Wavelet series representations for pathwise Young integrals

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Workshop on Complex Analysis and Operators Theory
Valencia, 27 – 29 October 2016

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Introduction

We are interested in the approximation of the Young integral

$$Y(t) := \int_0^t \sigma(s)\,\mathrm{d}X(s), \quad t \in I := [0,1]$$

General assumptions. There are $\alpha, \beta \in (0,1)$ such that for any compact interval $K \subset \mathbb{R}$,

$$\sigma \in C^\alpha(K)\,, \quad X \in C^\beta(K) \quad \text{and} \quad \alpha + \beta > 1.$$

Definition

For any $\theta \in [0,1)$, the Hölder space $C^\theta(K)$ is the Banach space of continuous functions $f : K \to \mathbb{R}$ such that

$$\|f\|_{C^\theta(I)} := \|f\|_{K,\infty} + \sup_{(x_1,x_2)\in K^2,\, x_1<x_2} \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\theta} < +\infty.$$

The Hölder conditions give the existence of $\zeta_t \in \mathbb{R}$ such that for any sequence

$$(\mathcal{D}_n)_{n\in\mathbb{Z}_+} = \Big(\{\delta_0^n, \delta_1^n, \ldots, \delta_{r_n}^n \,:\; r_n \in \mathbb{Z}_+,\; 0 = \delta_0^n < \delta_1^n < \ldots < \delta_{r_n}^n = t\}\Big)_{n\in\mathbb{Z}_+}$$

of partitions of the interval I for which $|\mathcal{D}_n| \to 0$, the Riemann-Stieltjes sum

$$\sum_{i=1}^{r_n} \sigma(\delta_{i-1}^n)\big(X(\delta_i^n) - X(\delta_{i-1}^n)\big)$$

converges to ζ_t. Therefore, one can define the integral $\int_0^t \sigma(s)\,\mathrm{d}X(s)$ by setting

$$\boxed{\int_0^t \sigma(s)\,\mathrm{d}X(s) := \zeta_t.}$$

Young - Loeve inequalities. There is a constant $\mathcal{K}_{\alpha+\beta} > 0$ such that for any $t_1 < t_2$,

$$\Big|\int_{t_1}^{t_2} \sigma(s)\,\mathrm{d}X(s) - \sigma(t_1)\big(X(t_2) - X(t_1)\big)\Big| \\ \leq \mathcal{K}_{\alpha+\beta}\|\sigma\|_{C^\alpha([t_1,t_2])}\|X\|_{C^\beta([t_1,t_2])}(t_2 - t_1)^{\alpha+\beta}.$$

In particular, $Y \in C^\beta(K)$ for any compact interval $K \subset \mathbb{R}$.

Approximation via Riemann sums

For $j \in \mathbb{Z}_+$ and $k \in \{0, \ldots, 2^j - 1\}$, it is natural to approximate

$$Y\left(\frac{k}{2^j}\right) = \int_0^{\frac{k}{2^j}} \sigma(s)\,\mathrm{d}X(s) = \sum_{l=0}^{k-1} \int_{\frac{l}{2^j}}^{\frac{l+1}{2^j}} \sigma(s)\,\mathrm{d}X(s)$$

by

$$Y_j\left(\frac{k}{2^j}\right) := \sum_{l=0}^{k-1} \sigma\left(s_{j,l}\right) \underbrace{\left(X\left(\frac{l+1}{2^j}\right) - X\left(\frac{l}{2^j}\right)\right)}_{:=\Delta_{j,l}(X) \text{ increments of order 1 of } X}, \quad s_{j,l} \in \left[\frac{l}{2^j}, \frac{l+1}{2^j}\right]$$

The Young-Loeve inequalities directly give

$$\begin{aligned} \left|Y\left(\frac{k}{2^j}\right) - Y_j\left(\frac{k}{2^j}\right)\right| &\leq \sum_{l=0}^{k-1} \left|\int_{\frac{l}{2^j}}^{\frac{l+1}{2^j}} \sigma(s)\,\mathrm{d}X(s) - \sigma\left(s_{j,l}\right)\Delta_{j,l}(X)\right| \\ &\leq \sum_{l=0}^{k-1} c_0 2^{-j(\alpha+\beta)} \leq c_0 2^{-j(\alpha+\beta-1)} \end{aligned}$$

Using linear interpolation, one gets for every $j \in \mathbb{Z}_+$, a function Y_j^{RS} which approximates Y :

$$Y_j^{RS}(t) := (2^j t - [2^j t])\sigma\left(s_{j,[2^j t]}\right)\Delta_{j,[2^j t]}(X) + Y_j\left(\frac{[2^j t]}{2^j}\right)$$

Proposition

There exists a constant $c > 0$ such that for all $\gamma \in [0, \beta)$ and $j \in \mathbb{Z}_+$, one has

$$\|Y - Y_j^{RS}\|_{C^\gamma(I)} \leq c 2^{-j \min(\beta-\gamma, \alpha+\beta-1)}. \tag{1}$$

Question. Is it possible to find approximation procedures for Y allowing to have better rates of convergence than the one provided by (1) ?

Content of the talk.

- The wavelet approximation (and the particular case of the Haar wavelet)
- Better rate of convergence under some Gaussian assumptions
- Examples of processes satisfying this assumption
- Discussion of the optimality of the improved rate of convergence

The wavelet approximation

We assume that the collection of functions, from $\mathbb{R}$ to itself,

$$\{\varphi(\cdot - l) : l \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot - k) : (j,k) \in \mathbb{Z}_+ \times \mathbb{Z}\}$$

satisfies one of the following two hypotheses :

($\mathcal{H}_1$) This collection is the Haar basis of $L^2(\mathbb{R})$, i.e.

$$\varphi := \mathbf{1}_{[0,1)} \quad \text{and} \quad \psi := \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}.$$

($\mathcal{H}_2$) This collection is an arbitrary compactly supported orthonormal wavelet basis of $L^2(\mathbb{R})$ such that the scaling function φ and the mother wavelet ψ are α-Hölder continuous on $\mathbb{R}$, i.e.

$$\sup_{(x_1,x_2)\in\mathbb{R}^2, x_1<x_2} \left\{ \frac{|\varphi(x_1) - \varphi(x_2)| + |\psi(x_1) - \psi(x_2)|}{|x_1 - x_2|^\alpha} \right\} < +\infty.$$

Definition

A multiresolution analysis of $L^2(\mathbb{R})$ is an increasing sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties :

- $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
- for every $j \in \mathbb{Z}$, $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$,
- for every $k \in \mathbb{Z}$, $f \in V_0$ if and only if $f(\cdot - k) \in V_0$,
- there exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ form an orthonormal basis of V_0.

For every $j \in \mathbb{Z}_+$, let W_j be the closed subspace of V_{j+1} such that $V_{j+1} = V_j \oplus W_j$. Then

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{j\in\mathbb{Z}_+} W_j \right)$$

and one can construct a function ψ whose translate form an orthonormal basis of W_0. Then, the functions $2^{j/2}\psi(2^j \cdot -k)$, $k \in \mathbb{Z}$, form an orthonormal basis of W_j.

For any fixed $t \in I$,

$$s \mapsto \sigma_t(s) := \sigma(s)\mathbf{1}_{[0,t]}(s)$$

belongs to $L^2(\mathbb{R})$. So,

$$\sigma_t = \sum_{l=-\infty}^{+\infty} b_{0,l}(t)\varphi(\cdot - l) + \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{j,k}(t) 2^{j/2} \psi(2^j \cdot - k)$$

which converges in $L^2(\mathbb{R})$, where

$$b_{0,l}(t) := \int_0^t \sigma(s)\varphi(s-l)\,\mathrm{d}s$$

and

$$a_{j,k}(t) := 2^{j/2} \int_0^t \sigma(s)\psi(2^j s - k)\,\mathrm{d}s.$$

For any fixed $t \in I$,

$$s \mapsto \sigma_t(s) := \sigma(s)\mathbf{1}_{[0,t]}(s)$$

belongs to $L^2(\mathbb{R})$. So, if supp $\varphi \subseteq [N_1, N_2]$ and supp $\psi \subseteq [N_1, N_2]$

$$\sigma_t = \sum_{l=1-N_2}^{[t]-N_1} b_{0,l}(t)\varphi(\cdot - l) + \sum_{j=0}^{+\infty} \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t) 2^{j/2} \psi(2^j \cdot -k)$$

which converges in $L^2(\mathbb{R})$, where

$$b_{0,l}(t) := \int_0^t \sigma(s)\varphi(s-l)\,\mathrm{d}s$$

and

$$a_{j,k}(t) := 2^{j/2} \int_0^t \sigma(s)\psi(2^j s - k)\,\mathrm{d}s.$$

$$\boxed{\sigma_t = \sum_{l=1-N_2}^{[t]-N_1} b_{0,l}(t)\varphi(\cdot - l) + \sum_{j=0}^{+\infty} \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t) 2^{j/2} \psi(2^j \cdot - k)}$$

For any $J \in \mathbb{N}$, we consider the partial sum

$$\sigma_{t,J} := \sum_{l=1-N_2}^{[t]-N_1} b_{0,l}(t)\varphi(\cdot - l) + \sum_{j=0}^{J-1} \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t) 2^{j/2} \psi(2^j \cdot - k)$$

Note that supp $\sigma_{t,J} \subseteq [Q_1, Q_2]$, where Q_1, Q_2 are independent of $t \in I$ and $J \in \mathbb{Z}_+$.

For any $t \in I$ and all $J \in \mathbb{N}$, one sets

$$\begin{aligned} Y_J^W(t) &:= \int_{Q_1}^{Q_2} \sigma_{t,J}(s)\,\mathrm{d}X(s) \\ &= \sum_{l=1-N_2}^{[t]-N_1} b_{0,l}(t) \int_{Q1}^{Q_2} \varphi(s-l)\,\mathrm{d}X(s) \\ &\qquad + \sum_{j=0}^{J-1} \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t) 2^{j/2} \int_{Q_1}^{Q_2} \psi(2^j s - k)\,\mathrm{d}X(s). \end{aligned}$$

Particular case of the Haar basis

In this case,

$$\varphi := \mathbf{1}_{[0,1)} \quad \text{and} \quad \psi := \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}.$$

Note that one has

$$\begin{aligned} \sigma_{t,J} &= b_{0,0}(t)\mathbf{1}_{[0,1)} + \sum_{j=0}^{J-1} \sum_{k=0}^{[2^j t]-1} a_{j,k}(t) 2^{j/2} \left(\mathbf{1}_{\left[\frac{k}{2^j}, \frac{2k+1}{2^{j+1}}\right)} - \mathbf{1}_{\left[\frac{2k+1}{2^{j+1}}, \frac{k+1}{2^j}\right)} \right) \\ &= \sum_{k=0}^{[2^J t]-1} b_{J,k}(t) 2^{J/2} \mathbf{1}_{\left[\frac{k}{2^J}, \frac{k+1}{2^J}\right)} \end{aligned}$$

where

$$b_{J,k}(t) := 2^{J/2} \int_0^t \sigma(s) \mathbf{1}_{\left[\frac{k}{2^J}, \frac{k+1}{2^J}\right)}(s)\, \mathrm{d}s.$$

Consequently,

$$Y_J^W(t) = \int_{Q_1}^{Q_2} \sigma_{t,J}(s)\, \mathrm{d}X(s) = \sum_{k=0}^{[2^J t]-1} b_{J,k}(t) 2^{J/2} \Delta_{J,k}(X).$$

$$Y_J^W(t) = \sum_{k=0}^{[2^J t]-1} b_{J,k}(t) 2^{J/2} \Delta_{J,k}(X) \,, \quad b_{J,k}(t) := 2^{J/2} \int_0^t \sigma(s) \mathbf{1}_{[\frac{k}{2^J}, \frac{k+1}{2^J})}(s) \, \mathrm{d}s$$

Remarks.

- This approximation procedure can be connected to the one with Riemann sums : Also assume that the $s_{J,l}$ used in the Riemann approximation are chosen so that

  $$\sigma(s_{J,l}) = 2^J \int_{2^{-J} l}^{2^{-J}(l+1)} \sigma(s) \, \mathrm{d}s \,, \quad \text{for every } l \in \{0, \dots, 2^J - 1\}.$$

  Then, one has $Y_J^W(2^{-J} l) = Y_J^{RS}(2^{-J} l)$.

- The same holds for the others wavelet basis :

  $$Y_J^W(t) = \sum_{k=1-N_2}^{[2^J t]-N_1} b_{J,k}(t) 2^{J/2} \int_{Q_1}^{Q_2} \varphi(2^J s - k) \, \mathrm{d}X(s)$$

  where

  $$b_{J,k}(t) := 2^{J/2} \int_0^t \sigma(s) \varphi(2^J s - k) \, \mathrm{d}s$$

Theorem

There is a constant $c > 0$ such that, for all $\gamma \in [0, \beta)$ and $J \in \mathbb{N}$, one has

$$\|Y - Y_J^W\|_{C^\gamma(I)} \leq c\, 2^{-J \min(\beta-\gamma, \alpha+\beta-1)}.$$

Key of the proof. For each $J \in \mathbb{N}$ and $t_1, t_2 \in I$ satisfying $t_1 < t_2$, we introduce

$$\mathbb{L}_{J,t_1,t_2} := \left\{ l \in \{1 - N_2, \dots, 2^J - N_1\} : \left[\frac{l + N_1}{2^J}, \frac{l + N_2}{2^J}\right] \subseteq [t_1, t_2] \right\},$$

and

$$\partial\mathbb{L}_{J,t_1,t_2} := \left\{ l \in \{1 - N_2, \dots, 2^J - N_1\} : l \notin \mathbb{L}_{J,t_1,t_2} \text{ and } \left[\frac{l + N_1}{2^J}, \frac{l + N_2}{2^J}\right] \cap [t_1, t_2] \neq \emptyset \right\}.$$

Note that there is $C > 0$ J, t_1 and t_2, such that

$$\mathrm{card}(\mathbb{L}_{J,t_1,t_2}) \leq C 2^J |t_1 - t_2| \quad \text{and} \quad \mathrm{card}(\partial\mathbb{L}_{J,t_1,t_2}) \leq C.$$

One has

$$Y_J^W(t) = \sum_{l=1-N_2}^{[2^J t]-N_1} b_{J,l}(t) 2^{J/2} \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \varphi(2^J s - l)\, \mathrm{d}X(s)$$

and

$$b_{J,l}(t_2) - b_{J,l}(t_1) = 2^{J/2} \int_{t_1}^{t_2} \sigma(s) \varphi(2^J s - l)\, \mathrm{d}(s)$$

Therefore, one gets

$$Y_J^W(t_2) - Y_J^W(t_1) = \sum_{l \in \mathbb{L}_{J,t_1,t_2}} \overline{\sigma}_{J,l} \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \varphi(2^J s - l)\, \mathrm{d}X(s)$$
$$+ \sum_{l \in \partial \mathbb{L}_{J,t_1,t_2}} 2^J \int_{t_1}^{t_2} \sigma(s) \varphi(2^J s - l)\, \mathrm{d}s \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \varphi(2^J s - l)\, \mathrm{d}X(s)$$

where

$$\overline{\sigma}_{J,l} := 2^J \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \sigma(s) \varphi(2^J s - l)\, \mathrm{d}s.$$

Moreover, it is known that integer translates of φ form "a partition of unity" in the sense that

$$\sum_{l=-\infty}^{+\infty} \varphi(x-l) = 1, \quad \text{for all } x \in \mathbb{R}.$$

Consequently,

$$\begin{aligned}
Y(t_2) - Y(t_1) &= \int_{t_1}^{t_2} \sigma(s)\,\mathrm{d}X(s) = \int_{t_1}^{t_2} \sigma(s)\Big(\sum_{l=-\infty}^{+\infty} \varphi(2^J s - l)\Big)\,\mathrm{d}X(s) \\
&= \int_{t_1}^{t_2} \sigma(s)\Big(\sum_{l\in\mathbb{L}_{J,t_1,t_2}} \varphi(2^J s - l) + \sum_{l\in\partial\mathbb{L}_{J,t_1,t_2}} \varphi(2^J s - l)\Big)\,\mathrm{d}X(s) \\
&= \sum_{l\in\mathbb{L}_{J,t_1,t_2}} \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \sigma(s)\varphi(2^J s - l)\,\mathrm{d}X(s) \\
&\quad + \sum_{l\in\partial\mathbb{L}_{J,t_1,t_2}} \int_{t_1}^{t_2} \sigma(s)\varphi(2^J s - l)\,\mathrm{d}X(s)\,.
\end{aligned}$$

Next, one gets that

$$\left|Y(t_2) - Y(t_1) - Y_J^W(t_2) + Y_J^W(t_1)\right| \leq \mathcal{A}_J^{(1)}(t_1, t_2) + \mathcal{A}_J^{(2)}(t_1, t_2),$$

where

$$\mathcal{A}_J^{(1)}(t_1, t_2) := \sum_{l \in \mathbb{L}_{J, t_1, t_2}} \left| \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \left(\sigma(s) - \overline{\sigma}_{J,l}\right) \varphi(2^J s - l) \, \mathrm{d}X(s) \right|$$

and

$$\mathcal{A}_J^{(2)}(t_1, t_2) := \sum_{l \in \partial\mathbb{L}_{J, t_1, t_2}} \left| \int_{t_1}^{t_2} \sigma(s) \varphi(2^J s - l) \, \mathrm{d}X(s) \right| + \left| 2^J \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \varphi(2^J s - l) \, \mathrm{d}X(s) \int_{t_1}^{t_2} \sigma(s) \varphi(2^J s - l) \, \mathrm{d}s \right|.$$

Lemma

($\mathcal{P}_1$) There is a constant $c_2 > 0$ such that, for every $J \in \mathbb{N}$ and every $l \in \{1 - N_2, \dots, 2^J - N_1\}$, one has

$$\Big| \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \big(\sigma(s) - \overline{\sigma}_{J,l}\big)\varphi(2^J s - l)\,\mathrm{d}X(s) \Big| \leq c_2\, 2^{-J(\alpha+\beta)}.$$

($\mathcal{P}_2$) There is a constant $c_1 > 0$ such that, for every $J \in \mathbb{N}$ and every $l \in \{1 - N_2, \dots, 2^J - N_1\}$, one has

$$\Big| \int_{2^{-J}(l+N_1)}^{2^{-J}(l+N_2)} \varphi(2^J s - l)\,\mathrm{d}X(s) \Big| \leq c_1 2^{-J\beta}.$$

($\mathcal{P}_3$) There is a constant $c_3 > 0$ such that, for every $t_1, t_2 \in I$ with $t_1 < t_2$, every $J \in \mathbb{N}$ and every $l \in \partial\mathbb{L}_{J,t_1,t_2}$, one has

$$\Big| \int_{t_1}^{t_2} \sigma(s)\varphi(2^J s - l)\,\mathrm{d}X(s) \Big| \leq c_3 \min\big(2^{-J\beta}, |t_1 - t_2|^{\beta}\big).$$

Better rate of convergence under the Gaussian condition ($\mathcal{G}$)

The Gaussian condition ($\mathcal{G}$)

- $\sigma \in C^{\alpha}(K)$ for any compact interval $K \subset \mathbb{R}$.
- $X := \{X(s)\}_{s\in\mathbb{R}}$ is a real-valued centered Gaussian process which is β_0-Hölder continuous in quadratic mean on any compact interval $K \subset \mathbb{R}$,

$$\mathbb{E}\left[\left|X(s_1) - X(s_2)\right|^2\right] \leq c|s_1 - s_2|^{2\beta_0} \quad \forall s_1, s_2 \in K$$

- $\alpha + \beta_0 > 1$.
- The wavelet coefficients

$$\lambda_{j,k} := 2^{j/2} \int_{2^{-j}(k+N_1)}^{2^{-j}(k+N_2)} \psi(2^j s - k)\,\mathrm{d}X(s)$$

have the following "short-range dependence" property :

$$\max_{1-N_2 \leq k_1 \leq 2^j - N_1} \left\{ \sum_{k_2=1-N_2}^{2^j - N_1} \left|\mathrm{Cov}\left[\lambda_{j,k_1}, \lambda_{j,k_2}\right]\right| \right\} \leq c 2^{-j(2\beta_0 - 1)}$$

Remarks.

- Using the equivalence of Gaussian moments and the Kolmogorov Hölder continuity theorem, one can derive that the paths of X belong to the Hölder spaces $C^\beta(K)$, for all $\beta \in (0, \beta_0)$ and all compact intervals K.

  $\longrightarrow$ The stochastic process

$$Y(t) = \int_0^t \sigma(s)\,\mathrm{d}X(s)$$

  can be defined pathwise and the previous result can be applied in this context. In particular, the stochastic processes Y_J^W converge to Y in $C^\beta(K)$.

- Using the fact that a pathwise Young integral is the limit of Riemann-Stieltjes sums, one can show that the processes $\{Y(t)\}_{t\in[0,1]}$, $\{\lambda_{j,k}\}_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}}$ and $\{Y_J^W(t)\}_{t\in[0,1]}$ are centered Gaussian processes.

- In the case of the Haar wavelets, the conditions of short-range dependence is a condition on the second order increments

$$\sum_{k_2=0}^{[2^j v]-1} \left|\mathrm{Cov}\left[\Delta^2_{j,k_1}(X), \Delta^2_{j,k_2}(X)\right]\right| \leq c 2^{-2j\beta_0}$$

This condition ($\mathcal{G}$) allows to improve the convergence rate :

Theorem

Under the condition ($\mathcal{G}$), for any fixed $\beta \in (1-\alpha, \beta_0)$ and non-negative real number $\gamma < \min(\beta, 1/2)$, there is a finite random constant $c > 0$ such that the inequality

$$\|Y - Y_J^W\|_{C^\gamma(I)} \leq c\, 2^{-J \min(\beta-\gamma, \alpha+\beta-1/2-\gamma)}$$

holds almost surely, for each $J \in \mathbb{N}$.

Lemma

It suffices to obtain the result for $\gamma = 0$, i.e. to prove that there is a finite random constant $c > 0$ such that the inequality

$$\|Y - Y_J^W\|_{I,\infty} \leq c\, 2^{-J \min(\beta, \alpha+\beta-1/2)}$$

holds almost surely, for each $J \in \mathbb{N}$.

Let us recall that

$$Y_J^W(t) = \sum_{l=1-N_2}^{[t]-N_1} b_{0,l}(t) \int_{Q1}^{Q_2} \varphi(s-l)\,\mathrm{d}X(s) + \sum_{j=0}^{J-1} \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t)\lambda_{j,k}$$

For every $j \in \mathbb{Z}_+$, we set

$$Z_j(t) := \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t)\lambda_{j,k}$$

so that

$$\|Y - Y_J^W\|_{I,\infty} = \|\sum_{j=J}^{+\infty} Z_j\|_{I,\infty} \leq \sum_{j=J}^{+\infty} \|Z_j\|_{I,\infty}$$

In order to get a rate of convergence of Y_J^W to Y, one has to estimate the norm $\|Z_j\|_{I,\infty}$.

Note that $\|Z_j\|_{I,\infty} := \sup_{t\in[0,1]} |Z_j(t)|$ is the supremum of infinitely many random variables. It is more convenient to work with a supremum of finite number of them.

Lemma

For each $j \in \mathbb{N}$, one sets $\nu(Z_j) := \sup_{l\in\{0,\dots,2^j\}} |Z_j(2^{-j}l)|$. Then, for any fixed $\beta \in (1-\alpha, \beta_0)$, one has almost surely

$$\sup_{j\in\mathbb{N}} \left\{ 2^{j\beta} \Big| \|Z_j\|_{I,\infty} - \nu(Z_j) \Big| \right\} < +\infty\,.$$

Idea. One has

$$\left|Z_j(t_0) - Z_j(2^{-j}[2^j t_0])\right| \leq \sum_{k=1-N_2}^{[2^j t_0]-N_1} \left|a_{j,k}(t_0) - a_{j,k}(2^{-j}[2^j t_0])\right| \underbrace{|\lambda_{j,k}|}_{\leq c2^{-(\beta-1/2)j}}$$

and

$$\left|a_{j,k}(t_0) - a_{j,k}(2^{-j}[2^j t_0])\right| \leq 2^{j/2}\left(t_0 - [2^j t_0]2^{-j}\right)\|\sigma\|_{I,\infty}\|\psi\|_{[N_1,N_2],\infty} \leq c2^{-j/2}$$

Notice that if one shows that

$$\sum_{j=1}^{+\infty} \mathbb{P}\Big(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j) > 1\Big) < +\infty\,,$$

then the Borel-Cantelli lemma entails that almost surely,

$$\sup_{j\in\mathbb{N}} \Big\{2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j)\Big\} < +\infty$$

and the Theorem will follows.

Using the Markov inequality, one has

$$\mathbb{P}\Big(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j) > 1\Big) \leq 2^{j\min(\beta,\alpha+\beta-1/2)}\,\mathbb{E}\left(\nu(Z_j)\right)$$

for every $j \in \mathbb{N}$

$\longrightarrow$ one has to estimate $\mathbb{E}\left(\nu(Z_j)\right)$

Lemma

There exists a universal deterministic finite constant $c > 0$, such that, for every centered Gaussian process $\{g_n\}_{n\in\mathbb{N}}$ and for all $N \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{1\le n\le N} |g_n|\right) \le c\big(1+\log N\big)^{\frac{1}{2}} \sup_{1\le n\le N} \Big(\mathbb{E}\left(|g_n|^2\right)\Big)^{\frac{1}{2}}.$$

The process $Z_j(t) := \sum_{k=1-N_2}^{[2^j t]-N_1} a_{j,k}(t)\lambda_{j,k}$ is Gaussian and centered. Consequently, for all $j \in \mathbb{N}$, one has

$$\mathbb{E}\left(\nu(Z_j)\right) \le c\big(2+j\big)^{\frac{1}{2}} \sup_{l\in\{0,\dots,2^j\}} \Big(\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right)\Big)^{\frac{1}{2}},$$

and the problem is now the computation of $\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right)$. One has

$$\mathbb{E}\left[|Z_j(2^{-j}l)|^2\right] = \sum_{k_1=1-N_2}^{l-N_1} \sum_{k_2=1-N_2}^{l-N_1} a_{j,k_1}(2^{-j}l)a_{j,k_2}(2^{-j}l)\mathrm{Cov}\left[\lambda_{j,k_1},\lambda_{j,k_2}\right].$$

$$\mathbb{E}\left[|Z_j(2^{-j}l)|^2\right] = \sum_{k_1=1-N_2}^{l-N_1} \sum_{k_2=1-N_2}^{l-N_1} a_{j,k_1}(2^{-j}l)a_{j,k_2}(2^{-j}l)\mathrm{Cov}\left[\lambda_{j,k_1}, \lambda_{j,k_2}\right].$$

Since σ is α-Hölder continuous, it is easy to see that there is a finite constant $c > 0$ such that, for every $t \in I$ and $j \in \mathbb{N}$, one has

$$\left|a_{j,k}(t)\right| \leq c2^{-j(\alpha+\frac{1}{2})}, \quad \text{for all } k \in \mathbb{L}_{j,0,t}\,,$$

and

$$\left|a_{j,k}(t)\right| \leq c2^{-\frac{j}{2}}, \quad \text{for all } k \in \partial\mathbb{L}_{j,0,t}\,.$$

Morever, condition ($\mathcal{G}$) gives that

$$\max_{1-N_2 \leq k_1 \leq 2^j - N_1} \left\{ \sum_{k_2=1-N_2}^{2^j-N_1} \left|\mathrm{Cov}\left[\lambda_{j,k_1}, \lambda_{j,k_2}\right]\right| \right\} \leq c_2 2^{-j(2\beta_0-1)}$$

Therefore, computing the cardinality of $\mathbb{L}_{j,0,t}$ and $\partial\mathbb{L}_{j,0,t}$, one gets that

$$\sup_{j\in\mathbb{N}} \sup_{l\in\{0,\ldots,2^j\}} \left\{ 2^{2j\min(\beta_0,\alpha+\beta_0-1/2)}\,\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right)\right\} < +\infty\,.$$

Total. We have proved that

$$\sup_{j\in\mathbb{N}} \sup_{l\in\{0,\dots,2^j\}} \left\{2^{2j\min(\beta_0,\alpha+\beta_0-1/2)}\,\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right)\right\} < +\infty\,.$$

Since

$$\mathbb{E}\left(\nu(Z_j)\right) \le c(2+j)^{\frac{1}{2}} \sup_{l\in\{0,\dots,2^j\}} \left(\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right)\right)^{\frac{1}{2}},$$

and

$$\mathbb{P}\left(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j) > 1\right) \le 2^{j\min(\beta,\alpha+\beta-1/2)}\,\mathbb{E}\left(\nu(Z_j)\right),$$

we get that $\mathbb{P}\left(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j) > 1\right)$ is the general term of a convergent series.

Consequently, the Borel-Cantelli lemma gives that almost surely,

$$\sup_{j\in\mathbb{N}} \left\{2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j)\right\} < +\infty$$

hence the result.

Examples of processes satisfying the condition ($\mathcal{G}$)

We consider stationary increments real-valued centered Gaussian processes $X := \{X(s)\}_{s\in\mathbb{R}}$ having, for all $(s_1, s_2) \in \mathbb{R}^2$, a covariance function of the following general form :

$$\begin{aligned} \mathrm{Cov}\,[X(s_1), X(s_2)] &= \mathbb{E}\,[X(s_1)X(s_2)] \\ &= \int_{-\infty}^{+\infty} (e^{is_1\xi} - 1)(e^{-is_2\xi} - 1) f(\xi)\,\mathrm{d}\xi\,, \end{aligned}$$

where the measurable nonnegative even function f satisfies the integrability condition :

$$\int_{-\infty}^{+\infty} \min\left(1, \xi^2\right) f(\xi)\,\mathrm{d}\xi < +\infty.$$

Example. The fractional Brownian motion : up to a multiplicative constant, $f(\xi) = |\xi|^{-2H-1}$ for all $\xi \neq 0$, where $H \in (0,1)$ denotes the Hurst parameter.

We will see how the conditions given by $(\mathcal{G})$ can be transformed into conditions on the function f.

Proposition

A sufficient condition for the process X to be β_0-Hölder continuous in quadratic mean is that there exist two positive finite deterministic constants c and ξ_0, such that

$$f(\xi) \leq c|\xi|^{-2\beta_0 - 1}, \quad \text{for almost all } \xi \in (-\infty, -\xi_0) \cup (\xi_0, +\infty).$$

Example. It is satisfied by the fractional Brownian motion of index $H = \beta_0$.

Proposition

The condition of "short-range dependence" holds as soon as f is twice continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfies the following condition :

($\mathcal{D}_1$) In the case of the Haar basis : there exist two finite deterministic constants $\beta_0' \in [\beta_0, 1)$ and $c > 0$ such that, for all $n \in \{0, 1, 2\}$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$\left|f^{(n)}(\xi)\right| \leq c \max\left(|\xi|^{-2\beta_0 - n - 1}, |\xi|^{-2\beta_0' - n - 1}\right).$$

($\mathcal{D}_M$) If the wavelet ψ is continuously differentiable on the real line and has at least M vanishing moments, that is

$$\int_{-\infty}^{+\infty} s^m \, \psi(s) \, \mathrm{d}s = 0, \quad \text{for all } m \in \{0, \ldots, M-1\} :$$

There exist two finite deterministic constants $\beta_0' \in [\beta_0, 1)$ and $c > 0$ such that, for all $n \in \{0, 1, 2\}$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has

$$\left|f^{(n)}(\xi)\right| \leq c \max\left(|\xi|^{-2\beta_0 - n - 1}, |\xi|^{-2\beta_0' - nM - 1}\right).$$

Example. Clearly, $(\mathcal{D}_1)$ and $(\mathcal{D}_M)$, for any $M \geq 1$, hold when $f(\xi) = |\xi|^{-2H-1}$, the two constants β_0 and β_0' being arbitrary and such that $0 < \beta_0 \leq H \leq \beta_0' < 1$.

Remarks.

- $(\mathcal{D}_M)$ is weaker than $(\mathcal{D}_{M'})$, for any $M' < M$.
- A major motivation for weakening the condition $(\mathcal{D}_1)$ to the condition $(\mathcal{D}_M)$ is the following : the behavior of the function f at low frequencies can then be much more singular, namely f can have infinitely many oscillations in the vicinity of 0. For instance, let $\tilde{f}_{u,v,w}$ be the function defined, for all $\xi \in \mathbb{R} \setminus \{0\}$, as

$$\tilde{f}_{u,v,w}(\xi) := |\xi|^{-2u-1} + |\xi|^{-2v-1} \sin^2\left(|\xi|^{-w}\right),$$

  where the three parameters u, v and w are arbitrary real numbers such that $0 < u \leq v < 1$ and $w > 0$. Observe that the larger is w the more oscillating is $\tilde{f}_{u,v,w}$ in the neighborhood of 0. This function fails to satisfy $(\mathcal{D}_1)$ but for any integer $M \geq 1 + w$, it satisfies $(\mathcal{D}_M)$ with $\beta_0 = u$ and $\beta_0' = v$.

Optimality of the improved rate of convergence

One denotes by $\overline{\alpha}$ and $\overline{\beta}$ the two critical exponents defined as

$$\overline{\alpha} := \sup\left\{\alpha \in [0,1) : \sigma \in C^{\alpha}(I)\right\} \text{ and } \overline{\beta} := \sup\left\{\beta \in [0,1) : X \in C^{\beta}(I)\right\}.$$

Proposition

Assume that $\overline{\alpha} \geq 1/2$, $\overline{\beta} < 1$, that the condition ($\mathcal{G}$) is satisfied for all $\beta_0 \in (1-\overline{\alpha}, \overline{\beta})$, and that the deterministic integrand σ vanishes nowhere on I. Then, for each fixed $\gamma \in [0, \min(\overline{\beta}, 1/2))$ and arbitrarily small $\epsilon > 0$, one has, almost surely,

$$\|Y - Y_J^W\|_{C^\gamma(I)} \asymp 2^{-J(\overline{\beta}-\gamma)},$$

i.e.

$$\sup_{J\in\mathbb{N}} \left\{2^{J(\overline{\beta}-\gamma-\epsilon)} \|Y - Y_J^W\|_{C^\gamma(I)}\right\} < +\infty$$

and

$$\sup_{J\in\mathbb{N}} \left\{2^{J(\overline{\beta}-\gamma+\epsilon)} \|Y - Y_J^W\|_{C^\gamma(I)}\right\} = +\infty.$$

Example. Assume that the integrator X is a fBm of an arbitrary Hurst parameter $H \in (0,1)$ and that the deterministic integrand σ is the positive (vanishing nowhere) Weierstrass type function defined as

$$\sigma(s) := \sigma_0 + \sum_{n=1}^{+\infty} b^{-na} \sin(b^n s) \quad \forall s \in \mathbb{R},$$

where a, b and σ_0 are parameters such that $a \in (0,1)$, $b > 1$ and $\sigma_0(b^a - 1) > 1$. Then, almost surely,

$$\overline{\alpha} = a \quad \text{and} \quad \overline{\beta} = H.$$

As soon as $a \geq 1/2$ and $a > 1 - H$, for each fixed $\gamma \in [0, \min(H, 1/2))$, one has

$$\|Y - Y_J^W\|_{C^\gamma(I)} \asymp 2^{-J(H-\gamma)}$$

References

A. Ayache, C. Esser and Q. Peng
Almost sure approximations in Hölder norms of a general stochastic process defined by a Young integral
Sumbitted for publication.

A. Bonami and A. Estrade.
Anisotropic Analysis of Some Gaussian Models.
J. Fourier Anal. Appl., 9 :215–236, 2003.

Z. Ciesielski, G. Kerkyacharian and B. Roynette.
Quelques Espaces Fonctionnels Associés à des processus Gaussiens.
Stud. Math., 107 :171–204, 1993.

I. Daubechies.
Ten Lectures on Wavelets.
SIAM Philadelphia, 1992.

M. Gubinelli, P. Imkeller and N. Perkowski.
A Fourier analytic approach to pathwise stochastic integration
Electron. J. Probab., 21 :1–37, 2016.

T. J. Lyons, M. Caruana and T. Lévy.
Differential Equations Driven by Rough Paths.
Springer. Editor : J. Picard, 2004.

Y. Meyer.
Wavelets and Operators
Cambridge University Press, 1992.