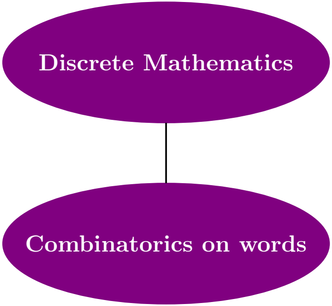
Generalized Pascal triangles for binomial coefficients of words: a short introduction

Joint work with Julien Leroy and Michel Rigo

Manon Stipulanti FRIA grantee

Sage Days 82 : Women in Sage
January 10, 2017

My research interests



Study of discrete structures

Study of words and formal languages

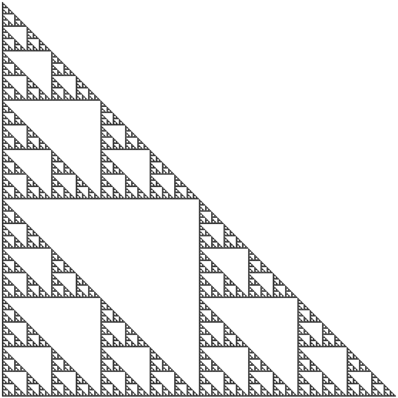
The classical Pascal triangle

		k							
		0	1	2	3	4	5	6	7
	0	1	0	0	0	0	0	0	0
	1	1	1	0	0	0	0	0	0
	2	1	2	1	0	0	0	0	0
m	3	1	3	3	1	0	0	0	0
	4	1	4	6	4	1	0	0	0
	5	1	5	10	10	5	1	0	0
	6	1	6	15	20	15	6	1	0
	7	1	7	21	35	35	21	7	1

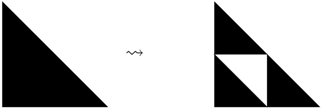
Usual binomial coefficients of integers:

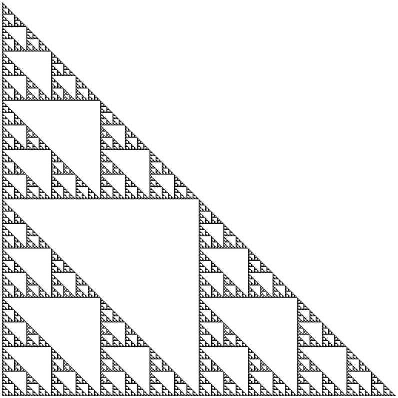
$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

The Sierpiński gasket



A way to build the Sierpiński gasket:





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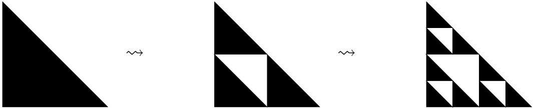
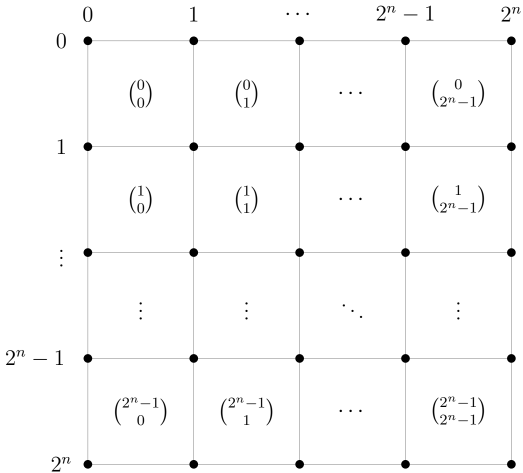


- Grid: intersection between $\mathbb{N}^2$ and $[0, 2^n] \times [0, 2^n]$



$\mathbb{N}^2 \cap [0, 2^n]^2$

- Color the grid:
  Color the first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0\leq m,k<2^n}$$

  in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
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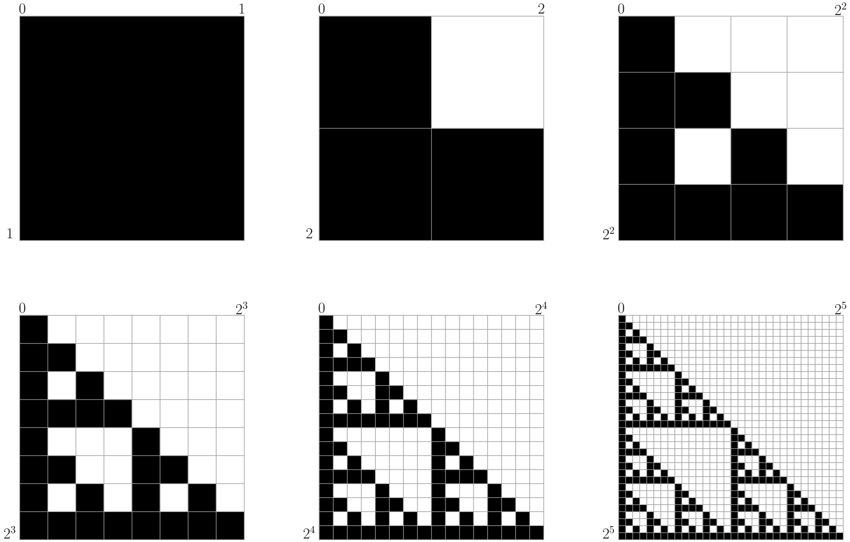
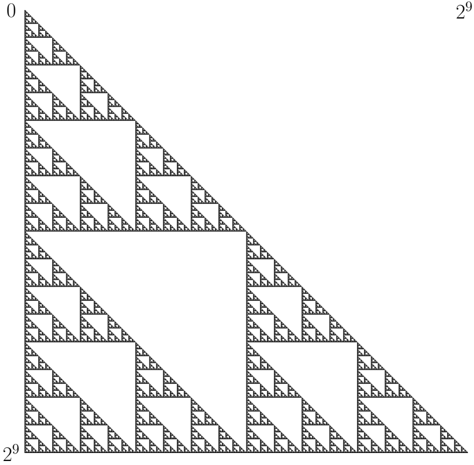
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$
  $\rightsquigarrow$ sequence belonging to $[0,1] \times [0,1]$

The first six elements of the sequence



The tenth element of the sequence



Folklore fact

This sequence converges to the Sierpiński gasket.

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called alphabet.

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

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Example: $u = 101001$ $\qquad v = 101$

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Example: $u = \mathbf{101}001$ $v = 101$ 1 occurrence

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Example: $u = \mathbf{10}100\mathbf{1}$ $\quad v = 101 \quad$ 2 occurrences

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Example: $u = 101001$ $v = 101$ 3 occurrences

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Example: $u = \mathbf{1}010\mathbf{01}$ $\quad v = 101 \quad$ 4 occurrences

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Example: $u = 10\mathbf{10}0\mathbf{1}$ $v = 101$ 5 occurrences

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Example: $u = 10\mathbf{1}\mathbf{0}0\mathbf{1}$ $\quad v = 101 \quad$ 6 occurrences

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Binomial coefficient of words

Let u, v be two finite words.
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Example: $u = 101001$ $\qquad v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:
Natural generalization of binomial coefficients of integers

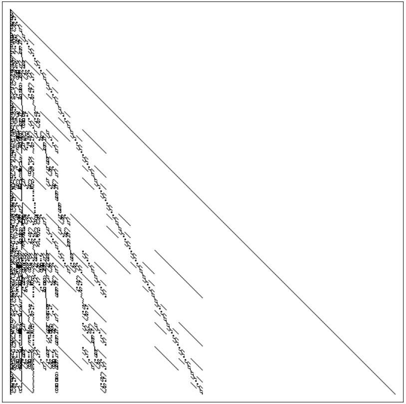
With a one-letter alphabet $\{a\}$

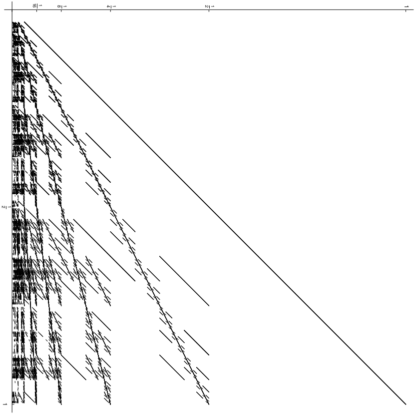
$$\binom{a^m}{a^k} = \binom{\overbrace{a\cdots a}^{m \text{ times}}}{\underbrace{a\cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m, k \in \mathbb{N}$$

What I do

Idea: replace binomial coefficients of **integers** by binomial coefficients of **words** and

- study a similar link
- extract specific sequences from generalized Pascal triangles and study their structural properties (automaticity, regularity, synchronicity, etc.)





What about Sage?

A lot of computations to test our results
$\rightsquigarrow$ usually *Mathematica*

Another way to test our results
$\rightsquigarrow$ become an independent user of *Sage*