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THE DYNAMICS OF FLEXIBLE BODIES

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Abstract—The motion of a flexible body undergoing arbitrarily large rotations with respect to an inertial frame is split into a mean rigid body motion, defining a dynamical reference frame, and a relative motion taking into account the deformations.

The mean motion is usually taken to satisfy the Tisserand conditions of zero relative momentum and angular momentum, a choice that, as shown in the paper, corresponds to a minimum value of the relative kinetic energy. The condition of zero angular momentum is however non linear and introduces discretization difficulties that can be overcome by another choice.

The choice proposed in the paper minimizes the mean square of relative displacements. It preserves the zero momentum condition but linearizes the angular momentum condition in such a way that the relative displacements are representable exactly by an expansion in natural elastic vibration modes.

Hamilton's principle is used to derive all the equations, including the mean motion ones, by using the concept of quasi-coordinates. Gravitational potential and thrust vectors, as locally oriented by the body motion and deformation, are accounted for. The equations are not limited to small distortion of the body, but to small strains.

1. THE DYNAMIC REFERENCE FRAME

THE FORMULATION of the motion of a flexible body as a continuum through inertial space is unsatisfactory from several viewpoints. One is usually not interested in the details of this motion but in its main characteristics such as motion of the center of mass and, under the assumption that the deformations remain small, the history of the average orientation of the body. This last information is of course essential to pilots, real or artificial, in order to implement guidance corrections. We therefore try to define a set of cartesian mean axes accompanying the body, or dynamic reference frame, with respect to which the relative displacement, velocities or accelerations of material points due to the deformations are minimum in some global sense. If the body does not deform, any set of axes fixed into the body is of course a natural dynamic reference frame.

In principle the transfer from inertial to dynamic axes is effected in two steps: a translation $\mathbf{a}(t)$ of the origin G_0 to the new origin G and a rotation of the translated axes. If we think of the reference configuration C_0 as mapped by these operations, we obtain (in dotted lines on Fig. 1) the dynamic reference configuration C_d. Any position vector $\mathbf{y}$ is carried into a corresponding new position vector $\mathbf{x}$ so that

$$\hat{y}_i = x_i \qquad (i = 1, 2, 3)$$

an equality between the inertial components of $\mathbf{y}$ and the dynamical components of $\mathbf{x}$. In matrix form we obtain the equality

$$\hat{y} = x \tag{1}$$

between the corresponding column matrices. The rotation of the axes is described by the matrix equation

$$x = U\hat{x} \qquad \text{or} \qquad \hat{x} = U^T x \tag{2}$$

relating the inertial and dynamic components of the same vector.

U is a rotation matrix; it is orthogonal

$$UU^T = U^T U = I \qquad \text{(identity matrix)} \tag{3}$$

and its determinant is unity

$$\det U = +1. \tag{4}$$

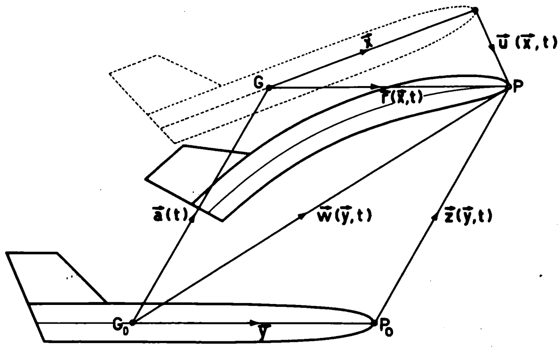

Fig. 1. Configurations of a flexible body.

The obvious vector relations illustrated on Fig. 1

$$\mathbf{w}=\mathbf{y}+\mathbf{z}=\mathbf{a}+\mathbf{r}=\mathbf{a}+\mathbf{x}+\mathbf{u}$$

can be stated either in terms of inertial components as

$$\hat{w}=\hat{y}+\hat{z}=\hat{a}+\hat{r}=\hat{a}+\hat{x}+\hat{u}$$

or in terms of dynamic components

$$w=y+z=a+r=a+x+u.$$

As $\mathbf{x}$ and the "relative displacement" vector $\mathbf{u}$ are normally observed in the dynamic reference frame, while $\mathbf{w}$ and $\mathbf{a}$ are given in the inertial frame, the most useful form of these relations is

$$x+u=U(\hat{w}-\hat{a})=U(\hat{y}+\hat{z}-\hat{a}). \tag{5}$$

Clearly the dynamic axes will be defined by a prescription of $\hat{a}(t)$ and $U(t)$; the central considerations for selecting particular prescriptions being the resulting properties of $u(x,t)$, the field of relative displacements.

2. MEASURE OF STRAIN IN THE DYNAMIC REFERENCE FRAME

The measure of strain provided by the Green tensor

$$\hat{g}_{mn}=\frac{1}{2}\left(\frac{\partial\hat{z}_m}{\partial\hat{y}_n}+\frac{\partial\hat{z}_n}{\partial\hat{y}_m}+\frac{\partial\hat{z}_i}{\partial\hat{y}_m}\frac{\partial\hat{z}_i}{\partial\hat{y}_n}\right)$$

in the inertial frame is identical to that

$$\gamma_{mn}=\frac{1}{2}\left(\frac{\partial u_m}{\partial x_n}+\frac{\partial u_n}{\partial x_m}+\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_n}\right) \tag{6}$$

in the dynamic frame; so that the elastic strain energy of the body with elastic moduli

$$C^{pq}_{mn}=C^{pq}_{nm}=C^{qp}_{mn}=C^{mn}_{pq}$$

is given by

$$W=\frac{1}{2}\int C^{pq}_{mn}\gamma_{mn}\gamma_{pq}\frac{\mathrm{d}\mu}{\rho} \tag{7}$$

ρ being the mass per unit volume and $\mathrm{d}\mu$ the element of mass. Denoting by

$$A=\left\{\frac{\partial u_m}{\partial x_n}\right\} \tag{8}$$

the matrix of displacement gradients, we may write

$$\Gamma = \{\gamma_{mn}\} = \frac{1}{2}(A + A^T + A^T A). \tag{9}$$

3. ABSOLUTE VELOCITY AND KINETIC ENERGY

If we differentiate (5) with respect to time, keeping $\hat{y} = x$ fixed,

$$\frac{\partial u}{\partial t} = \frac{dU}{dt}(\hat{y} + \hat{z} - \hat{a}) + U\left(\frac{\partial \hat{z}}{\partial t} - \frac{d\hat{a}}{dt}\right).$$

In this expression we identify

$$\hat{v}_a = \frac{\partial \hat{z}}{\partial t} \quad \text{and} \quad \hat{v}_g = \frac{d\hat{a}}{dt}$$

as respectively the absolute velocity of a particle and the velocity of the origin of the dynamic reference frame. If we substitute

$$\hat{y} + \hat{z} - \hat{a} = \hat{x} + \hat{u} = U^T(x + u)$$

and rearrange the terms, we obtain the classical decomposition of the absolute velocity in a moving reference frame

$$v_a = v_g + [\omega](x + u) + \frac{\partial u}{\partial t} \tag{10}$$

where

$$[\omega] = -\frac{dU}{dt}U^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = U\frac{dU^T}{dt} \tag{11}$$

is the skew symmetrix matrix of angular velocities of the dynamic axes. The skew symmetry is a direct consequence of the differentiation of (3)

$$U\frac{d}{dt}U^T + \frac{dU}{dt}U^T = 0.$$

The notation conveniently suggests that this matrix is built up from the components of the pseudo-vector "angular velocity"

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

so that the second term of (10) is that of the dynamic components of the vector product

$$\boldsymbol{\omega} \times \mathbf{r} = -\mathbf{r} \times \boldsymbol{\omega}.$$

It may thus be written indifferently as

$$[\omega](x + u) = -[x + u]\omega = [x + u]^T\omega = -[\omega]^T(x + u).$$

The inertial components of the velocity field induced by the rotation of the dynamical axes are given by

$$U^T[\omega]r = U^T[\omega]U\hat{r}$$

so that

$$[\hat{\omega}] = U^T[\omega]U = \frac{\mathrm{d}U^T}{\mathrm{d}t}U = -U^T\frac{\mathrm{d}U}{\mathrm{d}t} \tag{12}$$

is the skew symmetric matrix of inertial components of the angular velocities.

If the rotation operator U is decomposed in a succession of elementary rotations defining Euler angles, such as

$$U = CBA$$

$$A = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotation of a (yaw) angle ψ about the third inertial axis,

$$B = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

rotation of a (pitch) angle θ about the second inertial axis as already turned by operation A.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

rotation of a (roll) angle ϕ about the first inertial axis as already turned by operation BA, the application of (11) yields the well known relations between angular velocities about dynamical axes and velocities of Euler angles:

$$\begin{aligned} \omega_1 &= \dot{\phi} - \dot{\psi}\sin\theta \\ \omega_2 &= \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \\ \omega_3 &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta. \end{aligned} \tag{13}$$

The Euler angles, however, are not very convenient as degrees of freedom to use in a Hamiltonian formulation.

Quasi-coordinates are more symmetrical and easier to manipulate. From (10), after some manipulation like

$$(x+u)^T[\omega]^T[\omega](x+u) = \omega^T[x+u][x+u]^T\omega$$

the kinetic energy of the body

$$T = \frac{1}{2}\int \left(\frac{\partial\hat{z}}{\partial t}\right)^T \frac{\partial\hat{z}}{\partial t}\,\mathrm{d}\mu = \frac{1}{2}\int v_a^T v_a\,\mathrm{d}\mu$$

can be placed in the form

$$T = \frac{1}{2}v_g^T v_g \int \mathrm{d}\mu + v_g^T \int \frac{\partial u}{\partial t}\,\mathrm{d}\mu + v_g^T[\omega]\int (x+u)\,\mathrm{d}\mu$$
$$+\frac{1}{2}\int \frac{\partial u^T}{\partial t}\frac{\partial u}{\partial t}\,\mathrm{d}\mu + \omega^T\int [x+u]\frac{\partial u}{\partial t}\,\mathrm{d}\mu + \frac{1}{2}\omega^T R\omega \tag{14}$$

where

$$R = \int [x+u][x+u]^T\,\mathrm{d}\mu \tag{15}$$

is the matrix of inertia moments of the body about the dynamical axes in the actual configuration, that is taking into account the relative displacements u due to its flexibility.

$\int \frac{\partial u}{\partial t} \mathrm{d}\mu$ is the vector of relative momentum of the body,

$\int [x+u] \frac{\partial u}{\partial t} \mathrm{d}\mu$ is the vector of its relative angular momentum.

4. DECOMPOSITION OF VIRTUAL WORK. QUASI-COORDINATES

In terms of inertial components, the elementary virtual work accomplished by a force **f** on the virtual displacement $\delta \mathbf{z}$ of a material point is

$$\mathbf{f} \cdot \delta \mathbf{z} = \hat{f}^T \delta \hat{z}.$$

To obtain the expression in terms of dynamical components, begin with the differentiation of (5), noting that $\delta \hat{y} = \delta x = 0$, since the virtual displacement is that of a given material point, obtaining

$$\delta u = \delta U(\hat{y} + \hat{z} - \hat{a}) + U(\delta \hat{z} - \delta \hat{a}).$$

Hence

$$U \delta \hat{z} = U \delta \hat{a} + \delta u - \delta U U^T (x+u).$$

These dynamical components of the absolute virtual displacement of a material point, can of course be given the same structure of decomposition

$$\delta h = U \delta \hat{z} = \delta p + [\delta \alpha](x+u) + \delta u \tag{16}$$

as the absolute velocity (10). (In fact the virtual displacement considerations could be replaced by virtual velocity ones.)

$$\delta p = U \delta \hat{a} \tag{17}$$

are the dynamical coordinates of the virtual displacement of the origin of dynamical axes; they will also be termed the "quasi-coordinates of translation".

$$[\delta \alpha] = -\delta U \cdot U^T = \begin{pmatrix} 0 & -\delta\alpha_3 & \delta\alpha_2 \\ \delta\alpha_3 & 0 & -\delta\alpha_1 \\ -\delta\alpha_2 & \delta\alpha_1 & 0 \end{pmatrix} = U \delta U^T \tag{18}$$

are the "quasi-coordinates of rotation". We may note that instead of (17) we may write

$$\delta p = U \delta (U^T a) = U \delta U^T a + U U^T \delta a = \delta a + [\delta \alpha] a \tag{19}$$

in the same manner as

$$v_g = U \hat{v}_g = U \frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = U \frac{\mathrm{d}}{\mathrm{d}t}(U^T a) = \frac{\mathrm{d}a}{\mathrm{d}t} + [\omega] a. \tag{20}$$

Finally the virtual work, as seen from the dynamic reference frame, is, using $\hat{f}^T = f^T U$ and (16)

$$f^T(\delta p + [\delta \alpha](x+u) + \delta u) = (\delta p^T + \delta \alpha^T [x+u] + \delta u^T) f. \tag{21}$$

The interpretation of the quasi-coordinates of rotation is obtained by decomposing a perturbed rotation $U + \delta U$ into the product of the unperturbed rotation followed by the small additional rotation $I - [\delta \alpha]$. Thus we should have

$$U + \delta U = (I - [\delta \alpha]) U \qquad \text{or} \qquad \delta U = -[\delta \alpha] U$$

and this is clearly equivalent to the definition (18). The $\delta\alpha_i$ are consequently the small angles of rotation of the dynamic frame, about its own axes, when passing from the unperturbed to a perturbed position.

The usefulness of quasi-coordinates can be seen from the possibility of expressing the variations of the arguments v_g and ω of the kinetic energy, that are not easily related to displacement coordinates, in terms of the quasi-coordinates and their time derivatives. This permits a direct application of Hamilton's principle to derive the equations of motion of the dynamical axes themselves. The two essential formulas are

$$\delta v_g = \delta\left(U\frac{\mathrm{d}\hat{a}}{\mathrm{d}t}\right) = \delta U\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} + U\frac{\mathrm{d}}{\mathrm{d}t}\delta\hat{a} = \delta U U^T v_g + U\frac{\mathrm{d}}{\mathrm{d}t}(U^T\delta p)$$

or, finally,

$$\delta v_g = -[\delta\alpha]v_g + [\omega]\delta p + \frac{\mathrm{d}}{\mathrm{d}t}\delta p. \tag{22}$$

To obtain the second formula, take the time derivative of definition

$$\delta U = -[\delta\alpha]U$$

and compare it to the variation of definition

$$\frac{\mathrm{d}}{\mathrm{d}t}U = -[\omega]U$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\delta\alpha]U + [\delta\alpha]\frac{\mathrm{d}U}{\mathrm{d}t} = [\delta\omega]U + [\omega]\delta U.$$

Postmultiplication by U^T furnishes the desired result

$$\delta[\omega] = [\omega][\delta\alpha] - [\delta\alpha][\omega] + \frac{\mathrm{d}}{\mathrm{d}t}[\delta\alpha]$$

which is equivalent to

$$\delta\omega = \frac{\mathrm{d}}{\mathrm{d}t}\delta\alpha + [\omega]\delta\alpha. \tag{23}$$

5. DECOMPOSITION OF ABSOLUTE ACCELERATION

In inertial axes the absolute acceleration of a particle is

$$\hat{j}_a = \frac{\partial}{\partial t}\hat{v}_a = \frac{\partial}{\partial t}(U^T v_a) = \frac{\mathrm{d}}{\mathrm{d}t}U^T v_a + U^T\frac{\partial v_a}{\partial t}.$$

Hence, using definition (11), its dynamic components are given by

$$j_a = U\hat{j}_a = \frac{\partial v_a}{\partial t} + [\omega]v_a. \tag{24}$$

The substitution of (10) will produce the decomposition theorem of acceleration in moving axes

$$j_a = \frac{\mathrm{d}}{\mathrm{d}t}v_g + [\omega]v_g + [\omega][\omega](x+u) + \left[\frac{\mathrm{d}\omega}{\mathrm{d}t}\right](x+u) + 2[\omega]\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} \tag{25}$$

the first four terms being the acceleration experienced when the particle is frozen in the moving axes, the fifth being the complementary or Coriolis term, the sixth and last the relative acceleration.

The same result is obtained in the application of Hamilton's principle to the kinetic energy alone. We must have

$$\delta \int_{t_1}^{t_2} T \, dt = -\int_{t_1}^{t_2} \int j_a^T \delta h \, d\mu \tag{26}$$

where δh, the dynamic components of the absolute virtual displacement of a particle, were already seen to satisfy a decomposition (16) completely analogous to that of the absolute velocity.

Since

$$\delta T = \int v_a^T \delta v_a \, d\mu$$

we are led to compute the variation of the absolute velocity. This can be done by taking the variation of (10) and applying the formulas (22) and (23). The following derivation is shorter. Differentiate with respect to time the relation defining δh

$$\delta \hat{w} = \delta \hat{z} = U^T \delta h,$$

obtaining

$$\delta \hat{v}_a = \frac{d}{dt} U^T \delta h + U^T \frac{\partial}{\partial t} \delta h = \delta(U^T v_a) = \delta U^T v_a + U^T \delta v_a.$$

Premultiplication by U and use of definitions (11) and (18) produces

$$\delta v_a = \frac{\partial}{\partial t} \delta h + [\omega] \delta h - [\delta \alpha] v_a. \tag{27}$$

As $v_a^T[\delta\alpha]v_a = 0$ because of the skew symmetry of the central matrix, there comes

$$\delta \int_{t_1}^{t_2} T \, dt = \int v_a^T \delta h \, d\mu \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int \left(\frac{\partial v_a^T}{\partial t} + v_a^T [\omega]^T \right) \delta h \, d\mu$$

and, since δh must vanish at the time limits, the result indeed coincides with (26) on account of (24). Should we now make use of the decomposition (16) in (26),

$$\delta \int_{t_1}^{t_2} T \, dt = -\int_{t_1}^{t_2} \delta h^T j_a \, d\mu = -\int_{t_1}^{t_2} \left(\delta p^T \int j_a \, d\mu + \delta \alpha^T \int [x+u] j_a \, d\mu + \int \delta u^T j_a \, d\mu \right) dt.$$

This formula reveals the inertia forces attached to the different components of the motion of particles:

$-j_a \, d\mu$	to the relative motion of each particle,
$-\int j_a \, d\mu$	to the motion induced by a translation of the dynamical axes,
$-\int [x+u] j_a \, d\mu$	the inertial moment induced by the rotation of the dynamical axes.

The last two are dynamically equilibrated respectively by the vectorial sum of all extermal forces and their moment with respect to the origin. However the equations of momentum and angular momentum, or equations of the mean motion, are redundant. They will follow as consequence of the equations of motion of individual particles.

The interest of a decomposition into a mean motion and a relative motion lies in the simplifications that may be obtained for the relative motion by appropriate choices of $a(t)$ and $U(t)$. Clearly, if the body is rigid, the relative motion disappears entirely if the axes are fixed into the body. The interest is then entirely concentrated into the equations of mean motion, from which the motion of individual particles follows in a trivial manner. If the body is only slightly deformable a suitable choice of dynamical axes may reduce the local relative rotations to small quantities of the same order of magnitude as the strains, thus allowing simplifications to take place by linearization procedures.

But, even if the body is highly deformable, which can be true even for small strains, there are still choices of mean motion that are physically significant and conducive to simplifications of the equations of relative motion. Of the two main choices that will be investigated, the first minimizes the kinetic energy of the relative motion, the second minimizes the mean square of relative displacements.

6. MINIMUM KINETIC ENERGY IN RELATIVE MOTION

Assuming the absolute velocity of particles to be given, we analyze such motions of the dynamical axes that minimize the relative kinetic energy

$$T_r = \frac{1}{2}\int \left(\frac{\partial u}{\partial t}\right)^T \frac{\partial u}{\partial t}\,\mathrm{d}\mu.$$

The minimum property requires that

$$\int \left(\frac{\partial u}{\partial t}\right)^T \Delta \frac{\partial u}{\partial t}\,\mathrm{d}\mu = 0 \tag{28}$$

for all changes $\Delta(\partial u/\partial t)$ in relative velocity that are simply due to changes in the motion of the dynamical axes. The relative position of the dynamical axes at the epoch of comparison may be kept the same so that

$$\Delta u = 0 \qquad \Delta U = 0$$

and consequently

$$\Delta v_a = \Delta(U\hat{v}_a) = 0.$$

The non zero changes are contributed by

$$\Delta v_g = \Delta(U\hat{v}_g) = U\Delta\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} \qquad \text{and} \qquad \Delta\omega.$$

From (10) follows then

$$\Delta\frac{\partial u}{\partial t} = -\Delta v_g - [\Delta\omega](x+u) = -\Delta v_g + [x+u]\Delta\omega.$$

This result, substituted into (28), gives as minimum conditions

$$\int \frac{\partial u}{\partial t}\,\mathrm{d}\mu = 0 \tag{29}$$

$$\int [x+u]\frac{\partial u}{\partial t}\,\mathrm{d}\mu = 0. \tag{30}$$

The first requires the relative momentum of the body to be zero, the second requires the same for the relative angular momentum. To simplify further deductions we assume the more stringent condition

$$\int (x+u)\,\mathrm{d}\mu = 0 \tag{31}$$

at all times, which implies the satisfaction of (29) by differentiation with respect to time. It is equivalent to assume that the origin of dynamical axes is located at all times at the center of mass of the body. It is then also true that

$$\int [x+u]\,\mathrm{d}\mu = 0. \tag{32}$$

If we then take the relative velocity from (10) and substitute for it in the conditions of zero momentum and angular momentum, there comes

$$v_g \int \mathrm{d}\mu = \int v_a\,\mathrm{d}\mu \tag{33}$$

and, provided we replace

$$[x+u][\omega](x+u) = [x+u][x+u]^T\omega$$

and consider the definition (15)

$$R\omega = \int [x+u]v_a\,\mathrm{d}\mu. \tag{34}$$

Equation (33) fixes the dynamic components of the velocity of the origin of moving axes in terms of the absolute momentum of the body; for the inertial observer

$$\hat{v}_g \int \mathrm{d}\mu = \int \hat{v}_a\,\mathrm{d}\mu.$$

Equation (34) fixes the angular velocity of the moving axes capable of complying with the zero relative angular momentum condition. Its second member is the absolute angular momentum of the body. The angular velocities are uniquely defined as long as the matrix of inertia moments of the actual configuration is non singular. It is in fact positive definite unless the rather absurd situation is reached where all the particles are aligned on a straight line. For the inertial observer

$$\hat{R}\hat{\omega} = \int [\hat{w}-\hat{a}]\hat{v}_a\,\mathrm{d}\mu$$
$$\hat{R} = \int [\hat{w}-\hat{a}][\hat{w}-\hat{a}]^T\,\mathrm{d}\mu = U^T R U.$$

It should be observed that, while their motion is uniquely determined, the dynamic axis complying with the conditions of zero relative momentum and angular momentum, or Tisserand axes, are not unique. Any system of Tisserand axes may be deduced from another by a fixed (independent of time) rotation. In Tisserand axes the kinetic energy of the body is reduced from (14) to

$$T = \frac{1}{2}v_g^T v_g \int \mathrm{d}\mu + \frac{1}{2}\int \frac{\partial u^T}{\partial t}\frac{\partial u}{\partial t}\,\mathrm{d}\mu + \frac{1}{2}\omega^T R\omega. \tag{35}$$

7. MINIMUM SQUARE AVERAGE OF RELATIVE DISPLACEMENTS

In the case of a rigid body it is possible to let the relative displacement field vanish entirely by fixing the moving axes in the body. This suggests a choice of moving axes for a flexible body that minimizes at all times the functional

$$\frac{1}{2}\int u^T u\,\mathrm{d}\mu \qquad \text{minimum}. \tag{36}$$

The corresponding requirement $\int u^T \Delta u\,\mathrm{d}\mu = 0$ must hold for all

$$\Delta u = \Delta\{-x + U(\hat{w}-\hat{a})\} = \Delta U(\hat{w}-\hat{a}) - U\Delta\hat{a}$$

due to a modification of the position of the moving axes. In view of (17) and (18) this can be written

$$\Delta u = -[\Delta\alpha](x+u) - \Delta p = [x+u]\Delta\alpha - \Delta p.$$

As the variations Δp and $\Delta\alpha$ are free and independent we obtain the minimizing conditions

$$\int u \, \mathrm{d}\mu = 0 \tag{37}$$

$$\int [x+u]u \, \mathrm{d}\mu = \int [x]u \, \mathrm{d}\mu = 0 \qquad ([u]u \equiv 0) \tag{38}$$

The time derivative of the first condition coincides with the Tisserand condition (29) of zero relative momentum. It coincides also with the more stringent condition (31) provided the origin of inertial axes be taken at the center of mass of the reference configuration

$$\int x \, \mathrm{d}\mu = 0. \tag{39}$$

The time derivative of the second condition

$$\int [x] \frac{\partial u}{\partial t} \, \mathrm{d}\mu = 0 \tag{40}$$

is a linearized version of the Tisserand condition of zero relative angular moment, in which the angular momentum of particles is computed with the reference position vector, instead of the displaced one.

For the inertial observer, condition (37) is

$$U \int (\hat{w} - \hat{a}) \, \mathrm{d}\mu - \int x \, \mathrm{d}\mu = 0$$

and, in view of (39), yields the explicit value to be taken for the position vector of the origin of dynamical axes, given the actual configuration of the body

$$\hat{a} \int \mathrm{d}\mu = \int \hat{w} \, \mathrm{d}\mu. \tag{41}$$

Similarly condition (38) becomes an implicit equation for the computation of the angular position of the dynamical axes

$$\int [x] U \hat{w} \, \mathrm{d}\mu = 0. \tag{42}$$

(as the other terms $-a \int [x] \, \mathrm{d}\mu$ and $-\int [x]x \, \mathrm{d}\mu$ vanish).

The value to be found for U clearly depends only on the actual configuration of the body as measurable by the inertial observer.

The solution of this problem is surprisingly complicated and, in order to reduce it to a classical eigenvalue problem, it seems preferable to come back to a statement of the functional to be minimized in terms of quantities measured by the inertial observer. Thus we replace in functional (36)

$$u = U(\hat{w} - \hat{a}) - x$$

and expand the scalar product

$$\begin{aligned} u^T u &= (\hat{w} - \hat{a})^T (\hat{w} - \hat{a}) - 2x^T U (\hat{w} - \hat{a}) + x^T x \\ &= \hat{w}^T \hat{w} + x^T x + 2x^T U \hat{a} - 2\hat{a}^T \hat{w} + \hat{a}^T \hat{a} - 2x^T U \hat{w}. \end{aligned}$$

The contribution of the two first terms to the functional is constant and may be dropped. That of the third term vanishes because of (39). There remains

$$-2\hat{a}^T \int \hat{w} \, \mathrm{d}\mu + \hat{a}^T \hat{a} \int \mathrm{d}\mu - 2 \int x^T U \hat{w} \, \mathrm{d}\mu \qquad \text{minimum.}$$

Minimization with respect to $\hat{a}$ restitutes (41) and we are left with the problem

$$\phi = -\int x^T U \hat{w}\, d\mu \qquad \text{minimum.} \tag{43}$$

To solve it we now introduce a Rodrigues–Hamilton representation of the rotation operator

$$U = I + 2\beta[b] + 2[b][b]. \tag{44}$$

b is a vector invariant under the operator, since $[b]b \equiv 0$.

This shows b to be oriented as the axis of the rotation. For the same reasons $b^T U = b^T$ and, if u is any vector perpendicular to the axis of rotation,

$$b^T U u = b^T u = 0$$

showing that the rotated vector Uu is also perpendicular to the axis. Now, if θ denotes the angle through which the body has rotated, we should have

$$u^T U u = u^T u \cos\theta.$$

Computing the left-hand side ($u^T[b]u \equiv 0$)

$$\begin{aligned} u^T U u = u^T u + 2u^T[b][b]u &= u^T u + 2u^T(-b^T b I + b b^T)u \\ &= u^T u(1 - 2b^T b) \end{aligned}$$

and comparing, there follows

$$1 - 2b^T b = \cos\theta \qquad \text{or} \qquad b^T b = \sin^2\frac{\theta}{2}. \tag{45}$$

Finally the condition that $U^T U = I$, imposes the constraint

$$\beta^2 + b^T b - 1 = 0 \qquad \text{or} \qquad \beta^2 = \cos^2\frac{\theta}{2}. \tag{46}$$

When substituting (44) into the functional (43) we note that

$$x^T[b]\hat{w} = -b^T[x]\hat{w}$$

and

$$x^T[b][b]\hat{w} = -b^T[x][\hat{w}]^T b$$

so that the functional may be written as a quadratic form in the constrained variables b and β

$$\phi = -\int x^T \hat{w}\, d\mu + 2\beta b^T \hat{m} + b^T \hat{M} b \tag{47}$$

with

$$\hat{m} = \int [x]\hat{w}\, d\mu \tag{48}$$

and, having taken the symmetrical part,

$$\hat{M} = \int \{[x][\hat{w}]^T + [\hat{w}][x]^T\}\, d\mu. \tag{49}$$

The minimizing conditions obtained by equating to zero the partial derivatives of the augmented functional

$$\Phi = \phi - \lambda(\beta^2 + b^T b - 1)$$

take the form of a self-adjoint eigenvalue problem

$$\begin{pmatrix} \hat{M} & \hat{m} \\ \hat{m}^T & 0 \end{pmatrix}\begin{pmatrix} b \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} b \\ \beta \end{pmatrix}. \tag{50}$$

For any eigenvalue λ and associated eigenvector, normed by satisfying the constraint (46), we find from those equations,

$$\phi = -\int x^T \hat{w}\, \mathrm{d}\mu + \lambda$$

and the minimum of the functional is associated with the *smallest eigenvalue* of problem (50). In principle there are four eigenvalues: apart from the minimizing one, there is a maximizing one and two merely associated with a stationary value of the functional.

It is of course possible for this problem to degererate and the case that warrants further investigation is the one corresponding to a multiple root for the smallest eigenvalue. Because then, the choice of the dynamical axes implimenting (36) is not unique.

The discussion is simplified if we consider that the smallest eigenvalue and one associated eigenvector have been determined from (50). Denote by U the corresponding rotation and

$$w = U\hat{w} = a + x + u$$

the displacement as seen in this particular set of dynamic axes. We investigate the other choices of axes by considering a rotation operator V, applied *subsequently* to U, or if we prefer to start again from the reference configuration, the rotation operator VU. Following (43) the problem will then present itself as

$$-\int x^T V w\, \mathrm{d}\mu = -\int x^T V(x+u)\, \mathrm{d}\mu \qquad \text{minimum}$$

if due account is taken of (39).

The corresponding eigenvalue problem will be

$$\begin{pmatrix} M & m \\ m^T & 0 \end{pmatrix}\begin{pmatrix} b \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} b \\ \beta \end{pmatrix} \tag{51}$$

$$m = \int [x](x+u)\, \mathrm{d}\mu = \int [x]u\, \mathrm{d}\mu$$

$$M = \int \{[x][x+u]^T + [x+u][x]^T\}\, \mathrm{d}\mu.$$

From construction it is known that the values

$$b = 0 \qquad \beta = 1$$

corresponding to the identity operator $V = I$ represent an eigensolution associated with the smallest eigenvalue. This however implies

$$m = 0 \qquad 0 = \lambda$$

equations to which the system (51) is then reduced. In $m = 0$ we have rediscovered the minimizing property (38). The eigenvalue problem is then split into

$$Mb = \lambda b \tag{52}$$

and

$$\lambda\beta = 0. \tag{53}$$

If $\lambda = 0$, that was seen to be the smallest eigenvalue, is not a multiple root, problem (52) has positive eigenvalues and matrix M is positive definite. Conversely, if M is positive definite, the smallest eigenvalue of (50) will be simple and possess only one normed eigenvector. The solution to problem (36) will be unique.

This is the situation that prevails when the relative displacements are not unduly large, for M can be presented as the difference of two matrices of moments of inertia, one corresponding to a "half-way" configuration of the body, that is certainly positive definite, and one that takes only into account the displacements themselves and is only of second order of magnitude

$$M = 2\int \left[x + \frac{u}{2}\right]\left[x + \frac{u}{2}\right]^T \mathrm{d}\mu - \frac{1}{2}\int [u][u]^T \mathrm{d}\mu. \tag{54}$$

A consequence of equation (53) is that the other eigenvectors, for which $\lambda > 0$, have their β component equal to zero. In view of (46) this amounts to say that they correspond to rotations V of 180° amplitude. The three possible rotations occur about three mutually perpendicular axes, since the eigenvectors b of problem (52) are orthogonal. It is easy to see that the identity operator $V = I$ and the three rotations V_1, V_2, V_3 of 180° amplitude generated by problem (52) form an Abelian group. Assume now that $\lambda = 0$ is a double root. It implies that a unit vector n will exist such that

$$Mn = 0 \qquad n^T n = 1$$

n is parallel to the axis of the rotation V and since $\lambda = 0$, β may be taken arbitrarily. We may thus consider that

$$b = n \sin\frac{\theta}{2} \qquad \beta = \cos\frac{\theta}{2}$$

$$V = I + (nn^T - I)(1 - \cos\theta) + \sin\theta\,[n].$$

It is easily verified that ϕ remains minimum under rotations of this type *with an arbitrary value of* θ.

The example, illustrated on Fig. 2, shows the possibility of non uniqueness of the minimizing choice when large elastic displacements are involved. A satellite with a rigid central axisymmetric body has two flexible massless appendages terminated by concentrated masses. In its reference

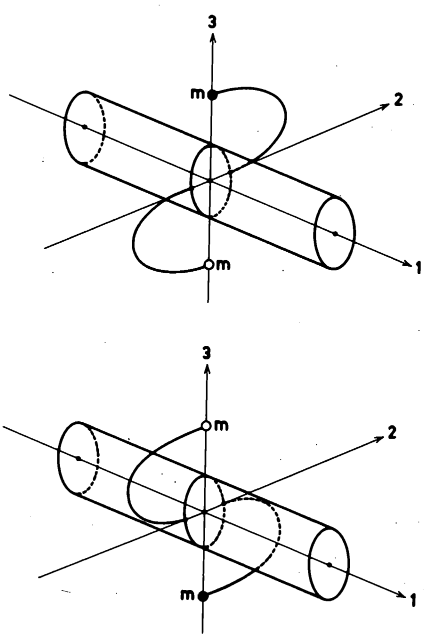

Fig. 2. Axisymmetric satellite with flexible appendages.

configuration the matrix of moments of inertia is

$$\begin{pmatrix} A+2mR^2 & 0 & 0 \\ 0 & B+2mR^2 & 0 \\ 0 & 0 & B \end{pmatrix}.$$

Suppose that in the deformed configuration the central body has not moved but that the masses of the appendages have been interchanged. We find from (48) and (49)

$$\hat{m} = 0 \qquad \hat{M} = 2\begin{pmatrix} A-2mR^2 & 0 & 0 \\ 0 & B-2mR^2 & 0 \\ 0 & 0 & B \end{pmatrix}.$$

The eigenvalues of problem (50) are:

$$\lambda_0 = 0 \qquad \text{(associated to } b = 0,\ \beta = 1 \text{ and the identity operator)}$$

$$\lambda_1 = 2(A - 2mR^2) \qquad \lambda_2 = 2(B - 2mR^2) \qquad \lambda_3 = 2B.$$

Take the case $B > A$, so that λ_1 is the smallest of the last three.

If $2mR^2 < A$ the smallest eigenvalue is λ_0 and is unique. The original axes are also the ones that minimize the functional. The rotations V_1, V_2 and V_3 are of 180° respectively about the first, second and third cartesian axis; V_3 maximizes the functional.

If $2mR^2 > A$, the smallest eigenvalue is λ_1 and is again unique. The rotation V_1 of 180° about the axis of symmetry of the central body brings the displacements of the concentrated masses of the appendages back to zero, but displaces all the masses of the central body. This operation achieves the relative configuration minimizing the functional.

In the limiting case $2mR^2 = A$, $\lambda_0 = \lambda_1 = 0$ and the smallest eigenvalue is a doublet. The functional remains invariant and minimum under rotations of arbitrary amplitude about the axis of symmetry.

8. DISCRETIZATION OF BODY FLEXIBILITY

A modal analysis of the small amplitude vibrations of the free body produces the following expansion for small relative displacements, in which the summation convention on repeated indices is used and e_{ipn} denotes the alternating tensor

$$u_i(x,t) = v_i(t) + e_{ipn}\alpha_p(t)x_n + q_\beta(t)f_i^\beta(x).$$

The unknowns are the rigid body translation amplitudes $v_i(t)$, the small rigid body rotation amplitudes $\alpha_p(t)$ and a denumerable set of vibration amplitudes $q_\beta(t)$.

As the set of functions describing the displacement field is complete, the expansion can be used even for large relative displacements, in which case large $\alpha_p(t)$ terms induce strains because of the non linear terms in the exact strain measures (6). However, the minimizing conditions (37) and (38) are precisely satisfied by keeping the q_β terms alone.

Indeed the modal functions $f_i^\beta(x)$ have the properties

$$\int f_i^\beta \,\mathrm{d}\mu = 0 \qquad (i = 1,2,3) \text{ all } \beta \tag{55}$$

$$e_{jqi}\int x_q f_i^\beta \,\mathrm{d}\mu = 0 \qquad (j = 1,2,3) \text{ all } \beta \tag{56}$$

expressing their inertial orthogonality with respect to the small rigid body modes (which are natural modes of zero frequency).

Thus, considering that the origin of the reference configuration is at the center of mass,

$$\int x_n \,\mathrm{d}\mu = 0 \qquad (n = 1,2,3)$$

the minimizing conditions (37)

$$\int u_i(x,t)\,\mathrm{d}\mu = 0 \qquad (i = 1,2,3)$$

reduce to

$$v_i(t)\int \mathrm{d}\mu = 0 \qquad (i = 1,2,3)$$

and are satisfied by setting $v_i(t) = 0$. The minimizing conditions (38)

$$e_{jqi}\int x_q u_i\,\mathrm{d}\mu = 0 \qquad (j = 1,2,3)$$

reduce to

$$\alpha_p(t)e_{ipn}e_{jqi}\int x_n x_q\,\mathrm{d}\mu = \alpha_p(t)\int (x_n x_n \delta_{jp} - x_j x_p)\,\mathrm{d}\mu = 0.$$

The matrix of integrals is that of the inertia moments of the body in its reference configuration. It is positive definite, and the conditions can only be satisfied by taking $\alpha_p(t) = 0$.

Thus, an expansion of relative displacements limited to the natural modes of non zero frequency

$$u_i(x,t) = q_\beta(t) f_i^\beta(x) \tag{57}$$

satisfies automatically the principle of minimum square average of relative displacements and the equations of mean motion will be those associated to the corresponding choice of mean axes.

Another advantage of the discretization in natural modes is of course the existence of the orthogonality properties

$$\int f_i^\beta f_i^\gamma\,\mathrm{d}\mu = \delta^{\beta\gamma}\int \mathrm{d}\mu \tag{58}$$

the squared norm of a mode, or generalized mass, being here conventionally equated to the total mass of the body, and then, with $D_m = \partial/\partial x_m$

$$\int C_{mn}^{pq} D_m f_n^\beta D_p f_q^\gamma \frac{\mathrm{d}\mu}{\rho} = 0 \qquad \text{if } \gamma \neq \beta$$

$$= \lambda_{(\beta)}^2 \int \mathrm{d}\mu \qquad \text{if } \gamma = \beta. \tag{59}$$

The natural circular frequencies $\lambda_{(\beta)}$ are assumed to be ordered by increasing values. In practice, as interest is primarily centered on the low frequency response of the body, the expansion is truncated, leaving only a finite number of degrees of freedom.

9. INERTIA TERMS OF THE EQUATIONS OF MOTION

The expansion (57) is used to compute the absolute velocity(10)

$$v_{ai} = v_{gi} + e_{imn}\omega_m(x_n + q_\beta f_n^\beta) + \dot{q}_\beta f_i^\beta \tag{60}$$

and produces for the kinetic energy the expression

$$\begin{aligned} 2T &= \int v_{ai} v_{ai}\,\mathrm{d}\mu \\ &= (v_{gi}v_{gi} + \dot{q}_\beta \dot{q}_\beta)\int \mathrm{d}\mu + \omega_m\omega_m\Big(\int x_n x_n\,\mathrm{d}\mu + 2q_\beta F_{nn}^\beta + q_\beta q_\beta \int \mathrm{d}\mu\Big) \\ &\quad - \omega_m\omega_n\Big(\int x_m x_n\,\mathrm{d}\mu + 2q_\beta F_{mn}^\beta + q_\beta q_\gamma S_{mn}^{\beta\gamma}\Big) + 2A_m^{\gamma\beta}\omega_m \dot{q}_\beta q_\gamma. \end{aligned} \tag{61}$$

This formula contains three types of coupling coefficients resulting from the modal analysis

$$F_{mn}^\beta = \int x_m f_n^\beta\,\mathrm{d}\mu = F_{nm}^\beta \tag{62}$$

the symmetry with respect to the lower indices being a consequence of (56),

$$S^{\beta\gamma}_{mn} = \frac{1}{2}\int (f^{\beta}_{m} f^{\gamma}_{n} + f^{\beta}_{n} f^{\gamma}_{m})\, \mathrm{d}\mu \tag{63}$$

symmetrical in both pairs of indices and finally a set of skew symmetric matrices, governing the gyroscopic terms

$$A^{\gamma\beta}_{m} = e_{imn}\int f^{\gamma}_{n} f^{\beta}_{i}\, \mathrm{d}\mu = -A^{\beta\gamma}_{m}. \tag{64}$$

The inertia terms of the equations of motion follow from the computation of the variation

$$\delta\int_{t_1}^{t_2} T\, \mathrm{d}t = \int_{t_1}^{t_2}\left(\frac{\partial T}{\partial v_{gi}}\delta v_{gi} + \frac{\partial T}{\partial \omega_i}\delta\omega_i + \frac{\partial T}{\partial q_\beta}\delta q_\beta + \frac{\partial T}{\partial \dot{q}_\beta}\delta\dot{q}_\beta\right)\mathrm{d}t.$$

Substitution of (22) and (23) and integration by parts yield the following inertia terms: *Mean translation* (coefficient of δp_i under the integral sign)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial v_{gi}} + e_{mni}\omega_n \frac{\partial T}{\partial v_{gm}} \qquad (i = 1, 2, 3). \tag{65}$$

Mean rotation (coefficient of $\delta\alpha_i$)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \omega_i} + e_{mni}\omega_n \frac{\partial T}{\partial \omega_m} \qquad (i = 1, 2, 3). \tag{66}$$

Deformation mode of index β (coefficient of δq_β)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\beta} + \frac{\partial T}{\partial q_\beta}. \tag{67}$$

10. ELASTIC RESTORING TERMS OF THE EQUATIONS OF MOTION

Under the expansion in modes the deformation tensor (6) becomes

$$2\gamma_{mn} = q_\beta (D_m f^{\beta}_{n} + D_n f^{\beta}_{m}) + q_\beta q_\gamma D_m f^{\beta}_{i} D_n f^{\gamma}_{i} \tag{68}$$

and the strain energy (7), due account being taken of (59)

$$W = \frac{1}{2} q_\beta q_\beta \lambda^2_{(\beta)} \int \mathrm{d}\mu + \frac{1}{2} q_\beta q_\eta q_\zeta \Gamma^{\beta}_{\eta\zeta} + \frac{1}{8} q_\beta q_\gamma q_\eta q_\zeta \Gamma^{\beta\gamma}_{\eta\zeta}. \tag{69}$$

The following coupling coefficients were introduced

$$\Gamma^{\beta}_{\eta\zeta} = \int C^{pq}_{mn} D_m f^{\beta}_{n} D_p f^{\eta}_{i} D_q f^{\zeta}_{i} \frac{\mathrm{d}\mu}{\rho} = \Gamma^{\beta}_{\zeta\eta} \tag{70}$$

$$\Gamma^{\beta\gamma}_{\eta\zeta} = \int C^{pq}_{mn} D_m f^{\beta}_{i} D_n f^{\gamma}_{i} D_p f^{\eta}_{j} D_q f^{\zeta}_{j} \frac{\mathrm{d}\mu}{\rho} \tag{71}$$

the last one presenting the same type of symmetry as the elastic moduli

$$\Gamma^{\beta\gamma}_{\eta\zeta} = \Gamma^{\beta\gamma}_{\zeta\eta} = \Gamma^{\gamma\beta}_{\eta\zeta} = \Gamma^{\eta\zeta}_{\beta\gamma}. \tag{72}$$

The generalized elastic restoring forces appear in the deformation mode equations only as

$$-\frac{\partial W}{\partial q_\beta} = -\lambda^2_{(\beta)} q_\beta \int \mathrm{d}\mu - \frac{1}{2} q_\eta q_\zeta (\Gamma^{\beta}_{\eta\zeta} + 2\Gamma^{\zeta}_{\beta\eta}) - \frac{1}{4} q_\gamma q_\eta q_\zeta (\Gamma^{\eta\zeta}_{\beta\gamma} + \Gamma^{\gamma\eta}_{\zeta\beta}). \tag{73}$$

For small deformations only the first term needs to be retained.

11. GRAVITATIONAL TERMS OF THE EQUATIONS OF MOTION

In the case of the gravitational potential use is made of the fact that the body dimensions are usually small compared to a characteristic length of the gravitational gradient. A truncated Taylor expansion of the specific gravitational potential is then considered, centered at the origin of dynamic axis

$$G = G(\hat{a}) - \hat{g}_m \hat{r}_m - \frac{1}{2} \hat{H}_{mn} \hat{r}_m \hat{r}_n \tag{74}$$

where

$$\hat{g}_m(\hat{a}) = -\frac{\partial G(\hat{a})}{\partial \hat{a}_m} \tag{75}$$

is the gravitational acceleration at the origin, and

$$\hat{H}_{mn}(\hat{a}) = -\frac{\partial^2 G(\hat{a})}{\partial \hat{a}_m \partial \hat{a}_n} \tag{76}$$

the local gravity gradient tensor. In dynamical axes this becomes

$$G = G(\hat{a}) - g_m(x_m + u_m) - \frac{1}{2} H_{mn}(x_m + u_m)(x_n + u_n)$$

and, integrated over the mass of the body, produces a potential

$$P = G(\hat{a}) \int \mathrm{d}\mu - H_{mn}\left\{\frac{1}{2} \int x_m x_n \, \mathrm{d}\mu + q_\beta F^\beta_{mn} + \frac{1}{2} q_\beta q_\gamma S^{\beta\gamma}_{mn}\right\}. \tag{77}$$

Unlike the inertial and strain-energy forces, the gravitational forces have fixed orientations in the inertial axes; in other terms the dynamic components H_{mn} depend on the orientation U of the dynamic axes

$$H = U\hat{H}U^T$$

and, using (18),

$$\delta H = \delta U \hat{H} U^T + U \hat{H} \delta U^T = -[\delta\alpha]H - H[\delta\alpha]^T$$

or

$$\delta H_{mn} = -e_{mij}\delta\alpha_i H_{jn} - e_{nij}\delta\alpha_i H_{jm}.$$

On the other hand from (17)

$$\delta G(\hat{a}) = -\hat{g}_i \delta\hat{a}_i = -g_i \delta p_i.$$

The contributions of the gravitational forces to the equations of motion are then established as the respective coefficients of δp_i, $\delta\alpha_i$ and δq_β appearing in $-\delta P$.

Mean translation

$$g_i \int \mathrm{d}\mu \qquad (i = 1, 2, 3) \tag{78}$$

Mean rotation

$$e_{mij} H_{jn}\left\{\int x_m x_n \, \mathrm{d}\mu + 2q_\beta F^\beta_{mn} + q_\beta q_\gamma S^{\beta\gamma}_{mn}\right\} \qquad (i = 1, 2, 3) \tag{79}$$

Deformation mode of index β

$$H_{mn}\{F^{\beta}_{mn}+q_{\gamma}S^{\beta\gamma}_{mn}\}. \tag{80}$$

In aircraft applications, as long as the velocity of flight V is small compared to the orbital velocity $\sqrt{(R_0 g_0)}$ (R_0 mean earth radius, g_0 modulus of gravitational acceleration at this distance from the center of the earth) it is common practice to accept a "flat earth" approximation. The gravitational field is considered to be uniform and oriented as the third inertial axis. In this case, with the Euler angles previously introduced, the dynamical components g_i are

$$-g_0\sin\theta,\qquad g_0\sin\phi\cos\theta,\qquad g_0\cos\phi\cos\theta$$

along the roll, pitch and yaw axes respectively.

In satellite applications the inertial axes are usually centered on the attracting body and oriented towards "fixed stars". Let R be the distance at which the gravitational acceleration has a known modulus g_R; then, neglecting harminics, the potential of a unit mass is

$$G(\hat{a}) = -g_R\frac{R^2}{\rho}\qquad \rho=\sqrt{(\hat{a}^T\hat{a})}=\sqrt{(a^Ta)}$$

From which follows easily

$$g_i = -\frac{a_i}{\rho^3}g_R R^2\qquad H_{mn}=\left\{\frac{3}{\rho^5}a_m a_n-\frac{1}{\rho^3}\delta_{mn}\right\}g_R R^2$$

12. THRUST FOLLOWER FORCES

Gravitational forces are so-called "dead" loads; they have components determined in inertial space and oriented independently of the deformation of the body. Propulsion forces generated by air-breathing engines or rocket thrusters are, generally speaking, "followers".

Attached rigidly to a rigid body their components remain fixed with respect to the dynamic axes. Mounted flexibly on a flexible body they are moreover influenced by the deformations. To take this into account, we assume that the thrust axis of a given propulsion unit passes through a given material point x of the body and is oriented by the local material rotation prevailing at this point.

Let $\mathrm{d}x$ denote a differential step taken from the point x of the thrust axis in the direction of the thrust in the reference configuration, so that

$$n=\frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x^T\mathrm{d}x)}}$$

are the direction cosines of the thrust vector f in the reference configuration. In the deformed state the convected unit vector will be

$$n'=\frac{\mathrm{d}x+\mathrm{d}u}{\sqrt{[(\mathrm{d}x+\mathrm{d}u)^T(\mathrm{d}x+\mathrm{d}u)]}}.$$

Introduce the matrix A of displacement gradients, as defined in (8)

$$\mathrm{d}u = A\,\mathrm{d}x$$

and

$$n'=\frac{(I+A)\,\mathrm{d}x}{\sqrt{(\mathrm{d}x^T\,\mathrm{d}x+2\,\mathrm{d}x^T\Gamma\,\mathrm{d}x)}}.$$

Under the assumption that the strains remain very small $2\,\mathrm{d}x^T\Gamma\,\mathrm{d}x$ is negligible before $\mathrm{d}x^T\,\mathrm{d}x$

and

$$n' = (I + A)n.$$

In other words the local Jacobian matrix, governing the local neighborhood transformation, represents with a good approximation the local rotation operator. The dynamic components of the thrust in the deformed configuration are thus given by

$$f' = (I + A)f.$$

The virtual work, computed from the general formula developed in Section 4, is now expressible in terms of the known thrust vector of the reference state

$$(\delta p^T + \delta\alpha^T[x + u] + \delta u^T)(I + A)f.$$

From this the contributions of a thrust follower to the different equations of motion is obtained; it requires the computation of the deformation modes and their derivatives at the local attachment point.

Mean translation

$$(\delta_{mi} + q_\beta D_m f_i^\beta)f_m \qquad (i = 1, 2, 3). \tag{81}$$

Mean rotation

$$e_{pin}(x_n + q_\beta f_n^\beta)(\delta_{pm} + q_\gamma D_m f_p^\gamma)f_m \qquad (i = 1, 2, 3). \tag{82}$$

If the deformations are small enough, only the linear terms in the amplitudes q_β may be retained.

Deformation mode of index β

$$f_i^\beta(\delta_{im} + q_\gamma D_m f_i^\gamma)f_m. \tag{83}$$

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