per unit area

If IAD = 1 E, NU, T, DA are read on one card.

2. ICHOI = 1 implies ISTRAX = 3 NAPPUI = 0 IANISO = 0
IAX = 0 IAD = 0, 1 or 2

If IAD = 0 the thickness is constant but the routines for variable thickness are used

If IAD = 1 the thickness at the 4 vertices are read on one card

If IAD = 2 the second card contains :
E, NU, blank, DA

3. ICHOI = 2 implies IAD = 1, 2, 3, 4, 5 or 6 (but not 0)
On the first card are read the variables :

E, NU, T (1), DA, IANISO, IAX (i = 1, 4), NAPPUI where T (1) is the thickness of the parent plate (eventually zero).
This card is followed by IANISO additional cards containing

NN1, NN2, T (anisotropic layer nbr i)
where NNI, NN2 are the 2 nodes defining the orientation of the layer and T its thickness. If IANISO = 0, these cards are omitted

If NAPPUI ≠ 0 another additional card follows with
NTRI, NELIM, NND
where - NTRI is the number (1, 2, 3 or 4) of the triangle in which the internal support is to introduce
- NELIM is the number (from 1 to 16) of the generalized displacement to replace by the internal support. This number refers to displacement sequence given above.
- NND is a node number which corresponds to the internal support (the 4 vertices have however to be specified in the element definition card)
If $\sum_i$ IAX (i) = 0, ISTRAX = 3. Otherwise ISTRAX = 1.

4. ICHOI = 3 implies identical options as for ICHOI = 2 but if
$\sum_i$ IAX (i) = 0, ISTRAX = 3. Otherwise ISTRAX = 4

Important : The element is used as an equilibrium membrane element by the stress function method if NU is negative. In this case however the number of options is limited to ICHOI = 0 which is the only possibility.

- Stress output

The bending moments are computed at the 4 vertices and at the diagonal intersection (reference point 5). They are the averages of the moments computed in each triangle at a reference point, inside the quadrilateral. In the stress function approach, the bending moments are replaced by stresses but the tittle in the output are not changed.

- Subroutines used

SP15	SP15B8	HOOK15	ELIM15
THC15	ABC15	COOR15	OBLI15
VAR15	LOC15	APUI15	TS15
			TS15PR
			T15VAR

(plus the matrix operation package)

TYPE 16 EQUILIBRIUM QUADRILATERAL MEMBRANE ELEMENT (LINEAR STRESS FIELD)

- References : 2.27, 2.26

- Assumptions : In each triangular region :

$$\sigma_x = \beta_1 + \beta_2 x + \beta_3 y$$

$$\sigma_y = \beta_4 + \beta_5 x + \beta_6 y$$

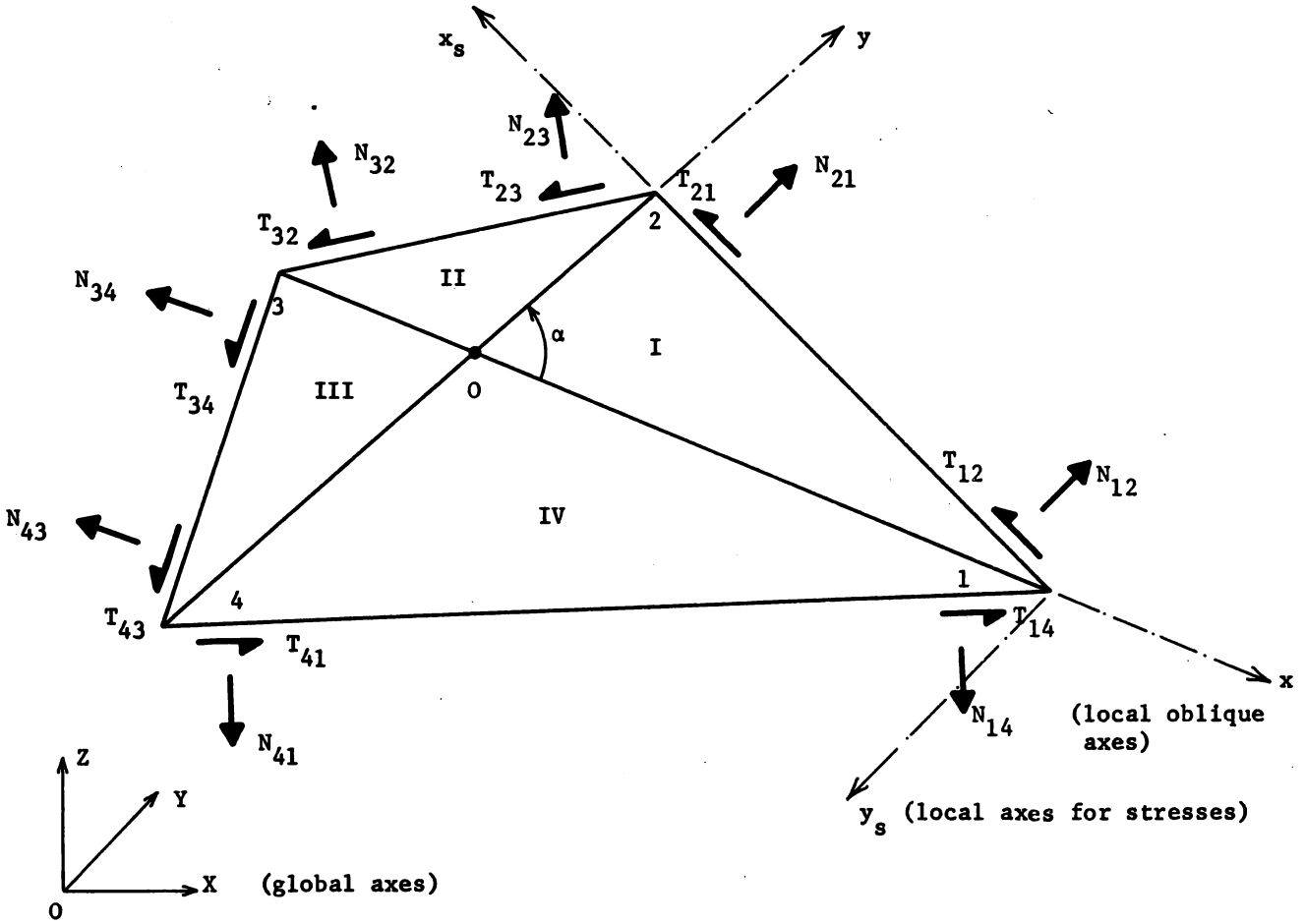
$$\tau_{xy} = \beta_7 - \beta_6 x - \beta_2 y$$

- Description

This equilibrium membrane is a super-element obtained by assembling the 4 triangular elements defined by the diagonals of the quadrilateral. In each of these triangles the stress field is linear as indicated in the assumptions. The 21 corresponding parameters are reduced to 13 by expressing the continuity of the normal and tangential surface tractions along the internal interfaces of the quadrilateral. The 16 generalized forces necessary to determine uniquely the linear variation of the surface tractions along the external interfaces are expressed in terms of the 13 parameters. As the 16 forces satisfy 3 global equilibrium equations, the element is free of spurious kinematic modes. The generalized forces are the local values of the surface tractions τ_{sn}, σ_n (times the thickness) at each vertex, along an interface. Note that although expressed at a vertex, they are interface variables.

The element is programmed with the possibility of using a linearly variable thickness.

- Degrees of freedom



The sequence of the generalized forces in the local axes Oxy is

$$g' = (N_{12}\ T_{12}\ N_{21}\ T_{21}\ N_{23}\ T_{23}\ N_{32}\ T_{32}\ N_{34}\ T_{34}\ N_{43}\ T_{43}\ N_{41}\ T_{41}\ N_{14}\ T_{14})$$

where N_{ij}, T_{ij} are cartesian normal and tangential components of the surface tractions multiplied by the length of the interface. These forces are transformed in global forces

$$g' = (F_{x_{12}}\ F_{y_{12}}\ F_{z_{12}}\ F_{x_{21}}\ F_{y_{21}}\ F_{z_{21}}\ F_{x_{23}} \ \ldots\ F_{z_{14}})$$

Eventually the phase 1 can decide to keep the local n, t axes on an interface in which case the 3rd component of the global forces at each point is given a zero stiffness. This is the case when only 2 membrane elements meet along an interface and are coplanar without being in a plane of coordinates.

- Correction of twist

If the 4 vertices of the element are not coplanar, a mean plane is determined and the 4 nodes projected in this plane. The importance of the warping is NOT checked.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD

All these variables are defined in section A3, FORM 3

ITYP = 16 NNO = 4 IAD = 0, 1 or 2

- Particular data

If IAD = 0 the general data are assumed

E is the Young's modulus

NU is the Poisson's ratio

T is the constant thickness

If IAD = 1 E, T1, T2, T3, T4 are read on one card

where Ti is the thickness at a vertex. The thickness at point 0 is the average of the 2 values computed at this point by assuming a linear variation along each diagonal independently.

If T2 is zero, the thickness is assumed to be constant and equals T1.

NU is assumed to have the value given in the general data.

If IAD = 2 NU is read on the second additional card.

- Stress output

The stresses are computed at the 4 vertices and at diagonals intersection in local axes $Ox_s y_s$ which have the Ox_s axis coinciding with the edge 1-2. The stresses are the averages of the values computed in each triangle meeting at the reference points.

In addition the normal and tangential fluxes are computed along each edge at the vertices.

- Subroutines used

SP16	MATS16	TCOR16
QEL16	BS2116	BCBI16
CONN16	GEOM16	DIV16
FLEX16	FLAN16	TS16PR

(plus the matrix operation package)

TYPE 17 EQUILIBRIUM BAR ELEMENT WITH LINEARLY VARYING SHEAR AND CROSS SECTION AREA

- Reference.: 2.27

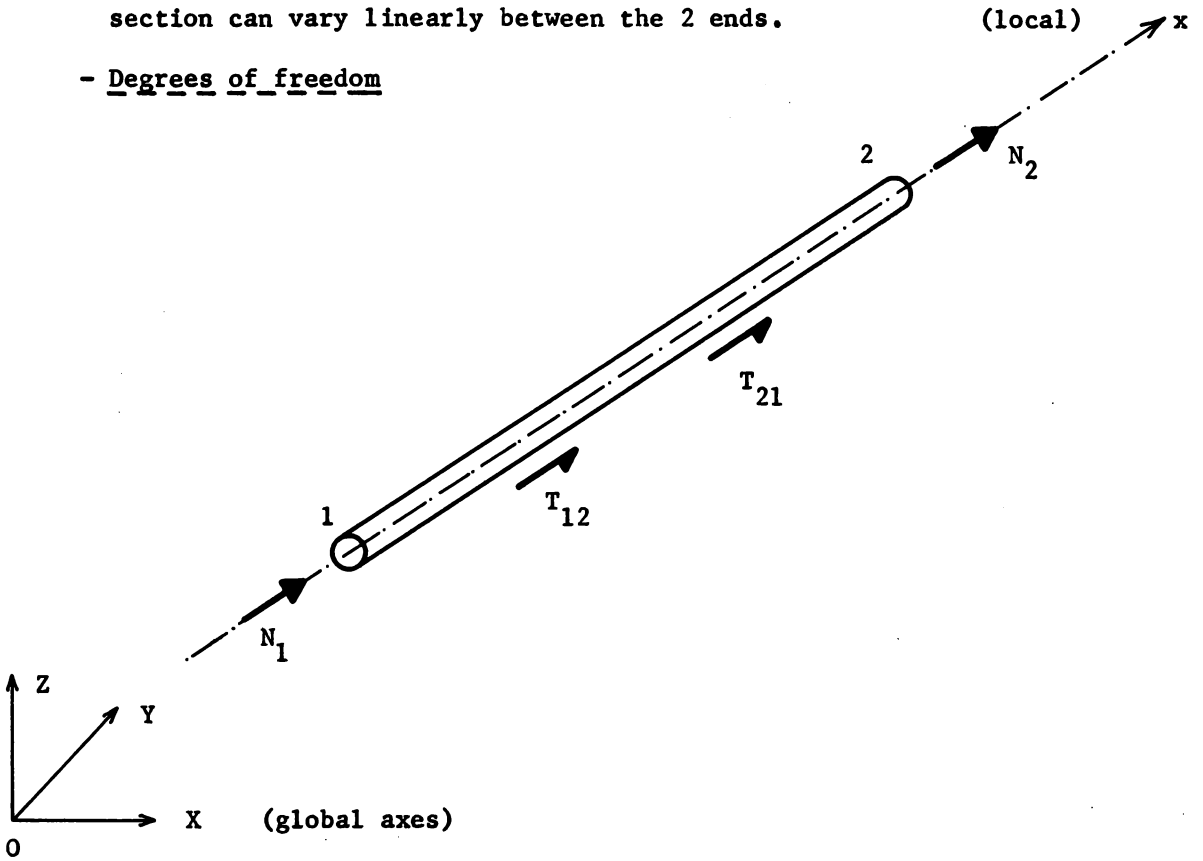
- Assumptions : $\tau(x) = \beta_1 + \beta_2 x$

$$\text{or } N(x) = \beta_0 + \beta_1 x + \beta_2 \frac{x^2}{2}$$

- Description

This bar element is intended to reinforce the membrane element of type 16. It is stresses by a linear axial shear and 2 tip loads. The cross section can vary linearly between the 2 ends.

- Degrees of freedom



The sequence of the generalized forces in the local axes is

$$g' = (N_1 \; N_2 \; T_{12} \; T_{21})$$

where N_i is a tip load and T_{ij} the local value of the shear at an end times the length of the bar.

These forces are expressed in the global axes in the sequence :

$g' = (F_{x_1} \; F_{y_1} \; F_{z_1} \; F_{x_2} \; F_{y_2} \; F_{z_2} \; T_{x_{12}} \; T_{y_{12}} \; T_{z_{12}} \; T_{x_{21}} \; T_{y_{21}} \; T_{z_{21}})$

Eventually the forces T_{12} and T_{21} are kept in local axes if this special choice of axes has been decided by the phase 1. In this case, zero stiffness is given to the components $T_{x_{12}}$ $T_{z_{12}}$ and T_{ij} replaces $T_{y_{ij}}$ in the sequence given above.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, LSIMPL, ISTRES, IAD

All these variables are defined in section A3, FORM 3.

ITYP = 17 NNO = 2 IAD = 0 or 1

- Particular data

If IAD = 0 the general data are assumed

E is the Young's modulus

T is the constant cross section

IF IAD = 1 the variables E, T1, T2 are read

T1, T2 are the cross sections at both ends

If T2 = 0, the cross section is assumed constant and equals T1.

- Stress output

The stresses are computed at 4 reference points which are : node 1, node 2, 1/3 and 2/3 of the length 1-2.

- Subroutines used

SP17 EVA17

BEL17

(plus the matrix operation package)

TYPE 20 CONFORMING PARALLELOGRAM ELEMENT WITH VARIABLE DEGREE AND BUBBLE FUNCTIONS

- Reference : A 11

- Assumption : u and v are of the form

$$P_1(x, y) + \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right) P_2(x, y)$$

where both functions P_1 and P_2 are

$$P_i = (\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots + \alpha_{n+1} x^n)(\beta_1 + \beta_2 y + \beta_3 y^2 + \ldots + \beta_{n+1} y^n)$$

truncated at the degree requested by the user.

- Description

This element is a generalization of the classical rectangular membrane elements in this sense that it is extended to variable degree polynomials (from 1 to 9) and that it includes bubble displacement modes of degree independently variable (from 0 to 9).

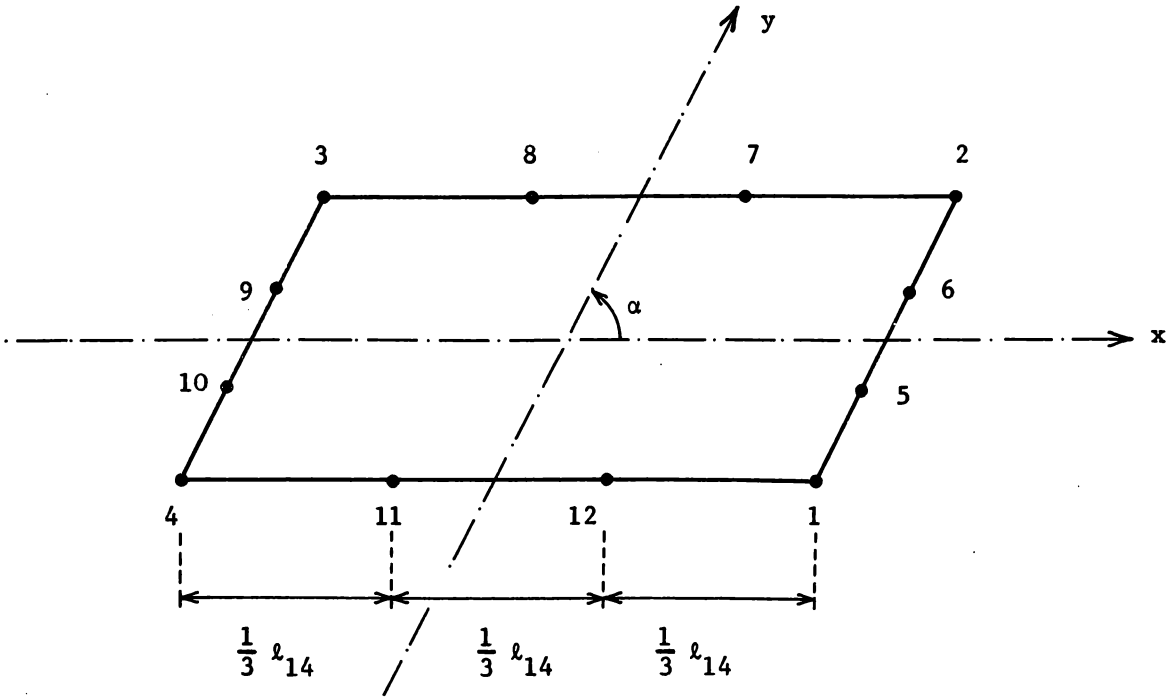
In this purpose the polynomials used to represent the displacements are split in two parts controlled by the functions P_1 and P_2. P_1 describes the basic field and the interface modes while P_2 represents only bubble displacement modes.

To allow easy comparison with other elements (type 4 or 14) the number of terms retained in the functions $P_1(x, y)$ corresponds exactly to the number of generalized displacements necessary along the interfaces. This reduction of parameters is obtained by dropping in P_1 the terms of power greater or equal 2 in x AND y and the terms of power greater than n in x OR y, n being the degree of the displacements in the direction of the edges.

The complete form given above is always used for $P_2(x, y)$.

The thickness is constant and the material isotropic. The element can be used as a plate bending equilibrium element by the Southwell analogies.

- Degrees of freedom



nodal point sequence for ND = 3

The generalized displacements are expressed at equidistant points along the interface, depending of the degree selected . The sequence of the displacements, the conventions for reducing the degree along an interface, the option of local normal-tangent axes are exactly the same as for elements type 4 or 14.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, N4, ISIMPL, ISTRES, IAD, ND, NB, ITENS

Most of the variables are defined in section A3, form 3

ITYP =20 NNO = 4 IAD = 0 or 1

ND is the degree of the basic field P_1 (x, y). If ND is zero the degree is that defined by NDC in the general data. The degree of the basic field is eventually reduced along these interfaces where the adjacent elements are of smaller degree.

NB is the degree of the bubble modes P_2 (x, y). If NDB is greater or equal 10, no bubble modes are included.

Note that NB = 0 corresponds to one bubble mode when P_2 = constant.

ITENS is an option for stress output composed of 5 digits :

1st digit If equal 1 the stress output are punched (or put out on unit 7)

2nd and 3rd digits : It specifies the mesh of reference points for stress output in the direction Ox.

4th and 5th digits : It specifies the mesh of reference points for stress output in the direction Oy.

If ITENS = 0, the mesh corresponds to ND - 1

- Particular_data

If IAD = 0 the general data are assumed

E is the Young's modulus

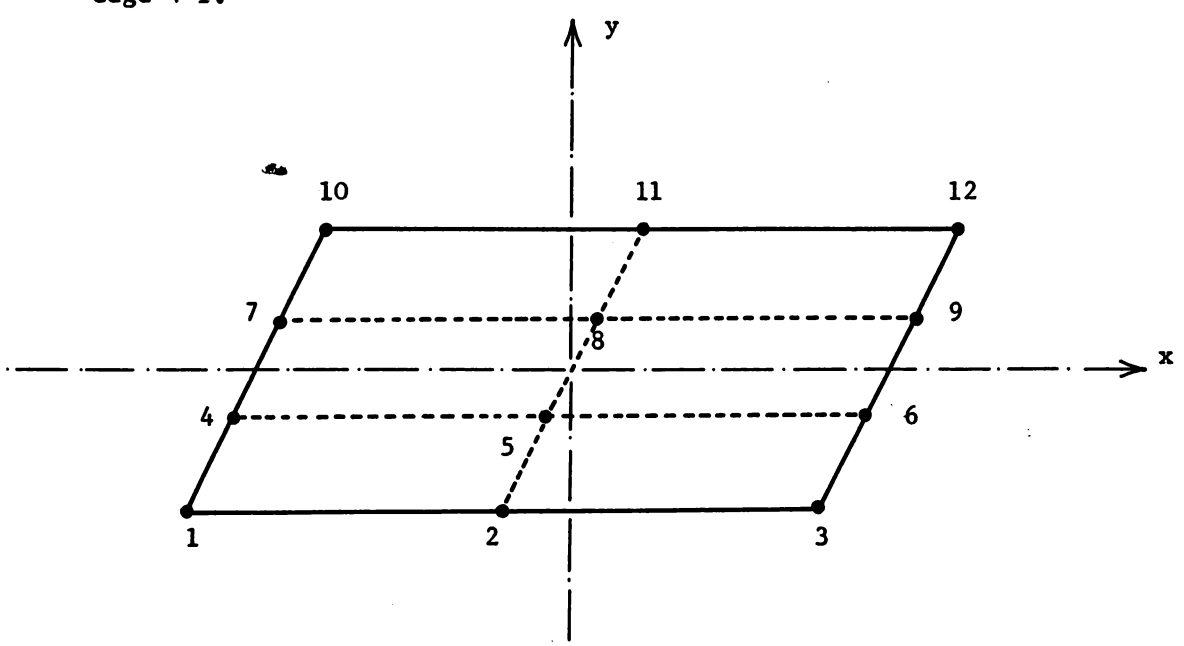
NU is the Poisson's ratio

T is the thickness

If IAD = 1 these variables are read on one card.

- Stress_output

The stresses are computed at reference points which form a regular mesh defined by ND-1 or ITENS and ordered in the sequence illustrated below. The local cartesian axes for stresses have the Ox axis parallel to the edge 4-1.



Reference points for stress output : 2nd degree in x and 3rd in y

When the element is used as an equilibrium element by the stress function method, bending moments are computed instead of stresses.

Note : the mesh specified for stress output is independent of the degree specified for the displacements. Redundant values are eventually interpolated.

- Subroutines used

SP20	POL203	BS2014	TC20
RECT20	POL204	BS2015	TS20
CONN20	POL205	BS2016	TE20
INT20	BS2004	BS2017	

(plus the matrix operation package)

- Reference : 2.12

- Form of the constraint

$$\alpha_j u_{ji} + \alpha_k u_{ki} = \alpha_\ell$$

when j is the index of node 1
k is the index of node 2
i is the index of the component of displacement
ℓ is the index of the node identifying the constraint

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD, I1, I2, I3, I4, I5, I6

where NOEL is the element number. It is a dummy variable
ITYP = 51
NNO = 3
N1 is the number of the first node subject to the constraint
N2 is the number of the second node subject to the constraint
N3 is the number of a dummy node necessary to identify the constraint in the phase 1.
ISIMPL ≠ 0 specifies that the constraint is identical to the preceding one
ISTRES is not used
IAD = 0 to 6
Ii are variables corresponding to the various degrees of freedom of the nodes.
If Ii = 0 the constraint does not affect the component i
If Ii ≠ 0 the constraint does affect the component i.
Therefore this element can generate up to 6 constraints at the same time.

- Particular data

If IAD = 0 the values of the coefficients for all the constraints generated by the element are :

$\alpha_j = A$ $\alpha_k = -A$ $\alpha_\ell = 0$

where A is the Young's modulus (as given in the general data) if the Poisson's ratio is positive and where A is the inverse of the Young's modulus if the Poisson's ratio is negative, that is if the stress function method is used.

If IAD = 1 the values of the coefficients α_j α_k α_ℓ are read on one card and are supposed to hold for all the constraints generated by the element

If IAD > 1 it must equal the number of degrees of freedom affected by the constraint at a node. The coefficients of constraint α_j α_k α_ℓ are read for each component affected on one card.

- Important remarks

1°/ If α_ℓ is not equal to zero, the general data IFO must be equal to 1 as this coefficient is treated as an implicit second member analog of a body force.

2°/ The constraints should be scaled in such a way that the coefficients are of the same order of magnitude as the diagonal terms of the stiffness matrix. This is the reason for chosing E or $\frac{1}{E}$ as the standard option. In case IAD $\neq$ 0 the scaling is under the responsability of the user.

3°/ The constraints must be specified after all the elements defined by the nodes N1 and N2. This is necessary to avoid numerical problems using the Cholesky decomposition.

4°/ The nodes corresponding to N3 have to be defined as the other nodes but their coordinates have no meaning. They should not be used as vertices of finite elements.

TYPE 52 CONSTRAINT BETWEEN THE DISPLACEMENTS AT A NODE AND THE FOUNDATION

- Reference : 2.12

- Form of the constraint

$$\alpha_j \, u_{ji} = \alpha_\ell$$

where j is the index of the node
i is the index of the component of displacement
ℓ is the index of the node identifying the constraint

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, ISIMPL, ISTRES, IAD, I1, I2, I3, I4, I5, I6

where most of the variables have the same meaning as for the constraint of TYPE 51, but

ITYP = 52 NNO = 2 IAD = 0 to 6

N1 is the number of the node subject to the constraint

N2 is the number of the dummy node necessary to identify the constraint during phase 1.

- Particular data

The conventions used in TYPE 51 are applicable to this element but only 2 coefficients α_j and α_ℓ are eventually read per card.

- Important remarks

1°/ The same 4 important remarks formulated for the constraint TYPE 51 hold.

2°/ When only one constraint 52 is used, it is equivalent to imposing the values of the displacement components at the node.

3°/ When 2 constraints 52 are used WITH THE SAME DUMMY NODE N3, they are additive. If j, k are 2 node numbers and

$$\alpha_j \, u_{ji} = \alpha_{\ell_1}$$

$$\alpha_k \, u_{ki} = \alpha_{\ell_2}$$

the two constraints identified by the same node N3, they are equivalent to one constraint of the form

$$\alpha_j \, u_{ji} + \alpha_k \, u_{ki} = \alpha_{\ell_1} + \alpha_{\ell_2}$$

which is a constraint of type 51 if α_ℓ is changed to $\alpha_{\ell_1} + \alpha_{\ell_2}$

4°/ More constraints of the type 51 and 52 can be combined following the same scheme.

- Reference : 2.12

- Form of the constraint

It is especially designed for the variable degree membrane element. Given the degree of the displacement along an interface it establishes the relationships necessary to reduce the degree of anyone or all the displacement components u, v, w to a prescribed degree.

- Element definition card : it contains

NOEL, ITYP, NNO, N1, N2, N3, ISIMPL, ISTRES, IAD, ND, IU, IV, IW

Most of these variables have the meaning given in section A3, FORM 3

ITYP = 53 NNO = 3 IAD = 0

N1, N2 define the interface

N3 is the dummy node identifying the interface

ND is the prescribed degree (≥ 1)

IU, IV, IW indicate on which component the constraint is effective

(= 1 means effective, = 0 otherwise)

Note that the initial degree along the interface is determined by the phase 1 and not given as a data.

- Particular data

None and IAD must be zero

A5. EXAMPLES

The program is provided with 17 short problems which yields examples of application of most of the finite elements and methods of solution.

They correspond to 3 structural problems illustrated in figure A10, A11, A12 and denoted :

TEST CASE NUMBER 1 : Cantilever thin walled U beam in bending

This problem is analysed in 10 different ways. In all cases, but in problem 6, the very crude idealisation illustrated on figure A10 is used. In problem 6, each element has been subdivided in four.

Problem nbr	Type of analysis	Energy	Bound
1	Conforming quadrilateral 1st degree	0.720	lower
2	Idem 2nd degree	1.482	lower
3	Idem 3rd degree	1.492	lower
4	Conforming parallelogram 2nd degree - bubble	1.473	lower
5	Equilibrium quadrilateral Constant stress	3.451	upper
6	Idem Constant stress - finer mesh	1.703	upper
7	Idem linear stress	1.500	upper
8	Idem linear stress solution by stress function	1.500	upper
9	Hybrid rectangle 2nd degree in displacement 2nd degree in stresses	1.628	none
10	Conforming quadrilateral + special bulkhead element	1.444	lower

TEST CASE NUMBER 2 : Cantilever skew plate in bending

This problem is analysed in 6 different ways for which the same idealisation presented on figure A11 is used. The element number from 101 to 136 and the deshed lines apply only in case of triangular elements.

Problem nbr	Type of analysis	Theory
11	Conforming quadrilateral TYPE 15	Kirchhoff
12	Equilibrium triangle TYPE 13	Kirchhoff
13	Equilibrium quadrilateral TYPE 9	Reissner
14	Conforming triangle TYPE 8	Hencky
15	Equilibrium triangle by stress function method	Kirchhoff
16	Conforming quadrilateral TYPE 15 anisotropic material	Kirchhoff

TEST CASE NUMBER 3 : Cylindrical shell reinforced by rings

It is analysed as a polygonal cylinder using conforming quadrilaterals of 2nd degree and beam elements for the rings.
It is denoted problem 17.

With the help of the preceding sections, the input and output data of these 17 problems should be self-explanatory.

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Report SF-15, LTAS, University of Liège

A9 Modèles d'éléments finis de membranes cinématiquement admissibles et codéformables - triangles, quadrilatères et barres -
P. BECKERS
Report SF-13, LTAS, University of Liège

A10 M. KIEFFER
Considérations et résultats relatifs aux éléments du type PIAN et ANTI-PIAN
Report SF-8, LTAS, University of Liège

A11 P. BECKERS
Introduction des modes de déplacements intérieurs de l'élément de parallélogramme de type cinématiquement admissible codéformable
Reprot SF-12, LTAS, University of Liège

GENERAL FLOW DIAGRAM

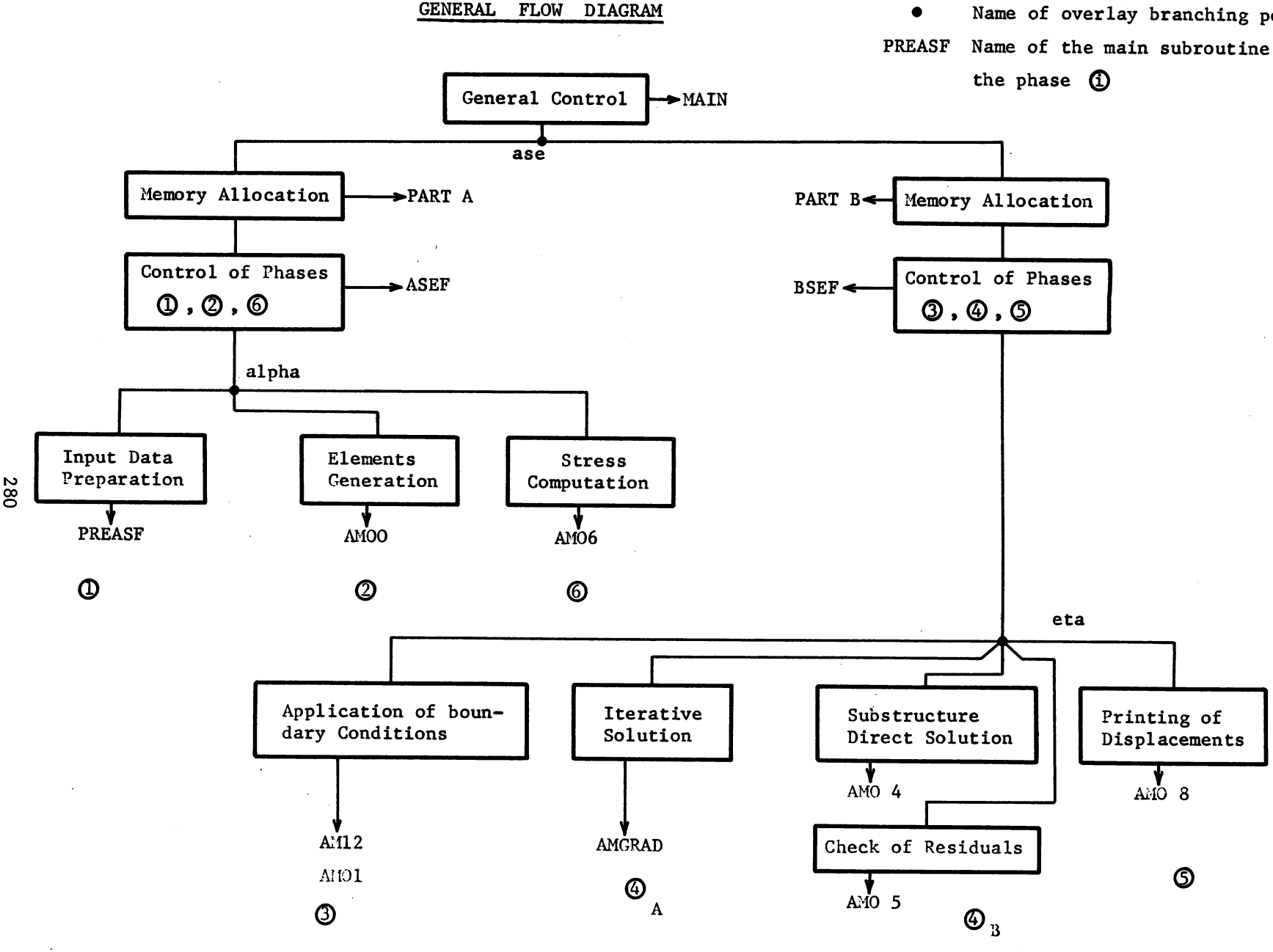


FIGURE A.1

```
            INSERT MAIN
            INSERT CHRONO,IMPMAT,IMPTIT,IMPSYM,MATINV
            INSERT MULMA2,ATBREC,ATSYRT,RTASYM,TRANSF
OVERLAY ASE
            INSERT PARTA,ASEF,ASEFCO
            INSERT MULMA1,MULMAR,MULMAT
OVERLAY ALPHA
            INSERT PREASF,PREP
OVERLAY BETA
            INSERT ELEM,BPAXE,PREAXE,BPLAN,DEPNEU,DECOT
            INSERT LL2,LL8,LL9,LL13,LL15
OVERLAY BETA
            INSERT CONDI,PREST,DECOUP,PRELOC
OVERLAY ALPHA
            INSERT AMOO,ELEMNT,IMP3,IMP4,IMP5,IMP6
            INSERT FORMUL,CONNEC,VECTPR,FOR
OVERLAY DELTA
            INSERT SP1,CP1,AUXPT
OVERLAY DELTA
            INSERT SP2,SPBAR2
OVERLAY DELTA
            INSERT SP3,SP4,DS21,BS2101,BS2102,BS2103,BS2104
            INSERT BS2105,BS2106,BS2107,BS2108,BS2109
            INSERT BS10,BS16,BS17,BS18,BS19
            INSERT BS01,BS02,BS03,BS04,BS05
OVERLAY DELTA
            INSERT SP5,CP5
OVERLAY DELTA
            INSERT SP6,LAND6,CD6,MC6,COR6,MP6,STA6,INT6
            INSERT PINT6,TA6,STS6
OVERLAY DELTA
            INSERT SP7,LCV7,ST7,MC7,CD7
OVERLAY DELTA
            INSERT SP8,COND8,SPL883
OVERLAY DELTA
            INSERT SP9,SPL9,FLEXI9,LOCEF9,CONN9,STREL9,COND9,FLEX
OVERLAY DELTA
            INSERT SP10
OVERLAY DELTA
            INSERT SP11,CP11
OVERLAY DELTA
            INSERT SP12,CP12,BS2112
OVERLAY DELTA
            INSERT SP13,CP13,COND13
OVERLAY DELTA
            INSERT SP14,MODE14,CONN14,SEXY14,CECI14,R14
            INSERT BS1404,BS1416,BS1417,TC14
OVERLAY DELTA
            INSERT SP15
OVERLAY ZETA
            INSERT THC15,VAR15
OVERLAY ZETA
            INSERT SP15B8,ABC15,LOC15,HOOK15
```


FIGURE A 2

```
                INSERT COOR15,APUI15,ELIM15,OBLI15
OVERLAY DELTA
                INSERT SP16,QEL16,CONN16,MATS16
                INSERT BS2116,GEOM16,TCOR16
OVERLAY THETA
                INSERT FLEX16
OVERLAY THETA
                INSERT FLAN16,BCBI16,DIV16
OVERLAY DELTA
                INSERT SP17,BEL17,EVA17
OVERLAY DELTA
                INSERT SP18
OVERLAY DELTA
                INSERT SP19
OVERLAY DELTA
                INSERT SP20,RECT20,CONN20,INT20,POL203,POL204,POL205
                INSERT BS2004,BS2014,BS2015,BS2016,BS2017,TC20
OVERLAY DELTA
                INSERT SP51,SP52,SP53,SQ53,COSD
OVERLAY DELTA
                INSERT SP54,SP55,SP56
OVERLAY ALPHA
                INSERT AM06,TSI,TSPRI,SAUPAG,PRIMOM,PRISTR
                INSERT TSBEAM,TSBAR
OVERLAY EPSILON
                INSERT TS3,DS21TS,BS11,BS12,BS13,BS14,BS15
                INSERT TS2105,TS2106,TS2108,TS2110
OVERLAY EPSILON
                INSERT TS5,TE5,TS5PR
OVERLAY EPSILON
                INSERT TS6PR,TS8PR,TS9PR,TS13PR,TS16PR,TS18PR
OVERLAY EPSILON
                INSERT TS10
OVERLAY EPSILON
                INSERT TS14,TE14
OVERLAY EPSILON
                INSERT TS15,TS15PR,T15VAR
OVERLAY EPSILON
                INSERT TS20,TE20
OVERLAY ASE
                INSERT PARTB,BSEF,BSEFCO
OVERLAY ETA
                INSERT AM12,AM13
OVERLAY ETA
                INSERT AM01,AM07
OVERLAY ETA
                INSERT AMGRAD,XSOLVE,SCALE,DSCALE,XKQ,RESIDU
                INSERT LECT3,IMPVEC
OVERLAY ETA
                INSERT AM04,AM02,AM03,SI210,SAVEKR
OVERLAY ETA
                INSERT AM05
OVERLAY ETA
                INSERT AM08,IMPDEP
```


FIGURE A 2 (end)

ASEF-20

MAIN CONTROL CARD

1 card FORMAT (3I5)

ISTART		ISTOP		IU5	

TITLE OF THE PROBLEM

2 cards FORMAT (20A4)

1 80

GENERAL INPUT DATA

1 card FORMAT (15I5)

NDC	IFO	IGRAD	NITER	LPREC	IDEP	NS	NTO	NCL	NCSOPT	NOP1	NOP2	NOP3	NOP4	NOP5	

5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80

1 card FORMAT (5E15.6 or 5F15.6)

E	NU	T	DA	DB	

15 30 45 60 75 80

FIGURE A 3

ASEF-20

FORM 2
PAGE :

NODAL POINTS COORDINATES

NODE NBR 5	X 20	Y 35	Z 50	FORMAT (5E15.6 or 5F15.6)

NB : Last node must be 99999

FIGURE A 4

ASEF-20

FORM 3
PAGE :

DEFINITION OF THE SUBSTRUCTURES AND ELEMENTS

FORMAT (15I5)

NOS NOEL 5	NEL ITYP 10	NNO 15	NODE NUMBERS, ISIMPL, ISTRES, IAD etc (see manual) 20 25 30 35 40 45 50 55 60 65 70 75 80

NB : Last element must be 99999

FIGURE A 5

ASEF-20

FORM 4
PAGE :

ELASTIC AND GEOMETRIC DATA OF THE ELEMENTS

FORMAT (I5, 5E15.6 or I5, 5F15.6) (For TYPE 15, also 15I5)

ELEM. NBR	DATA (see manual)				
5	20	35	50	65	80

NB : Last element must be 99999

FIGURE A 6

ASEF-20

FORM 5
PAGE :

ZERO PRESCRIBED DISPLACEMENTS

FORMAT (15I5)

NODE 1	NODE 2	COMPONENTS FIXED													
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	

NB : Last card must start by 88888 if non zero prescribed displacements follow, otherwise 99999

FIGURE A 7

ASEF-20

FORM 6
PAGE :

NON ZERO PRESCRIBED DISPLACEMENTS OR LOAD COMPONENTS

FORMAT (3I5, E15.6 or F15.6)

NODE 1 5	NODE 2 10	COMPO-NENTS 15	INTENSITY 30	

NB : Last card must be 99999 for fixation AND for loads. Different loading cases are separated by 88888.

FIGURE A 8

INPUT DATA FLOW CHART

(for a normal exécution with ISTART = 1)

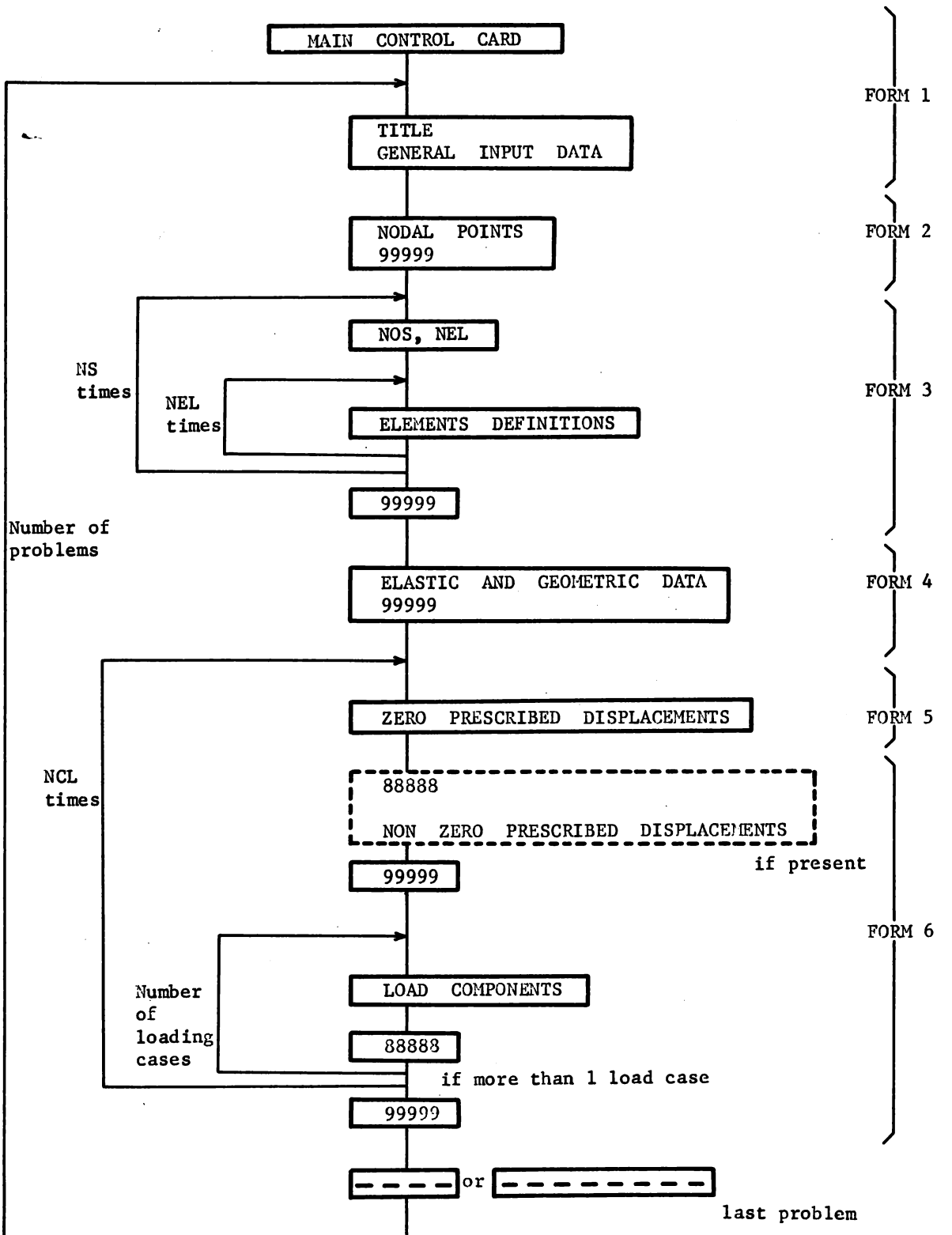
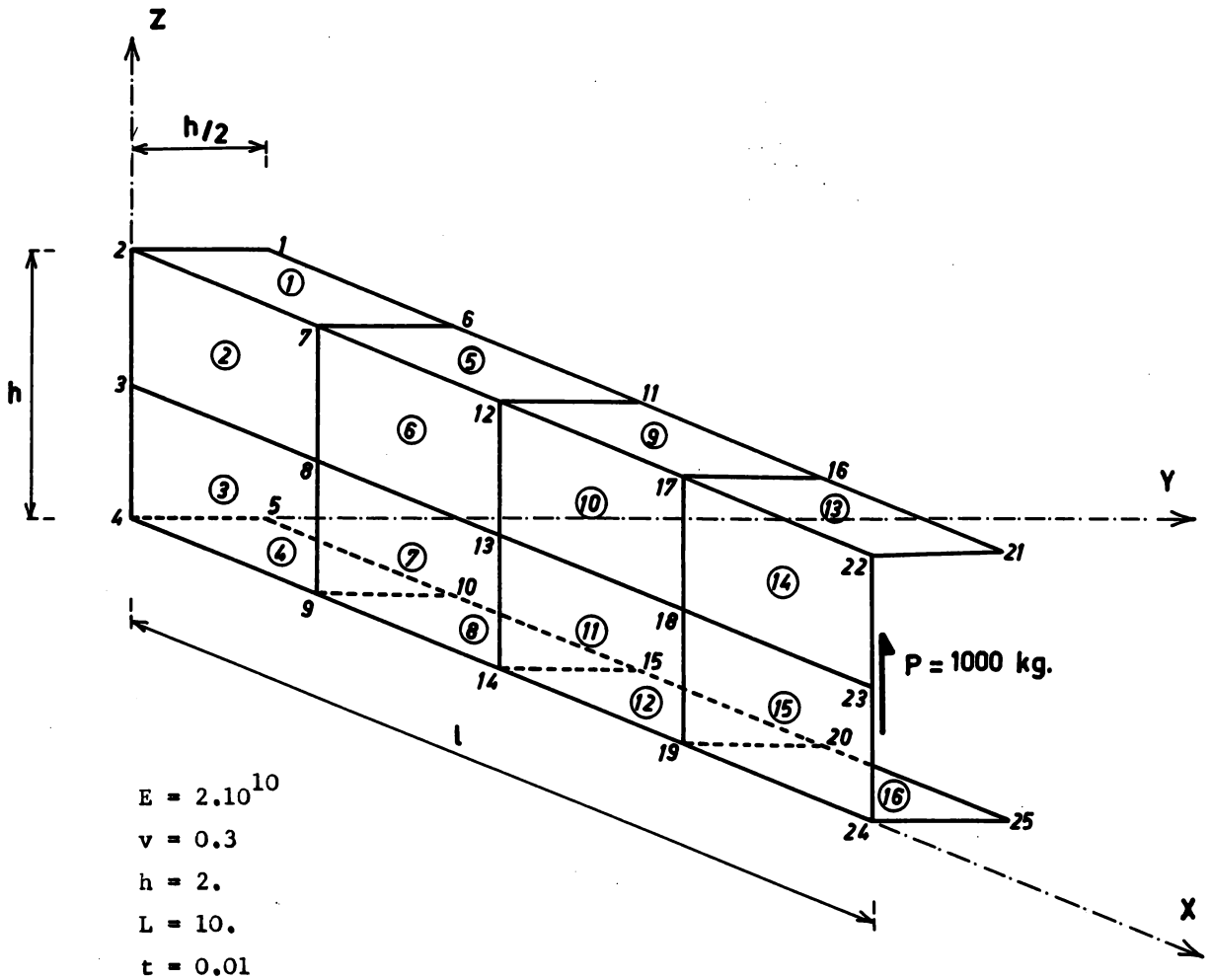
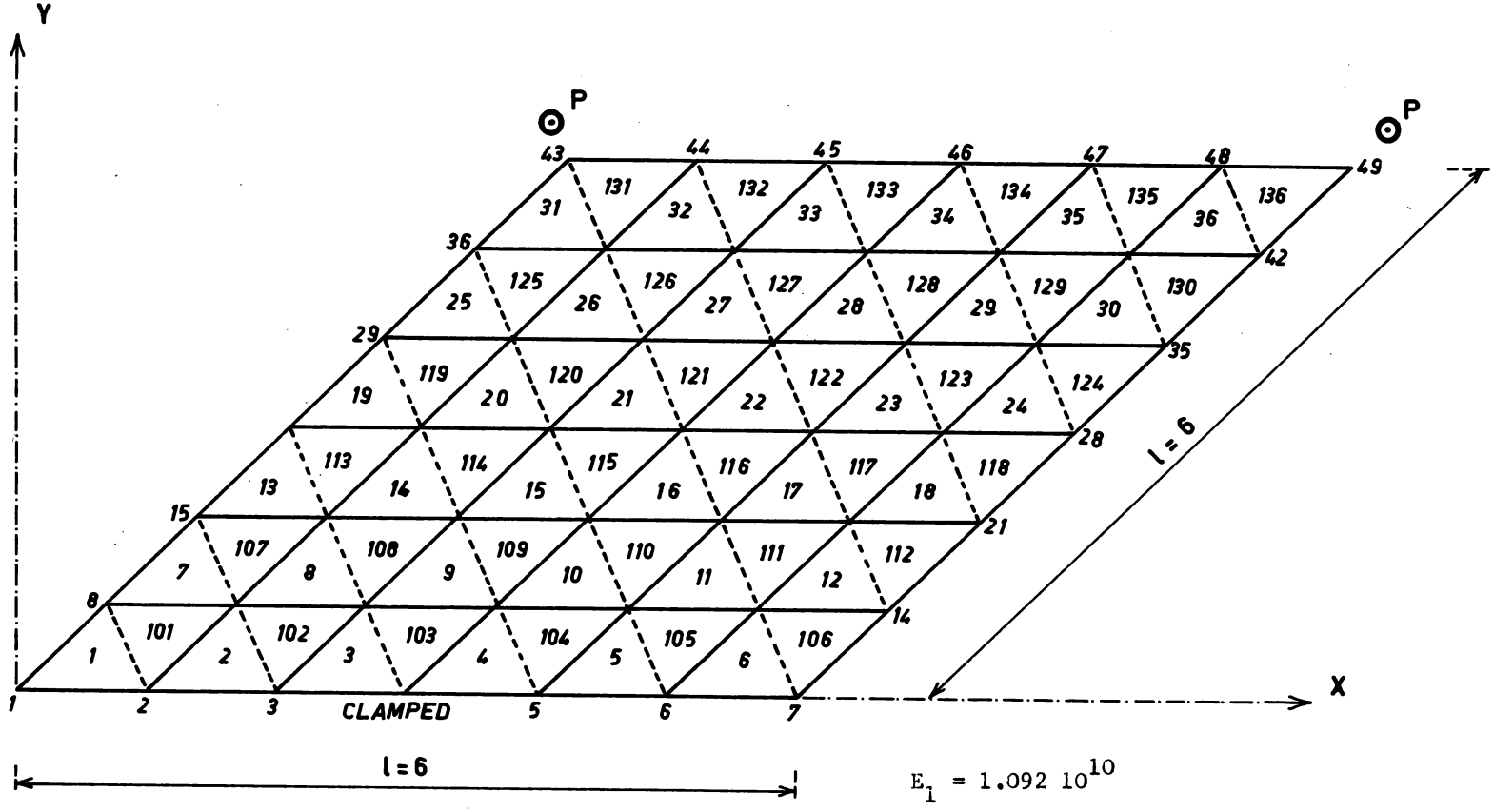


FIGURE A 9



TEST CASE NBR 1 : CANTILEVER U BEAM

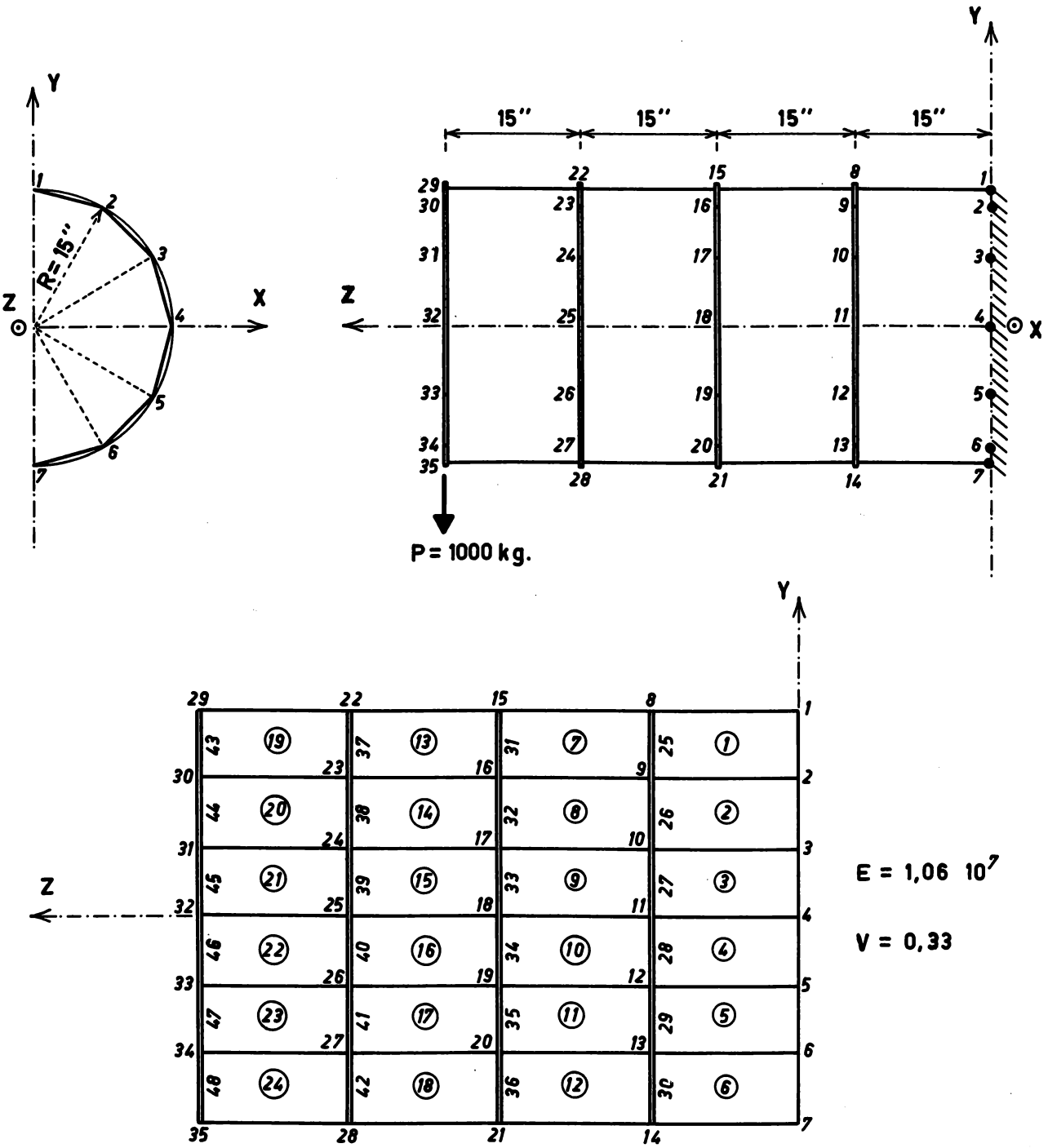
Figure A10



TEST CASE NBR 2 : CANTILEVER SKEW PLATE

$E_1 = 1.092\ 10^{10}$
$\nu = 0.3$
$p = 10^3$
$E_2 = 10^8$
$t = 0.1$
$P = 10^4$

Figure A11



TEST CASE NBR 3 : REINFORCED CYLINDRICAL SHELL

Figure A12

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Security Classification

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
University of Liege Liege, Belgium	Unclassified
	2b. GROUP

3. REPORT TITLE

"Dual Analysis by Finite Elements Linear and Non Linear Applications"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Scientific Report October 1968 - September 1970

5. AUTHOR(S) (Last name, first name, initial)

B. Fraeijs de Veubeke, G. Sander, P. Beckers

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
Dec 1972	287	

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
F61052-69-C-0004	
b. PROJECT NO.	SA-22
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	AFFDL-TR-72-93

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Air Force Flight Dynamics Laboratory Wright-Patterson AFB, OH 45433

13. ABSTRACT

This document presents the results obtained during the aforementioned period concering the linear and non linear applications of the dual analysis technique. New formulation for finite element derivation are proposed including equilibrium, conforming and hybrid elements. A new method of solution of the structural problem is based on the stress function formulation. The slab analogies are used to discover additional finite element families.

A useful complementary energy principle is derived for large deformations. The stability criterion is obtained from the second variation of the total energy.

A non linear shell theory is presented which avoids the difficulties of defining the constitutive equations.

A computer program for linear applications is documented.

DD FORM 1 JAN 64 1473

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14. KEY WORDS		LINK A		LINK B		LINK C	
		ROLE	WT	ROLE	WT	ROLE	WT
Dual Analysis of Structure	Shell Theory						
Finite Elements	Computer Program ASEF						
Stress Function	Energy Methods						
Force Method	Variational Principles						
Combined Method							
Numerical Analysis							
Large Deformations							
Stability							
Shell Theory							
Computer Program (ASEF)							

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