STABILITY ANALYSIS BY FINITE ELEMENTS

by

DR. A. P. KABAILA

University of Sydney, Australia

and

B. FRAEIJS de VEUBEKE

Professor, Universities of Liege and Louvain

Laboratoire de Techiques Aeronautiques et Spatiales, Universite de Liege 75, Rue du Val-Benoit, Liege, Belgium

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FOREWORD

This report was prepared by the Aeronautics and Space Laboratory, University of Liege, Belgium, under Contract F61052-69-C-0004, Project No. 1467, "Structural Analysis Methods", Task No. 146705, "Automatic Computer Methods of Analysis for Flight Vehicle Structures". The work was administered under the direction of the Air Force Flight Dynamics Laboratory by Mr. James R. Johnson, Project Engineer, and through the European Office of Aerospace Research (OAR), United States Air Force. Lt. Colonel Richard T. Boverie is the Project Officer for the European Office of Aerospace Research.

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This report has been reviewed and is approved.

FRANCIS J. JANIK, JR.
Chief, Solid Mechanics Branch
Structures Division

ABSTRACT

The application of finite elements to the analysis of structural stability problems is examined. A variational criterion for stability, namely the criterion that for stable equilibrium the second variation of the total energy must be positive definite, is used to develop a quadrilateral plate element, as well as an element for a prismatic member. The theory is presented in such a form that other elements can be derived therefrom with ease.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS AND SYMBOLS

Superscript (o) indicates energy, stress or strain components which are independent of ε

Superscript (1) indicates energy, stress or strain components with a linear term of ε

Superscript (2) indicates energy, stress or strain components with a quadratic term of ε, i.e. ε^2

A = sectional area

a_i = coordinates of a point before deformation (tensor notation)

B = matrix, defined in the text

b = vector, defined in the text

$C_{mn\,pq}$ = material constant (tensor notation)

D_m = differential operator, $\frac{\partial}{\partial a_m}$

D = flexural rigidity of a plate

E = Young's modulus

H = matrix, defined in the text

I = moment of inertia (second moment of area)

K = stiffness matrix

N = axial force

P = parameter which defines the applied loads

P(u) = potential energy

q = generalized displacements, a vector

R = domain of integration

S = stability matrix, or geometric stiffness matrix

T = matrix connecting α's to q

U = strain energy

(U + P) = total energy

$\delta(U + P)$ = first variation of the total energy

$\delta^2(U + P)$ = second variation of the total energy

$\Delta(U + P)$ = complete increment of the total energy

u_i = displacements (tensor notation)

$\hat{u}_m$ = perturbation of displacements (tensor notation)

u, v, w = displacement perturbations in x, y and z directions (scalar notation)

$\bar{x}$, $\bar{y}$ = orthogonal coordinate axes

x, y = oblique coordinate axes

z = coordinate axis, orthogonal to the $x\,y$ plane

W = strain energy per unit volume

α_{mn} = strains (tensor notation)

α_1, α_2 = constants, defined in the text

σ_{ij} = stresses (tensor notation)

σ_{xx}, σ_{yy} = normal stress (scalar notation)

τ_{xy} = shear stress

ε = arbitrarily small scalar quantity

γ = matrix, defined in the text

Summary.

A variational criterion of stability, namely the requirement that for stable equilibrium the second variation of the total energy, $\delta^2(U + P)$, must be positive definite, is examined. As a simple example the critical load of a prismatic column is calculated with a finite element method. A quadrilateral element is presented for the stability analysis of plates subjected to membrane stresses. This element is based on a complete cubic deflection field, which satisfies continuity of deflections and slopes at all interfaces.

I. Introduction.

One of the variational statements of equilibrium can be written as

$$\delta(U + P) = 0 \qquad \text{(I)}$$

where U is the strain energy and P is the potential energy. $\delta(U + P)$ is the first variation of the total energy $(U + P)$. Equation (I) is a first order approximation of the change of the total energy, $\Delta(U + P)$, resulting from a small disturbance from the equilibrium configuration. It does not yield any information about the stability of the structure. To examine the stability, $\Delta(U + P)$ must be expanded to the second order terms :

$$\Delta(U + P) = \varepsilon\ \delta(U + P) + \varepsilon^2\ \delta^2(U + P) + \ldots, \ldots \qquad (2)$$

where ε is an arbitrarily small quantity.

If the configuration of deformation is such that the structure is in equilibrium then $\delta(U + P) = 0$, and if the equilibrium is a stable one, then

$$\delta^2(U + P) > 0 \tag{3}$$

since additional energy is required in order to produce any kinematically admissible disturbance of the structure. Conversely, if for some kinematically admissible disturbance $\delta^2(U + P) < 0$, then energy is released during this disturbance, and the equilibrium configuration is an unstable one.

If an equilibrium configuration is a neutral one, then

$$\delta^2(U + P) = 0 \tag{4}$$

for at least one mode of kinematically admissible disturbance.

The stability criterion of eq. (3) is valid for both the bifurcation and the snap-through phenomena [6] . This generality is one of the advantages of this variational criterion.

In this paper an application of the finite element method to the bifurcation problems is examined. Stability requirement of eq. (3) is used as the basis of the method.

No attempt is made to survey existing literature. Not only is the literature on elastic stability very extensive, but there is no lack of recent reviews, e.g. ref. [5] . The authors believe that both the method of solution and the development of a compatible quadrilateral element for the stability analysis are new.

2. Derivation of an Expression for the Second Variation of Total Energy.

A linear elastic material is considered. Let the coordinates of a point before deformation be denoted by a_i , the coordinates after deformation by x_i , and the displacement components by u_i , where index $i = 1, 2, 3$, and where it refers to three orthogonal directions. Then

$$x_i = a_i + u_i(a_j) , \tag{5}$$

Let α_{mn} be an element of Green's strain tensor.
It can be shown [3] that

$$2\,\alpha_{mn} = D_m u_n + D_n u_m + D_m u_i D_n u_i , \tag{6}$$

where D_m is a differential operator $\frac{\partial}{\partial a_m}$.

The strain energy per unit volume can be written as

$$W = \alpha_{mn}\,\alpha_{pq}\,C_{mn\,pq} \tag{7}$$

where $C_{mn\,pq}$ is a material constant which relates strains to stresses and which satisfies the following conditions of symmetry :

$$C_{mn\,pq} = C_{nm\,pq} = C_{mn\,qp} = C_{nm\,qp} = C_{pq\,nm} = C_{pq\,mn} = C_{qp\,mn} =$$
$$= C_{pq\,nm} = C_{qp\,nm} ; \tag{8}$$

The stresses can be expressed in terms of strain energy as follows :

$$\sigma_{ij} = \frac{\partial W}{\partial \alpha_{ij}} = 2\,C_{ij\,pq}\,\alpha_{pq} , \tag{9}$$

Equation (6) is valid for large displacements as its derivation does not require an assumption of magnitude of displacements or strains. At this point we postulate that, whilst the displacements can be large, the strains and stresses are small, so that equation (9) is the usual engineering equation between the stresses and the strains.

An increment of u_m from an initial value u_m^o to

$$u_m = u_m^o + \varepsilon\, \hat{u}_m \qquad (I0)$$

is considered. Here $\hat{u}_m$ is an arbitrary disturbance, whilst ε is an arbitrarily small constant.

As the second variation of the total energy, (U + P) , is required, the strain energy, U , and the potential energy, P(u) , will be expanded to the second order terms of ε . The expansion for strains can be written as

$$\alpha_{mn} = \alpha_{mn}^{(o)} + \varepsilon\ \alpha_{mn}^{(1)} + \frac{1}{2}\varepsilon^2\ \alpha_{mn}^{(2)} \qquad (II)$$

Substitution of (II) into (7) yields

$$W = W^{(o)} + \varepsilon\ (\alpha_{mn}^{(o)} C_{mn\ pq}\ \alpha_{pq}^{(1)} + \alpha_{mn}^{(1)} C_{mn\ pq}\ \alpha_{pq}^{(o)})$$
$$+ \frac{1}{2}\varepsilon^2\ (\alpha_{mn}^{(2)} C_{mn\ pq}\ \alpha_{pq}^{(o)} + \alpha_{mn}^{(o)} C_{mn\ pq}\ \alpha_{pq}^{(2)} + 2\ \alpha_{mn}^{(1)} C_{mn\ pq}\ \alpha_{pq}^{(1)}) + \dots ; \qquad (I2)$$

Here $W^{(o)}$ denotes $\alpha_{mn}^{(o)} \alpha_{pq}^{(o)}\ C_{mn\ pq}$, the strain energy in terms of strains before the disturbance. Let the corresponding stresses be denoted by $\sigma_{mn}^{(o)}$, and let $W^{(1)}$ denote the strain energy in terms of $\alpha_{mn}^{(1)}$.
With this notation and with substitution of equations (9) and (7), equation (I2) can be written as

$$W = W^{(o)} + \varepsilon\ \sigma_{mn}^{(o)}\ \alpha_{mn}^{(1)} + \varepsilon^2 (W^{(1)} + \frac{1}{2}\sigma_{mn}^{(o)}\ \alpha_{mn}^{(2)}) + \dots \qquad (I3)$$

With equation (I3), equations for $\delta(U + P)$ and $\delta^2(U + P)$ can be written immediately as follows :

$$\delta(U + P) = \int_R \sigma_{mn}^{(o)}\ \alpha_{mn}^{(1)}\ dR - P(\hat{u}_m) = 0\ ; \dots \qquad (I4)$$

for every perturbation $\hat{u}_m$ compatible with kinematical boundary conditions, provided that the potential energy is a linear function of u_m . This is the classical large displacement condition of equilibrium. Similarly,

$$\delta^2(U + P) = \int_R \left(W^{(1)} + \frac{1}{2}\sigma_{mn}^{(o)}\ \alpha_{mn}^{(2)} \right) dR > 0\ ; \qquad (I5)$$

for every perturbation $\hat{u}_m$ compatible with kinematical boundary conditions. This is the condition of stability.

In bifurcation problems, equation (I5) can often be interpreted in terms of classical expressions for strain energy.

For instance, consider buckling of a plate, which lies in plane 1-2 and is subjected to loads which, in a stable equilibrium state, produce membrane stresses only. One can postulate a priori that :

(1) All displacement gradients are $<< 1$.

(2) Displacements and their gradients in the plane 1-2, i.e. $D_i\ u_1$ and $D_j\ u_2$, are of the same order of magnitude.

(3) Displacements and their gradients associated with bending deformations, i.e. $D_k\ u_3$, may be of a larger or the same order of magnitude as $D_i\ u_1$ and $D_j\ u_2$.

With these postulations, which are justified by physical conditions, one can write

$$\alpha_{11} = D_1\ u_1 + \frac{1}{2}(D_1\ u_3)^2$$

$$\alpha_{22} = D_2\ u_2 + \frac{1}{2}(D_2\ u_3)^2 \qquad \text{(I6)}$$

$$2\ \alpha_{12} = D_1\ u_2 + D_2\ u_1 + D_1\ u_3\ D_2\ u_3$$

$$\alpha_{33} = D_3\ u_3$$

$$2\ \alpha_{13} = D_1\ u_3 + D_3\ u_1 \qquad \text{(I7)}$$

$$2\ \alpha_{23} = D_2\ u_3 + D_3\ u_2$$

It is known a priori that there are equilibrium states with $u_3 \equiv 0$. Substitution of eq. (I0) into (I6) and (I7), and collection of terms of first order in ε , yields

$$\alpha^{(1)}_{11} = D_1\ \hat{u}_1$$

$$\alpha^{(1)}_{22} = D_2\ \hat{u}_2 \qquad \text{(I8)}$$

$$2\ \alpha^{(1)}_{12} = D_1\ \hat{u}_2 + D_2\ \hat{u}_1$$

$$\alpha_{33}^{(1)} = D_3\, \hat{u}_3$$

$$2\, \alpha_{13}^{(1)} = D_1\, \hat{u}_3 + D_3\, \hat{u}_1 \qquad (19)$$

$$2\, \alpha_{23}^{(1)} = D_2\, \hat{u}_3 + D_3\, \hat{u}_2$$

It is seen that equations (18) and (19) have the same form in terms of $\hat{u}_i$, as the linearized small displacement equations in terms of u_i. Similarly, the terms of second order in ε, are

$$\alpha_{11}^{(2)} = (D_1\, \hat{u}_3)^2$$

$$\alpha_{22}^{(2)} = (D_2\, \hat{u}_3)^2 \qquad (20)$$

$$2\, \alpha_{12}^{(2)} = D_1\, \hat{u}_3\, D_2\, \hat{u}_3$$

$$\alpha_{33}^{(2)} = \alpha_{23}^{(2)} = \alpha_{13}^{(2)} = 0$$

Implying substitution of eq. (18), we can write the first variation of $(U + P)$ of eq. (14) in full, as follows :

$$\int_R \sigma_{mn}^{(o)}\, \alpha_{mn}^{(1)}\, dR - \int_R X_i\, u_i^{(1)}\, dR - \int_{\partial R} p_i\, u_i^{(1)}\, d\partial R = 0 \quad ; \qquad (21)$$

where X_i are body forces and p_i are surface tractions over ∂R. This is the classical linearized form of the condition of equilibrium configuration in the 1-2 plane. Similarly, implying substitution of (18) and (20) into (15), we can write the stability condition as

$$\int_R W^{(1)}\, dR + \frac{1}{2} \int_R \sigma_{mn}^{(o)}\, \alpha_{mn}^{(2)}\, dR > 0 \quad ; \qquad (22)$$

With equation (18), the first integral of equation (22) has the classical linearized form for strain energy, with displacements u_i replaced by the perturbations $\hat{u}_i$.
The second integral contains only the stresses in the plane 1-2, associated with the transverse displacements.
When the stress resultants are used instead of the stresses, equations (21)

and (22) can be applied directly in some other notation replacing the strains by the generalized displacements conjugate to the stress resultants.

3. Application to a closed form solution.

As a demonstration of stability criterion of eq. (22), the critical load of a cantilever column of a constant section (ref. Fig. I) will be determined analytically. For the investigation of a straight equilibrium state, eq. (22) yields immediately

$$\frac{1}{2}\int_0^\ell \left(EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{dv^2}{dx^2}\right)^2\right)dx + \frac{1}{2}\int_0^\ell N\left(\frac{dv}{dx}\right)^2 dx > 0 \quad ; \qquad (23)$$

where u and v are the displacement perturbations and where N is the axial force in the strut in the straight equilibrium configuration, which in the present example is constant.

Since the disturbances u and v are not coupled and since disturbance u always yields a positive contribution to $\delta^2(U + P)$, one can put $u \equiv 0$. Thus equation (23) is simplified to

$$\frac{1}{2}\int_0^\ell EI\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^\ell N\left(\frac{dv}{dx}\right)^2 dx > 0 \quad ; \qquad (24)$$

Evidently the potential energy, $P(u)$, does not appear directly in the equation of $\delta^2(U + P)$, eq. (24). The potential energy is accounted for by the second term of equation (24).

When the critical load is sought by considerations of equilibrium, with the assumption that a bent equilibrium state of the column is a state of neutral equilibrium, an assumption which is based on engineering intuition and experience, an engineering linearized strain equations lead to an equation of a similar form to eq. (24). In such intuitive approach the axial deformations are ignored, and, with disturbance v replaced by actual displacement $\bar{v}$, the first integral (in terms of $\bar{v}$) is obtained from the strain energy. The second integral (in terms of $\bar{v}$) is obtained as the column shortening. With $N = -P$, we seek the least value of P which satisfies equation (24). Since eq. (24) is homogeneous, a norm

$\int_0^\ell \left(\frac{dv}{dx}\right)^2 dx = 1$, is introduced. Thus, eq. (24) becomes

$$-P\int_0^\ell \left(\frac{dv}{dx}\right)^2 dx + EI\int_0^\ell \left(\frac{d^2v}{dx^2}\right)^2 dx + \lambda\left(\int_0^\ell \left(\frac{dv}{dx}\right)^2 dx - 1\right) =$$

$$= K^2\int_0^\ell \left(\frac{dv}{dx}\right)^2 dx + \int_0^\ell \left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{\lambda}{EI} > 0 \quad ; \qquad (25)$$

where λ is a lagrangian multiplier and where $K^2 = (\lambda - P)/EI$. The smallest value of P is obtained when the augmented functional is a minimum. Variation δv yields :

$$K^2 \int_o^\ell v \frac{d}{dx} \delta v \, dx + \int_o^\ell \frac{d^2v}{dx^2} \frac{d^2}{dx^2} \delta v \, dx = 0 \quad ; \tag{26}$$

Integration of eq. (26) by parts yields the governing differential equation

$$K^2 \frac{d^2v}{dx^2} + \frac{d^3v}{dx^3} = 0 \quad ; \tag{27}$$

together with the boundary conditions

$$\begin{aligned}
&\text{(i)} && v = 0 \quad \text{at} \quad x = 0 \\
&\text{(ii)} && \frac{dv}{dx} = 0 \quad \text{at} \quad x = 0 \; . \\
&\text{(iii)} && K \frac{dv}{dx} - \frac{d^3v}{dx^3} = 0 \quad \text{at} \quad x = \ell \\
&\text{(iv)} && \frac{d^2v}{dx^2} = 0 \quad \text{at} \quad x = \ell
\end{aligned} \tag{28}$$

The general solution of eq. (27) is

$$v = A \sin Kx + B \cos Kx + Cx + D \quad ; \tag{29}$$

Substitution of boundary conditions (i), (ii) and (iii) into eq. (29) yields

$$v = B(\cos Kx - 1) \quad ; \tag{30}$$

From boundary condition (iv) one finds that either $B = 0$, in which case $v \equiv 0$ and the solution is trivial, or

$$K\ell = \frac{2\,n + 1}{2} \pi \quad , \qquad n = 0, 1, 2 \tag{3I}$$

or

$$K_n = \frac{2\,n + 1}{2\,\ell} \pi \quad , \qquad n = 0, 1, 2 \tag{32}$$

Hence,

$$v = B_n(\cos K_n\, x - 1) \qquad (33)$$

B_n can be determined by substitution of equation (33) into the adopted norm, which yields

$$1 = \int_0^{\ell} \left(\frac{dv}{dx}\right)^2 dx = B_n^2\, K_n^2 \int_0^{\ell} \sin^2 K_n\, x\, dx = B_n^2\, K_n^2\, \frac{\ell}{2} \quad ; \qquad (34)$$

Substitution of (33) and (34) into eq. (24) yields

$$-\frac{1}{2}\,\frac{P}{EI}\,\frac{\ell}{2} + \frac{K_n^2}{\ell} \int_0^{\ell} \cos^2 K_n\, x\, dx = -\frac{P}{EI} + K_n^2 > 0 \quad ; \qquad (35)$$

It is seen from eq. (35) that for $\delta^2(U + P)$ to be a minimum, K_n^2 must be a minimum. Hence the lowest critical load is found as $P_c = \pi^2\ EI/4\ \ell^2$.

4. Finite element for column.

The stability criterion, $\delta^2(U + P) > 0$ can readily be used in the finite element method. To demonstrate this, a finite element for a column will be developed.

It is postulated that the axial loads are applied at the junction of two elements. Let the applied loads be expressed in terms of a parameter P , and let $n = N/P$. Then equation (24) can be written as

$$\frac{EI}{2} \int_0^{\ell} \left(\frac{d^2v}{dx^2}\right)^2 dx + P\,\frac{n}{2} \int_0^{\ell} \left(\frac{dv}{dx}\right)^2 dx > 0 \quad ; \qquad (36)$$

Assume

$$v = \alpha_1 + \alpha_2\, x + \alpha_3\, x^2 + \alpha_4\, x^3 \quad ; \qquad (37)$$

Let $\qquad \alpha = (\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\ )^T$

Hence $\qquad \dfrac{dv}{dx} = b^T\alpha = (\ 0 \quad 1 \quad 2\,x \quad 3\,x^2\ )\ \alpha \quad ; \qquad (38)$

$$n \int_0^{\ell} \left(\frac{dv}{dx}\right)^2 dx = n\ \ \alpha^T \left\{\int_0^{\ell} b\, b^T\ dx\right\} \alpha \quad ; \qquad (39)$$

Let generalized displacements, q , be equal to

$$q = \left(v_1 \quad (\frac{dv}{dx})_1 \quad v_2 \quad (\frac{dv}{dx})_2 \right)^T \quad ; \tag{40}$$

From (37),

$$q = T^{-1} \alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \alpha \quad ; \tag{4I}$$

Substitution of (4I) into (39) yields

$$P \frac{1}{2} q^T S q = P \frac{1}{2} q^T n T^T \left\{ \int_0^\ell b b^T dx \right\} T q \quad ; \tag{42}$$

Here, the "stability matrix", S , is found after integration to be equal to :

$$S = n T^T \left\{ \int_0^\ell b b^T dx \right\} T = \frac{n}{30 \ell} \begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ & 4\ell^2 & -3 & -\ell^2 \\ & & 36 & -3\ell \\ & (SYM) & & 4\ell^2 \end{bmatrix} \tag{43}$$

Similarly, the first integral of equation (3I) yields the usual form of stiffness matrix :

$$\frac{1}{2} EI \int_0^\ell \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{1}{2} q^T K q \qquad \text{with}$$

$$K = \frac{2 EI}{\ell^3} \begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ & 2\ell^2 & -3\ell & \ell^2 \\ & & 6 & -3\ell \\ & (SYM) & & 2\ell^2 \end{bmatrix} \quad ; \tag{44}$$

The gross stiffness of the whole structure, K_G , is assembled in the usual way. The gross stability matrix, S_G , is assembled in an identical way. Thus, in terms of the generalized displacements of the whole structure, the stability condition, $\delta^2(U + P) > 0$, can be written as follows :

$$\frac{1}{2} q_G^T K_G q_G + P \frac{1}{2} q_G^T S_G q_G > 0 \quad ; \tag{45}$$

To find the minimum value of P which satisfies equation (45), a variation δq_G in equation (45) is considered and the resulting equation is set to zero. Hence,

$$(K_G + P S_G) q_G = 0 \quad ; \tag{46}$$

whence the lowest eigenvalue of P can be found by iteration.
The expression for the stability matrix, S , of eq. (43) is the same as that reported by Martin [7] , though the derivation is different.
When the member coordinate axes do not coincide with the structure coordinate axes, the axes transformation for the stability matrix, S , is the same as that usually employed for the stiffness matrix, K . Extension for the stability of three dimensional structures made up of prismatic axially loaded members is easily achieved.
The convergence of the numerically computed critical load towards the exact solution is remarkably good, as indicated by the following table, which shows the results of a calculation of the critical load of a cantilever column :

Number of elements	Number of generalized displacements	Degrees of freedom	Computed critical load
1	4	2	$1.0075 \frac{\pi^2 EI}{\ell^2}$
2	6	4	$1.0005 \frac{\pi^2 EI}{\ell^2}$
3	8	6	$1.0001 \frac{\pi^2 EI}{\ell^2}$

5. Buckling of Plates.

A plate subjected to membrane stresses $\sigma^{(o)}_{\bar{x}\bar{x}}$, $\tau^{(o)}_{\bar{x}\bar{y}}$, $\sigma^{(o)}_{\bar{y}\bar{y}}$ is considered. The plate lies in the plane of orthogonal axes $\bar{x}$ and $\bar{y}$. In this section the orthogonal axes will be denoted by $\bar{x}$, $\bar{y}$ and $\bar{z}$, whilst the oblique axes will be denoted by x , y and z . The displacements in $\bar{x}$, $\bar{y}$ and $\bar{z}$ directions are denoted by u , v and w respectively, so that a straight equilibrium state is defined by $w = 0$.

Equation (22) with Kirchhoff's theory of plate bending yields the following stability criterion

$$\int_R W^{(1)}\, dR + \frac{t}{2} \iint \left(\sigma^{(o)}_{\bar{x}\bar{x}} \left(\frac{\partial w}{\partial \bar{x}}\right)^2 + 2\, \tau^{(o)}_{\bar{x}\bar{y}} \left(\frac{\partial w}{\partial \bar{x}} \frac{\partial w}{\partial \bar{y}}\right) + \sigma^{(o)}_{\bar{y}\bar{y}} \left(\frac{\partial w}{\partial \bar{y}}\right)^2 \right) d\bar{x}d\bar{y} > 0 \tag{47}$$

It is evident that the first integral of equation (47) yields, in view of eq. (18), the usual stiffness matrix.

The second integral yields the stability matrix S . As in prismatic members (section 3), this integral is similar in form to the usual expression for potential energy [8] , and has been used for the development of a rectangular non-conforming element [4] for stability analysis.

It will be noted that in the second integral of eq. (47) the shear stresses are multiplied by 2, in order to allow for both $\tau^{(o)}_{\bar{x}\bar{y}}$ and $\tau^{(o)}_{\bar{y}\bar{x}}$. Equation (47) is used here to find the stability matrix of a quadrilateral element, with the use of a conforming displacement field. The stiffness matrix for such an element has been developed and reported in literature [1] . Operations which have been described in detail in ref. (1) will only be mentioned briefly in this paper.

We consider oblique coordinate axes x, y , shown in figure 3. It can be shown that the expression for the stability matrix, S , is the same in the skew coordinate axes as that in the orthogonal coordinate axes, provided that the membrane stresses of the orthogonal reference axes are replaced by their equivalents in the oblique axes. Thus, in terms of the generalized displacements, q , one can write in the oblique axes x , y ,

$$\frac{1}{2} q^T S q = \frac{1}{2}\frac{t}{P} \iint \left(\frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y}\right) \begin{vmatrix} \sigma^{(o)}_{xx} & \tau^{(o)}_{xy} \\ \tau^{(o)}_{xy} & \sigma^{(o)}_{yy} \end{vmatrix} \begin{vmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{vmatrix} dxdy \ ; \tag{48}$$

where

$$\sigma_{xx}^{(o)} = \sigma_{\overline{xx}}^{(o)} \sin\alpha - 2\,\tau_{\overline{xy}}^{(o)} \cos\alpha + \sigma_{\overline{yy}}^{(o)} \frac{\cos^2\alpha}{\sin\alpha} \;;$$

$$\tau_{xy}^{(o)} = \tau_{\overline{xy}}^{(o)} - \sigma_{\overline{yy}}^{(o)} \cot\alpha \qquad (49)$$

$$\sigma_{yy}^{(o)} = \sigma_{\overline{yy}} \frac{1}{\sin\alpha}$$

In the derivation of stability matrix, uniform membrane stresses in each of the four triangles of a quadrilateral (figure 4) will be considered. Modification for a varying stress field can be readily accomplished at the cost of more complicated expressions. A uniform stress field is consistent with finite element stress analysis when simple triangular elements (2) are used. Let

$$\gamma = \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \;; \qquad H = \begin{bmatrix} h_1 & h_3 \\ h_3 & h_2 \end{bmatrix} = \begin{bmatrix} \sigma_{xx}^{(o)} & \tau_{xy}^{(o)} \\ \tau_{xy}^{(o)} & \sigma_{yy}^{(o)} \end{bmatrix} \frac{t}{P} \qquad (50)$$

With this notation eq. (48) can be written as

$$\frac{1}{2} q^T S\, q = \frac{1}{2} \iint \gamma^T H \gamma \, dxdy \;; \qquad (51)$$

The following complete cubic deflection field is considered in the first triangle of figure 4 :

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + 2\,\alpha_5 xy + \alpha_6 y^2$$

$$+ 4(\alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3) \;; \qquad (52)$$

Hence,

$$\gamma = B\,\alpha = \begin{bmatrix} 0 & 1 & 0 & 2x & 2y & 0 & 12x^2 & 8xy & 4y^2 & 0 \\ 0 & 0 & 1 & 0 & 2x & 2y & 0 & 4x^2 & 8xy & 12y^2 \end{bmatrix} \alpha \qquad (53)$$

where $\alpha^T = (\alpha_1\ \alpha_2\ ..\ \alpha_{10})$; In the first triangle the coefficients α

can be expressed in terms of the generalized displacements, q_1 , as follows

$$\alpha = T_1 \; q_1 \tag{54}$$

where T_1 matrix is taken from reference 1 and shown in table 1, and where

$$q_1^T = (\; w_o \; \phi_o \; \psi_o \; w_1 \; \phi_1 \; \psi_1 \; w_2 \; \phi_2 \; \psi_2 \; \phi_{12} \;) \tag{55}$$

Equation (5I) thus becomes, for the first triangle

$$\frac{1}{2} q_1^T \; S \; q_1 = \frac{1}{2} q_1^T \; T_1^T \; \overline{S}_1 \; T_1 \; q_1 \tag{56}$$

where

$$\overline{S}_1 = \iint \; B_1^T \; H_1 \; B_1 \quad dxdy \tag{57}$$

with integration over the first triangle. The subscript 1 indicates that the expressions are valid in the first triangle only. The product $B_1^T H_1 B_1$ is formed algebraically and the integration is performed analytically. The resulting matrix $\overline{S}_1$ is shown in table 2. To obtain S_1 , premultiplication by T_1^T and postmultiplication by T_1 is performed numerically. For the second triangle, as shown in reference (1), continuity of deformation along the interface $x = 0$ is maintained if α_4 , α_7 and α_8 are replaced by independent coefficients α_4' , α_7' and α_8' , whilst the other α coefficients are unaltered. With

$$q_2^T = (\; w_o \; \phi_o \; \psi_o \; w_3 \; \phi_3 \; \psi_3 \; w_2 \; \phi_2 \; \psi_2 \; \phi_{23} \;) \tag{58}$$

T_2 and $\overline{S}_2$ can be obtained by changing a into $- c$ in the expressions for T_1 and $\overline{S}_1$ respectively, and by changing signs of all elements of $\overline{S}_2$. Similarly, in triangle 3, the new coefficients are α_6' , α_9' and α_{10}' . With

$$q_3^T = (\; w_o \; \phi_o \; \psi_o \; w_1 \; \phi_1 \; \psi_1 \; w_4 \; \phi_4 \; \psi_4 \; \phi_{41} \;) \tag{59}$$

T_3 and $\overline{S}_3$ are obtained by changing b into $- d$ in the expressions for T_1 and $\overline{S}_1$ respectively, and by changing signs of all elements of $\overline{S}_3$.

In triangle 4, $\alpha_4^T = (\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4' \; \alpha_5 \; \alpha_6' \; \alpha_7' \; \alpha_8' \; \alpha_9' \; \alpha_{10}')$
With

$$q_4^T = (w_o \; \phi_o \; \psi_o \; w_3 \; \phi_3 \; \psi_3 \; w_4 \; \phi_4 \; \psi_4 \; \phi_{34}) \tag{60}$$

T_4 and $\overline{S}_4$ are computed from expressions for T_1 and $\overline{S}_1$, changing a into $-b$ and b into $-d$.
In each triangle different values of matrix H are used, depending on the membrane stresses.
The assembly of the partial stability matrices S_i into element stability matrix before condensation, S^*, which is in terms of

$$p^T = (w_o \; \phi_o \; \psi_o \; w_1 \; \phi_1 \; \psi_1 \; w_2 \; \phi_2 \; \psi_2 \; w_3 \; \phi_3 \; \psi_3 \; w_4 \; \phi_4 \; \psi_4 \; \phi_{12} \; \phi_{23} \; \phi_{34} \; \phi_{41}) \tag{6I}$$

is achieved by the addition

$$\sum_1^4 q_i^T \, S_i \, q_i = p^T \, S^* \, p \tag{62}$$

where the elements of S_i are addressed into their proper place in S^*.
The condensation of S^* (19 x I9) into S (16 x 16) by elimination of w_o, ϕ_o and ψ_o, as well as the slope transformation from the local to the global axes, is identical to the same operations on the stiffness matrix. It is described in detail in reference (1), sections 4 and 5.
With the element stiffness matrix, K, and the element stability matrix, S, known, gross stiffness matrix K_G is assembled, and, in the same manner, gross stability matrix S_G is also assembled. Thus, eq. (46) is obtained, whence the critical load intensity is determined.

6. Examples of Plate Buckling.

To check the accuracy of the critical load computed with the quadrilateral element developed in this paper, several examples, with known analytical solutions, were evaluated. For comparison, the results obtained by Carson and Newton [9], who developed and used a rectangular element, are also shown. For a simply supported rectangular plate under uniaxial compression the analytical solution yields $k = 4.0$, where

$$k = N \, b^2/\pi^2 D \tag{63}$$

where N is the edge loading per unit length, b is the side dimension of the plate and D is the flexural rigidity of the plate. The following values of k were computed numerically :

Simply supported square plate under uniaxial compression.

Grid size	Rectangular element of Carson et Newton [9]		Quadrilateral element	
	k	error %	k	error %
2 × 2	4.01575	0.39	4.02964	0.741
4 × 4	4.00104	0.03	4.00224	0.056

The agreement with theoretical values is good for both the rectangular [9] and the quadrilateral elements.

Under a uniform edge shear the following results are obtained :

Uniform edge shear, simply supported quadratic plate, analytical

$k = 9.34$

Grid size	Rectangular element [9]		Quadrilateral element	
	k	error %	k	error %
2 × 2	10.016	7.2	9.6360	3.17
3 × 3	9.577	2.5	9.5073	1.79

Uniform edge shear, clamped edges, quadratic plate, analytical

$k = 14.71$

Grid size	Rectangular element [9]		Quadrilateral element	
	k	error %	k	error %
2 × 2	23.264	58.2	22.954	56.0
3 × 3	16.046	9.1	15.516	5.5

The agreement with the theoretical values is better for the quadrilateral element than for the rectangular element. A possible explanation for this is that the rectangular element is based on the incomplete polynomial of the type [9]

$$w = \sum_{i=o}^{3} \sum_{j=o}^{3} a_{ij} x^i y^j \tag{64}$$

The complete polynomial used for the derivation of the quadrilateral element may be expected to better represent twisting of a section.
The quadrilateral element presented here is fully compatible, therefore the upper bounds of the critical load are obtained.

7. Conclusion.

Stability matrices presented in this paper enables one to determine critical loads of a variety of problems. Quadrilateral element is particularly suitable when the boundaries of the plate are of an irregular shape. The convergence of the numerical results in the fact examples investigated is very satisfactory.

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Table 1 : Matrix T_1 which relates coefficients α to generalized displacements q

1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
$-3/a^2$	$-2/a$	0	$3/a^2$	$-1/a$	0	0	0	0	0
$3/ab$	$-1/b$	$-1/a$	$3/ab$	$-1/2\ b$	$1/a$	0	$1/2\ b$	0	$-2/b$
$-3/b^2$	0	$-2/b$	0	0	0	$3/b^2$	0	$-1/b$	0
$1/2\ a^3$	$1/4\ a^2$	0	$-1/2\ a^3$	$1/4\ a^2$	0	0	0	0	0
$3/2\ a^2b$	$1/2\ ab$	$1/4\ a^2$	$-3/2\ a^2b$	$1/4\ ab$	$-1/4\ a^2$	0	$-1/4\ ab$	0	$1/ab$
$3/2\ ab^2$	$1/4\ b^2$	$1/2\ ab$	$-3/2\ ab^2$	$1/4\ b^2$	$-1/2\ ab$	0	0	0	$+1/b^2$
$1/2\ b^3$	0	$1/4\ b^2$	0	0	0	$-1/2\ b^3$	0	$1/4\ b^2$	0

$\frac{h_1}{2} a b$	$\frac{h_3}{2} a b$	$\frac{h_1}{3} a^2 b$	$\frac{h_3}{3} a^2 b + \frac{h_1}{3} a b^2$	$\frac{h_3}{3} a b^2$	$h_1 a^3 b$	$\frac{h_3}{3} a^3 b + \frac{h_1}{3} a^2 b^2$	$\frac{h_3}{3} a^2 b^2 + \frac{h_1}{3} a b^3$	$h_3 a b^3$
	$\frac{h_2}{2} a b$	$\frac{h_3}{3} a^2 b$	$\frac{h_2}{3} a^2 b + \frac{h_3}{3} a b^2$	$\frac{h_2}{3} a b^2$	$h_3 a^3 b$	$\frac{h_2}{3} a^3 b + \frac{h_3}{3} a^2 b^2$	$\frac{h_2}{3} a^2 b^2 + \frac{h_3}{3} a b^3$	$h_2 a b^3$
		$\frac{h_1}{3} a^3 b$	$\frac{h_3}{3} a^3 b + \frac{h_1}{6} a^2 b^2$	$\frac{h_3}{6} a^2 b^2$	$\frac{6}{5} a^4 b h_1$	$\frac{2}{5} h_3 a^4 b + \frac{4}{15} h_1 a^3 b^2$	$\frac{4}{15} h_3 a^3 b^2 + \frac{2}{15} h_1 a^2 b^3$	$\frac{6}{15} h_3 a^2 b^3$
			$\frac{h_2}{3} a^3 b + \frac{h_3}{3} a^2 b^2 + (\frac{h_1}{3}) a b^3$	$\frac{h_3}{3} a b^3 + \frac{h_2}{6} a^2 b^2$	$\frac{2}{5} h_1 a^3 b^2 + \frac{6}{5} h_3 a^4 b$	$\frac{2}{5} h_2 a^4 b + (\frac{2}{5}) h_3 a^3 b^2 + \frac{4}{15} h_1 a^2 b^3$	$\frac{4}{15} h_2 a^3 b^2 + \frac{2}{5} h_3 a^2 b^3 + \frac{2}{5} h_1 a b^4$	$\frac{6}{5} h_3 a b^4 + \frac{2}{5} h_2 a^2 b^3$
SYM.				$\frac{h_2}{3} a b^3$	$\frac{2}{5} h_3 a^3 b^2$	$\frac{2}{15} h_2 a^3 b^2 + \frac{4}{15} h_3 a^2 b^3$	$\frac{4}{15} h_2 a^2 b^3 + \frac{2}{5} h_3 a b^4$	$\frac{6}{5} h_2 a b^4$
					$\frac{24}{5} h_1 a^5 b$	$\frac{8}{5} h_3 a^5 b + \frac{4}{5} h_1 a^4 b^2$	$\frac{4}{5} h_3 a^4 b^2 + \frac{4}{15} h_1 a^3 b^3$	$\frac{12}{15} h_3 a^3 b^3$
						$\frac{8}{15} h_2 a^5 b + \frac{8}{15} h_3 a^4 b^2 + \frac{16}{45} h_1 a^3 b^3$	$\frac{4}{15} h_2 a^4 b^2 + \frac{4}{9} h_3 a^3 b^3 + \frac{4}{15} h_1 a^4 b^2$	$\frac{4}{5} h_3 a^2 b^4 + \frac{4}{15} h_2 a^3 b^3$
							$\frac{16}{45} h_2 a^3 b^3 + \frac{8}{15} h_3 a^2 b^4 + \frac{8}{15} h_1 a b^5$	$\frac{8}{5} h_3 a b^5 + \frac{4}{5} h_2 a^2 b^4$
								$\frac{24}{5} h_2 a b^5$

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13. ABSTRACT

The application of finite elements to the analysis of structural stability problems is examined. A variational criterion for stability, namely the criterion that for stable equilibrium the second variation of the total energy must be positive definite, is used to develop a quadrilateral plate element, as well as an element for a prismatic member. The theory is presented in such a form that other elements can be derived therefrom with ease.

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