THE NUMERICAL INTEGRATION OF LAMINAR BOUNDARY LAYER EQUATIONS

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Abstract—Self-similar solutions of boundary layer equations obey non-linear differential equations, automorphic under certain continuous transformation groups. Changes of variables suggested by the theory of continuous LIE groups may reduce the problem to the integration of a first order non-linear differential equation, followed by quadratures, thereby greatly simplifying computer integration.

The famous Blasius equation, governing the asymptotic laminar boundary layer flow over a semi-infinite plate is presented as a typical example.

1. POSITION OF THE PROBLEM

The problem is that of the two-dimensional steady flow of an incompressible Newtonian fluid of density \( \rho \) along a semi-infinite plate, whose trace is the \([0, \infty)\) segment of the \( x \) axis. At infinity upstream the flow has the uniform velocity \((U, 0)\). Reduced co-ordinates

\[ \xi = R_x = xU/v \quad \eta = yU/v \quad (1) \]

where \( v \) is the kinematic viscosity, combined with the use of \( U \) as the velocity unit and \( \rho U^2 \) as the pressure unit, yield the following Navier–Stokes and volume conservation equations

\[ \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} = - \frac{\partial p}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \quad (2) \]

\[ \frac{\partial v}{\partial \xi} + u \frac{\partial v}{\partial \eta} = - \frac{\partial p}{\partial \eta} + \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} \quad (3) \]

\[ \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0. \quad (4) \]

An asymptotic solution (valid for sufficiently high \( \xi \) values) is found, following Prandtl[1], by neglecting \( \frac{\partial p}{\partial \xi} \) and \( \frac{\partial^2 u}{\partial \xi^2} \) in equation (2).

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The problem is then to solve the system of two equations in the unknowns \((u, v)\) formed by (4) and

\[
\frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2}
\]  

subject to the boundary conditions

\[
\begin{align*}
u &= 0, v = 0 \quad \text{for } \eta = 0 \\
u &= 1 \quad \text{for } \eta = \infty.
\end{align*}
\]

Solving (2') for \(v\) and substituting into (4), furnishes a single partial differential equation for \(u\). For an asymptotic solution of type

\[
u = \lambda g(\beta) \quad \beta = \eta \phi(\xi)
\]

it reduces to the form with separated variables (\(c\) is the separation constant)

\[
\phi^{-3} \frac{d\phi}{d\xi} = (\lambda g)^{-1}(\ddot{g} / \dot{g})' = -c.
\]

This solution is self-similar; that is, the velocity profile \(u\) against the distance \(\eta\) to the plate is only subject to a change of scale when the distance \(\xi\) to the leading edge of the plate is altered. For such a solution (2') gives

\[
v = \phi(\ddot{g} / \dot{g} + \lambda c \beta g).
\]

It follows from (7) that

\[
\phi = \frac{1}{\sqrt{2c\xi}}
\]

and that the function \(g\) obeys the differential equation

\[
(\ddot{g} / \dot{g})' + \lambda cg = 0
\]

with the following boundary conditions stemming from (5)

\[
\begin{align*}g(0) &= 0 \\
\ddot{g}(0) &= 0 \\
g(\infty) &= \lambda^{-1}.
\end{align*}
\]

2. AUTOMORPHISM AND NORMALIZATION

Self-similar solution (6) contains two arbitrary parameters, \(\lambda\) and the separation constant \(c\). By fixing the product \(\lambda c\), the differential equation to be solved (10) is "normalized". Here we make the choice \(\lambda c = 2\).

There remains one degree of freedom. Either we may choose \(c\) independently and normalize the function \(\phi(\xi)\), hence also the variable \(\beta\) in (6). Or we may choose \(\lambda\) independently, which would allow a normalization of the third of the boundary conditions (11). Our choice will be guided here by the elegant modification of the boundary conditions due to Toepfer[2].
3. TRANSFER OF THE THIRD BOUNDARY CONDITION

Numerical integration of differential equation (10) could be achieved by a marching procedure, provided all the boundary conditions were known in \( \beta = 0 \). This can be obtained precisely, even after normalization of the differential equation, by the existence of its remaining automorphism.

Imagine the conditions (11) be replaced by

\[
g(0) = 0 \quad \dot{g}(0) = \frac{1}{2} \quad \ddot{g}(0) = 0
\]  

(12)

under which the normalized differential equation

\[
(\ddot{\bar{g}}/\dot{\bar{g}}) + 2g = 0
\]

(13)

would yield an asymptotic value

\[
g(\infty) = \frac{m}{2}.
\]

By comparison, the previous boundary conditions (11) are now satisfied by the choice \( \lambda = 2/m \), giving explicitly

\[
u = \frac{2}{m} g(\beta) \quad \beta = \frac{n}{\sqrt{(2m\xi)}},
\]

(15)

\[
v = \frac{1}{\sqrt{(2m\xi)}} (\ddot{\bar{g}}/\dot{\bar{g}} + 2\beta g).
\]

(16)

4. THE BLASIUS EQUATION

From (13), integrating from \( \beta = 0 \) and noting that \( \ddot{g}(0) = 0 \)

\[
\ddot{\bar{g}}/\dot{\bar{g}} + 2\int_0^\beta g(\beta')d\beta' = 0.
\]

Hence introducing the new function

\[
f(\beta) = 2\int_0^\beta g(\beta')d\beta'
\]

a normalized form of the Blasius equation

\[
\ddot{\bar{f}} + f\dot{\bar{f}} = 0
\]

(17)

with normalized boundary conditions

\[
f(0) = 0 \quad \dot{f}(0) = 0 \quad \ddot{f}(0) = 1
\]

(18)

and asymptotic value

\[
f(\infty) = m.
\]

(19)
This equivalent mathematical form, due to Blasius\cite{3}, is directly related to his use of a stream function \( \psi \)

\[
u = \partial \psi / \partial \eta \quad v = -\partial \psi / \partial \xi
\]
to satisfy immediately the incompressibility condition (4). In terms of our self-similar solution, there comes

\[
\psi = \sqrt{\left( \frac{2\zeta}{m} \right)} f(\beta) \quad u = \left( \frac{1}{m} \right) \dot{f} \quad v = \frac{1}{\sqrt{(2m\zeta)}} (\beta \dot{f} - f).
\]

\begin{equation}
(20)
\end{equation}

5. FIRST REDUCTION OF THE DIFFERENTIAL EQUATION

New independent variable: \( mu = 2g = w \).
New unknown function: \( 2\dot{g} = dw/d\beta = p \).
As

\[
\frac{dp}{dw} = \frac{dp}{d\beta} \frac{d\beta}{dw} = \frac{1}{p} \frac{dp}{d\beta}
\]

\[
\frac{d^2p}{dw^2} = \frac{d}{d\beta} \left( \frac{1}{p} \frac{dp}{d\beta} \right) \frac{d\beta}{dw} = \frac{1}{p} \frac{d}{d\beta} \left( \frac{1}{p} \frac{dp}{d\beta} \right)
\]
equation (13), can also be written,

\[
\left( \frac{2\dot{g}}{2\ddot{g}} \right) = \frac{d}{d\beta} \left( \frac{1}{p} \frac{dp}{d\beta} \right) = -2g = -w
\]
and is split into the pair

\[
\frac{d^2p}{dw^2} + w = 0; \quad \frac{d\beta}{dw} = \frac{1}{p}.
\]
\begin{equation}
(21)
\end{equation}

From the definition of \( w \)

\[
w = 0 \quad \text{for } \beta = 0; \quad w = m \quad \text{for } \beta = \infty.
\]
\begin{equation}
(22)
\end{equation}
While from the definition of \( p \)

\[
p(0) = 2\dot{g}(0) = 1
\]
\begin{equation}
(23)
\end{equation}

\[
p'(0) = \frac{1}{p(0)} \left( \frac{dp}{d\beta} \right)_0 = 2\ddot{g}(0) = 0.
\]
\begin{equation}
(24)
\end{equation}
These results establish the initial values and the domain of integration of the differential system. The existence of an asymptotic value of \( g \) when \( \beta \to \infty \), leads to

\[
\dot{g}(\infty) = 0; \quad \text{hence } p(m) = 0.
\]
\begin{equation}
(25)
\end{equation}
6. Second Reduction of the Differential Equation

The first of differential equations (21) has itself an automorphism. It remains invariant under the continuous group of transformations

$$\hat{p} = \gamma^{3/2} p; \quad \hat{w} = \gamma w.$$  \hspace{1cm} (26)

Setting $r = dp/dw$, the extended group of infinitesimal transformations is easily found to be

$$\frac{\delta w}{w} = \frac{\delta p}{2p} = \frac{\delta r}{2r} = \delta \gamma,$$

and the following first integrals are available:

$$wp^{-2/3} = c_1; \quad rp^{-1/3} = c_2.$$  

This suggests a solution of the form

$$\frac{dp}{dw} = -p^{1/3} F(\omega); \quad \omega = wp^{-2/3}$$  \hspace{1cm} (27)

which is equivalent to $c_2 + F(c_1) = 0$.

From (27) the required computations can be conducted as follows:

$$\frac{d^2 p}{dw^2} = -\frac{1}{3} p^{-2/3} \frac{dp}{dw} F - p^{1/3} \frac{dF}{d\omega} \frac{d\omega}{dw};$$

but

$$\frac{d\omega}{dw} = p^{-2/3} - \frac{3}{5} p^{-5/3} \frac{dp}{dw} = p^{-2/3}(1 + \frac{3}{5} \omega F);$$

whence

$$\frac{d^2 p}{dw^2} = \frac{1}{5} p^{-1/3} F^2 - p^{-1/3} \frac{dF}{d\omega} (1 + \frac{3}{5} \omega F)$$

and the first of differential equations (21) splits into

$$\frac{dF}{d\omega} = \frac{F^2 + 3\omega}{3 + 2\omega F}$$  \hspace{1cm} (28)

$$\frac{1}{p} \frac{dp}{d\omega} = -\frac{3F}{3 + 2\omega F}$$  \hspace{1cm} (29)

with as boundary conditions,

for $\beta = 0$, \hspace{0.5cm} $w = 0$ and \hspace{0.5cm} $p = 1$, \hspace{0.5cm} hence $\omega = 0$ \hspace{1cm} (9)

for $\beta = \infty$, \hspace{0.5cm} $w = m$ and \hspace{0.5cm} $p = 0$, \hspace{0.5cm} hence $\omega = \infty$

$$\frac{dp}{dw} = 0, \hspace{0.5cm} \text{for} \hspace{0.5cm} w = 0, \hspace{0.5cm} \hspace{0.5cm} \text{hence} \hspace{0.5cm} F(0) = 0$$  \hspace{1cm} (31)

$$\text{and} \hspace{0.5cm} p(0) = 1$$
The differential equation (28) itself shows that \( F'(0) = 0 \) and an extremely accurate starting solution is obtained by (alternating) power series

\[
F = \omega^2 \left( \frac{1}{2} - \frac{1}{20} \omega^3 + \frac{1}{80} \omega^6 - \frac{59}{13 \cdot 200} \omega^9 + \frac{151}{92 \cdot 400} \omega^{12} - \frac{16 \cdot 539}{25 \cdot 132 \cdot 800} \omega^{15} + \ldots \right).
\]

After the numerical integration of \( F \), we have from (29) and the boundary conditions a quadrature for the computation of \( p \):

\[
p = \exp \left( -\int_0^\omega \frac{3F \omega'}{3 + 2\omega' F} \right). \tag{32}
\]

Similarly, from the second of equations (27) and the previous result, a quadrature for the computation of the horizontal velocity

\[
w = 2g = mu = \omega p^{2/3} = \omega \exp \left( -\int_0^\omega \frac{2F \omega'}{3 + 2\omega' F} \right) \tag{33}
\]

Finally a second quadrature is required to obtain the co-ordinate \( \beta \). Using the second of equations (21) and the second of equations (27),

\[
\frac{d\beta}{d\omega} = \frac{dw}{d\omega} = p^{-1/3}(1 + \frac{2}{3} \omega F)^{-1} \tag{34}
\]

the starting value of which is \( \beta(0) = 0 \).

The asymptotic value \( m \) of \( w \) is one of the essential numerical results. As equation (33) yields in the limit \( \omega \to \infty \), an indeterminate product, the following transformation is indicated

\[
\omega = \exp \ln \omega = \exp \int_1^\omega \frac{d\omega'}{\omega'}
\]

and (33) is modified for \( \omega > 1 \) into

\[
w = \exp \left( -\int_0^1 \frac{2F \omega}{3 + 2\omega F} \right) \cdot \exp \left( \int_1^\omega \frac{d\omega'}{\omega'} - \frac{2F \omega'}{3 + 2\omega' F} \right).
\]

After reduction of the second integral, that becomes a convergent one

\[
w = w(1) \exp \int_1^\omega \frac{3d\omega'}{\omega'(3 + 2\omega' F)} \tag{35}
\]

\[
w(1) = \exp \left( -\int_0^1 \frac{3F \omega}{3 + 2\omega F} \right). \tag{36}
\]

The asymptotic behavior of \( F \) for large \( \omega \) is obtainable from the approximate differential equation

\[
\frac{dF}{d\omega} = \frac{F}{2\omega} + \frac{3}{2F}
\]

the two contributions to the derivative having the same order of magnitude if \( F \) is of the order of \( \sqrt{\omega} \).
Setting

\[ F = \sqrt{(\omega)H} \]

the resulting approximate differential equation

\[ 2H \, dH = \frac{3 \, d\omega}{\omega} \]

has the exact solution

\[ H^2 = K + 3 \ln \omega \]

and an asymptotic value of \( F \) is given by

\[ F = \sqrt{\omega} \sqrt{(K + 3 \ln \omega)} \]

The numerical integrations were carried out on the IBM 370–155 computer of the University by the junior author. They are in complete agreement with the numerical results obtained by Smith[5]; in particular for the asymptotic value

\[ m^3 = 4.53465 \]

whence the friction coefficient \( m^{-3/2} \) in the tangent stress formula

\[ \tau = \rho U^2 \sqrt{\frac{v}{2U_x}} (m^{-3/2}) \]

receives the already widely accepted value of 0.664.

REFERENCES