On the Zero Defect Conjecture

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Abstract

Brlek et al. conjectured in 2008 that any fixed point of a primitive morphism with finite palindromic defect is either periodic or its palindromic defect is zero. Bucci and Vaslet disproved this conjecture in 2012 by a counterexample over ternary alphabet. We prove that the conjecture is valid on binary alphabet. We also describe a class of morphisms over multiliteral alphabet for which the conjecture still holds. The proof is based on properties of extension graphs.

Keywords: palindromic defect, marked morphism, special factor, extension graph
2000 MSC: 68R15, 37B10

1 Introduction

Palindromes — words read the same from the left as from the right — are a favorite pun in various languages. For instance, the words ressasser, ťahať, and šílíš are palindromic words in the first languages of the authors of this paper. The reason for a study of palindromes in formal languages is not only to deepen the theory, but it has also applications.

The theoretical reasons include the fact that a Sturmian word, i.e., an infinite aperiodic word with the least factor complexity, can be characterized using the number of palindromic factors of given length that occur in a word, see [10]. The application motives include the study of the spectra of discrete Schrödinger operators, see [12, 13].

In [9], the authors provide an elementary observation that a finite word of length n cannot contain more than $n+1$ (distinct) palindromic factors, including the empty word as a palindromic factor. We illustrate this on the following 2 examples of words of length 9:

$$w^{(1)} = 010010100 \quad \text{and} \quad w^{(2)} = 011010011.$$

The word $w^{(1)}$ is a prefix of the famous Fibonacci word and w_2 is a prefix of (also famous) Thue–Morse word. There are 10 palindromic factors of $w^{(1)}$: 0, 1, 00, 010, 101, 1001, 01010, 010010, 0010100, and the empty word. The word $w^{(2)}$ contains only 9 palindromes: 0, 1, 11, 0110, 101, 010, 00, 1001, and the empty word.

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The existence of the upper bound on the number of distinct palindromic factors lead to the definition of *palindromic defect* (or simply *defect*) of a finite word w, see [5], as the value

$$D(w) = n + 1 - \text{ the number of palindromic factors of } w$$

with n being the length of w. Our examples satisfy $D(w^{(1)}) = 0$, i.e., the upper bound is attained, and $D(w^{(2)}) = 1$. The notion of palindromic defect naturally extends to infinite words. For an infinite word $\mathbf{u}$ we set

$$D(\mathbf{u}) = \sup\{D(w) \colon w \text{ is a factor of } \mathbf{u}\}.$$

In this paper, we deal with infinite words that are generated by a primitive morphism of a free monoid $\mathcal{A}^*$ with $\mathcal{A}$ being a finite alphabet. A morphism φ is completely determined by the images of all letters $a \in \mathcal{A} \colon a \mapsto \varphi(a) \in \mathcal{A}^*$. A morphism is *primitive* if there exists a power k such that any letter $b \in \mathcal{A}$ appears in the word $\varphi^k(a)$ for any letter $a \in \mathcal{A}$.

The two mentioned infinite words can be generated using a primitive morphism. Consider the morphism φ_F over $\{0,1\}^*$ determined by $0 \mapsto 01$ and $1 \mapsto 0$. By repeated application of φ_F, starting from 0, we obtain

$$0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto 01001010\ldots$$

Since $\varphi_F^n(0)$ is a prefix of $\varphi_F^{n+1}(0)$ for all $n \in \mathbb{N}$, there exists an infinite word $\mathbf{u}_F$, called the Fibonacci word, such that $\varphi_F^n(0)$ is its prefix for all n. Consider a natural extension of φ_F to infinite words, we obtain that $\mathbf{u}_F$ is a fixed point of φ_F since

$$\mathbf{u}_F = \varphi_F(\mathbf{u}_F) = \varphi_F(u_0 u_1 u_2 \ldots) = \varphi_F(u_0)\varphi_F(u_1)\varphi_F(u_2)\ldots$$

where $u_i \in \{0,1\}$.

Similarly, let φ_{TM} be a morphism determined by $0 \mapsto 01$ and $1 \mapsto 10$. By repeated application of φ_{TM}, starting again from 0, we obtain

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110\ldots$$

The infinite word having $\varphi_{TM}^n(0)$ as a prefix for each n is the Thue–Morse word, sometimes also called Prouhet–Thue–Morse word.

The present article focuses on palindromic defect of infinite words which are fixed points of primitive morphisms. In order for the palindromic defect of such an infinite word to be finite, the word must contain an infinite number of palindromic factors. This property is satisfied by the two mentioned words $\mathbf{u}_F$ and $\mathbf{u}_{TM}$. However, for their palindromic defect, we have $D(\mathbf{u}_F) = 0$, whilst $D(\mathbf{u}_{TM}) = +\infty$.

There exist fixed points $\mathbf{u}$ of primitive morphisms with $0 < D(\mathbf{u}) < +\infty$, but on a two-letter alphabet, only ultimately periodic words are known. In [5], examples of such words are given by Brlek, Hamel, Nivat and Reutenauer as follows: for any $k \in \mathbb{Z}, k \geq 2$ denote by z the finite word

$$z = 01^k 01^{k-1} 001^{k-1} 01^k 0\,.$$

Then the infinite periodic word z^ω has palindromic defect k. Of course, the periodic word z^ω is fixed by the primitive morphism $0 \mapsto z, 1 \mapsto z$. In [4], the authors stated the following conjecture:

Conjecture (Zero Defect Conjecture). *If $\mathbf{u}$ is a fixed point of a primitive morphism such that $D(\mathbf{u}) < +\infty$, then $\mathbf{u}$ is periodic or $D(\mathbf{u}) = 0$.*

In 2012, Bucci and Vaslet [7] found a counterexample to this conjecture on a ternary alphabet. They showed that the fixed point of the primitive morphism determined by

$$a \mapsto aabcacba, b \mapsto aa, c \mapsto a$$

has finite positive palindromic defect and is not periodic.

In this article, we show that the conjecture is valid on a binary alphabet. Then we generalize the method used for morphisms on a binary alphabet to marked morphisms on a multiliteral alphabet. The main result of the article is the following theorem.

Theorem 1. *Let φ be a primitive marked morphism and let $\mathbf{u}$ be its fixed point with finite defect. If all complete return words of all letters in $\mathbf{u}$ are palindromes or there exists a conjugate of φ distinct from φ itself, then $D(\mathbf{u}) = 0$.*

Observe that in the case of primitive marked morphisms, as it was noted in [15, Cor. 30, Cor. 32], there is no ultimately periodic infinite word $\mathbf{u}$ fixed point of a primitive marked morphisms such that $0 < D(\mathbf{u}) < \infty$.

The main ingredients for the presented proofs of Theorem 1 and Theorem 24 are the following:

1. description of bilateral multiplicities of factors in a word with finite palindromic defect ([1]),

2. description of the form of marked morphisms with their fixed points containing infinitely many palindromic factors ([15]).

3. observation that non-zero palindromic defect of a word implies an existence of a factor with a specific property, see Lemma 23 for the binary case and Theorem 26 for the multiliteral case.

The paper is organized as follows: First we recall notions from combinatorics on words and we list known results that we use in the sequel. In Section 3, the properties of marked morphisms are studied. In Section 4, we introduce the notion of a graph of a factor and we interpret bilateral multiplicity of factors in the language of graph theory. Section 5 is focused on properties of a graph of a factor in the case of language having finite palindromic defect. The validity of the Zero Defect Conjecture on binary alphabet is demonstrated in Section 6 (Theorem 24). Section 7 contains the proof of Theorem 1. Comments on counterexamples to two conjectures concerning palindromes form the last Section 8.

2 Preliminaries

An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A *finite word* $w = w_0 w_1 \cdots w_{n-1}$ is a finite sequence over $\mathcal{A}$, i.e., $w_i \in \mathcal{A}$. The *length* of w is n and is denoted by $|w|$. An *infinite word* is an infinite sequence over $\mathcal{A}$. Given words p, f, s with p and f being finite such that $w = pfs$, we say that p is a *prefix* of w, f is a *factor* of w, and s is a *suffix* of w.

2.1 Language of an infinite word

Consider an infinite word $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ over the alphabet $\mathcal{A}$. An index $i \in \mathbb{N}$ is an occurrence of a factor $w = w_0 w_1 \cdots w_{n-1}$ of $\mathbf{u}$ if $u_i u_{i+1} \cdots u_{i+n-1} = w$, in other words w is prefix of the infinite word $u_i u_{i+1} u_{i+2} \cdots$. The set of all factors of $\mathbf{u}$ is referred to as the *language* of $\mathbf{u}$ and denoted

$\mathcal{L}(\mathbf{u})$. The mapping $\mathcal{C}(n)$, defined by $\mathcal{C}(n) = \#\mathcal{L}(\mathbf{u}) \cap \mathcal{A}^n$, is the *factor complexity* of $\mathbf{u}$. A word $\mathbf{u}$ is called recurrent if any factor $w \in \mathcal{L}(\mathbf{u})$ has infinitely many occurrences. If $i < j$ are two consecutive occurrences of the factors w, then the factor $u_i u_{i+i} \cdots u_j u_{j+1} \cdots u_{j+n-1}$ is the *complete return word* to w in $\mathbf{u}$. If any factor of a recurrent word $\mathbf{u}$ has only finitely many complete return words, then $\mathbf{u}$ is called *uniformly recurrent*.

Reversal of a finite word $w = w_0 w_1 \cdots, w_{n-1}$ is the word $\widetilde{w} = w_{n-1} w_{n-2} \cdots w_0$. A word w is a *palindrome* if $w = \widetilde{w}$. The language of $\mathbf{u}$ is said to be *closed under reversal* if $w \in \mathcal{L}(\mathbf{u})$ implies $\widetilde{w} \in \mathcal{L}(\mathbf{u})$; $\mathbf{u}$ is said to be *palindromic* if $\mathcal{L}(\mathbf{u})$ contains infinitely many palindromes. If a uniformly recurrent word $\mathbf{u}$ is palindromic, then its language is closed under reversal. The mapping counting the palindromes of length n in $\mathcal{L}(\mathbf{u})$ is the *palindromic complexity* and is denoted by $\mathcal{P}(n)$, i.e., we have $\mathcal{P}(n) = \#\{w \in \mathcal{L}(\mathbf{u}) \colon |w| = n \text{ and } w = \widetilde{w}\}$.

A letter $b \in \mathcal{A}$ is called *right* (resp. *left*) *extension* of w in $\mathcal{L}(\mathbf{u})$ if wb (resp. bw) belongs to $\mathcal{L}(\mathbf{u})$. In a recurrent word $\mathbf{u}$ any factor has at least one right and at least one left extension. A factor w is *right special* (resp. *left special*) if it has more than one right (resp. left) extension. A factor w which is simultaneously left and right special is *bispecial*. To describe one-sided and both-sided extensions of a factor w we put

$$E^+(w) = \{b \in \mathcal{A} \colon wb \in \mathcal{L}(\mathbf{u})\}, \quad E^-(w) = \{a \in \mathcal{A} \colon aw \in \mathcal{L}(\mathbf{u})\},$$

$$\text{and} \quad E(w) = \{(a,b) \in \mathcal{A}^2 \colon awb \in \mathcal{L}(\mathbf{u})\}.$$

The *bilateral multiplicity* $m(w)$ of a factor $w \in \mathcal{L}(\mathbf{u})$ is defined as

$$m(w) = \#E(w) - \#E^+(w) - \#E^-(w) + 1.$$

Under the assumption of recurrent language, the second difference of the factor complexity may be expressed using bilateral multiplicities as follows:

$$\Delta^2 \mathcal{C}(n) = \mathcal{C}(n+2) - 2\mathcal{C}(n+1) + \mathcal{C}(n) = \sum_{\substack{w \in \mathcal{L}(\mathbf{u}) \\ |w|=n \\ w \text{ is bispecial}}} m(w) = \sum_{\substack{w \in \mathcal{L}(\mathbf{u}) \\ |w|=n}} m(w). \tag{1}$$

(See [8], Section 4.5.2 for the equation (1) and Section 4 for a recent reference on factor complexity in general.) Note that the last equality in (1) follows from the fact that $m(w)$ is nonzero only for bispecial factors in the case of a recurrent language.

A bispecial factor $w \in \mathcal{L}(\mathbf{u})$ is said to be *strong* if $m(w) > 0$, *weak* if $m(w) < 0$ and *neutral* if $m(w) = 0$.

2.2 Palindromic defect

As shown in [9] finite words with zero defect can be characterized using palindromic suffixes. More specifically, a word $w = w_0 w_1 \cdots w_{n-1}$ has defect 0 if and only if for any $i = 0, 1, \ldots, n-1$ the longest palindromic suffix of $w_0 w_1 \cdots w_i$ is unioccurrent in w. To illustrate this important property, consider the words

$$w^{(1)} = 010010100 \quad \text{and} \quad w^{(2)} = 011010011.$$

mentioned in Introduction. The longest palindromic suffix of $w^{(1)}$ is 0010100 and it is unioccurrent in $w^{(1)}$, whereas the longest palindromic suffix of $w^{(2)}$ is 11 and occurs in $w^{(2)}$ twice. The index i for which the longest palindromic suffix is not unioccurrent is called a *lacuna* and the number of lacunas equals the palindromic defect of w.

Since the number of palindromes in w and in its reversal $\widetilde{w}$ is the same, we have $D(w) = D(\widetilde{w})$. Therefore, instead of the longest palindromic suffix one can consider the longest palindromic prefix as well.

The complete return words were applied in [11] to characterize infinite words with zero defect.

Theorem 2 ([11]). $D(\mathbf{u}) = 0$ *if and only if for all palindromes* $w \in \mathcal{L}(\mathbf{u})$ *all complete return words to w in $\mathbf{u}$ are palindromes.*

Before stating a generalization of the previous result we need a new notion.

Definition 3. *Let* $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ *and* $w \in \mathcal{L}(\mathbf{u})$. *A word* $c = c_1 c_2 \cdots c_n \in \mathcal{L}(\mathbf{u})$ *is a* complete mirror return *to w in $\mathbf{u}$ if*

1. *neither w nor $\widetilde{w}$ is a factor of $c_2 \cdots c_{n-1}$, and*

2. *either w is a prefix of c and $\widetilde{w}$ is suffix of c, or $\widetilde{w}$ is a prefix of c and w is a suffix of c.*

Note that c is a complete mirror return to w if and only if it is a complete mirror return to $\widetilde{w}$.

Example 4. We illustrate the notion of complete mirror return word on the Fibonacci word $\mathbf{u}_F$. The factors r_1, r_2 and r_3 are complete mirror returns to $w_1 = 0101$, $w_2 = 001$ and $w_3 = 00$ respectively.

$$\mathbf{u}_F = 010 \underbrace{01010}_{r_1} 0100101 \underbrace{0010100}_{r_2} 1 \underbrace{0010100}_{r_3} 10 \cdots$$

Note that if $w = \widetilde{w}$, then the complete mirror return words of w and $\widetilde{w}$ coincide with complete return words of w.

Using the notion of complete mirror return word we can reformulate Proposition 2.3 from [6].

Proposition 5 ([6]). *Let* $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$. *We have* $D(\mathbf{u}) = 0$ *if and only if for each factor* $w \in \mathcal{L}(\mathbf{u})$ *any complete mirror return word to w in $\mathbf{u}$ is a palindrome.*

A generalization of the previous statement to finite defect follows from [2, Cor. 5 and Lemma 14].

Theorem 6 ([2]). *Let* $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ *be aperiodic and have its language closed under reversal.* $D(\mathbf{u}) < +\infty$ *if and only if there exists a positive integer K such that for every factor w of length at least K the occurrences of w and $\widetilde{w}$ alternate and every complete mirror return to w in $\mathbf{u}$ is a palindrome.*

2.3 Morphisms

In this section we concentrate on primitive morphisms. For a morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ consider the maps $\mathrm{FST}(\varphi), \mathrm{LST}(\varphi) : \mathcal{A} \to \mathcal{A}$ defined by the formula

$$\mathrm{FST}(\varphi)(a) = \text{the first letter of } \varphi(a) \qquad \text{and} \qquad \mathrm{LST}(\varphi)(a) = \text{the last letter of } \varphi(a)$$

for all $a \in \mathcal{A}$. A morphism φ may have more fixed points, see for example the Thue–Morse morphism. The number of fixed points of a primitive morphism φ is the number of letters for which $\mathrm{FST}(\varphi)(a) = a$. It is easy to see that the languages of all fixed points of a primitive morphism coincide and therefore all its fixed points have the same defect.

Recall from Lothaire [17] (Section 2.3.4) that a morphism ψ is a *left conjugate* of φ, or that φ is a *right conjugate* of ψ, denoted $\psi \triangleright \varphi$, if there exists $w \in \mathcal{A}^*$ such that

$$\varphi(x)w = w\psi(x), \quad \text{for all words } x \in \mathcal{A}^*, \tag{2}$$

or equivalently that $\varphi(a)w = w\psi(a)$, for all letters $a \in \mathcal{A}$. We say that the word w is the *conjugate word of the relation* $\psi \triangleright \varphi$. If, moreover, the map $\text{FST}(\psi)$ is not constant, then ψ is the *leftmost conjugate* of φ. Analogously, if $\text{LST}(\varphi)$ is not constant, then φ is the *rightmost conjugate* of ψ.

Example 7. Let

$$\varphi : \begin{array}{ll} a & \mapsto abab \\ b & \mapsto abb \end{array} \quad \text{and} \quad \psi : \begin{array}{ll} a & \mapsto baba \\ b & \mapsto bba \end{array} .$$

We have $\psi \triangleright \varphi$ and the conjugate word of the relation is $w = a$. The leftmost conjugate of φ (and of ψ) is the morphism

$$a \mapsto abab \quad \text{and} \quad b \mapsto bab.$$

If φ is a primitive morphism, then any of its (left or right) conjugate is primitive as well and the languages of their fixed points coincide.

A morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is *cyclic* [16] if there exists a word $w \in \mathcal{A}^*$ such that $\varphi(a) \in w^*$ for all $a \in \mathcal{A}$. Otherwise, we say that φ is *acyclic*. If φ is cyclic, then the fixed point of φ is $wwww \dots$ and is periodic. Remark that the converse does not hold. For example, the morphism determined by $a \mapsto aba$ and $b \mapsto bab$ is acyclic but its fixed point is periodic. Obviously, a morphism is cyclic if and only if it is conjugate to itself with a nonempty conjugate word. If a morphism is acyclic, it has a leftmost and a rightmost conjugate. See [15] for a proof of these statements on cyclic morphisms.

3 Marked morphisms

A morphism φ over binary alphabet has a trivial but important property: the leftmost conjugate of φ maps both letters to words with a distinct starting letter and analogously for the rightmost conjugate. The notion of *marked morphism* extends this important property to any alphabet.

Definition 8. *Let φ be an acyclic morphism. We say that φ is* marked *if*

$$\text{FST}(\varphi_L) \text{ and } \text{LST}(\varphi_R) \text{ are injective}$$

and that φ is well-marked *if*

$$\text{it is marked and if } \text{FST}(\varphi_L) = \text{LST}(\varphi_R)$$

where φ_L (resp. φ_R) is the leftmost (resp. rightmost) conjugate of φ.

Remark 9. Any injective mapping f on a finite set is a permutation and thus there exists a power k such that f^k is the identity map. It implies that for any marked morphism φ there exists a power k such that φ^k is well-marked and moreover $\text{FST}(\varphi_L^k) = \text{LST}(\varphi_R^k) = \text{Id}$.

Theorem 10 ([15]). *Let φ be a primitive well-marked morphism and $\mathbf{u}$ be its palindromic fixed point. The conjugacy word w of $\varphi_L \triangleright \varphi_R$ is a palindrome and*

$$\widetilde{\varphi_R(a)} = \varphi_L(a) \quad \text{for all } a \in \mathcal{A}.$$

We are interested in the defect of fixed points of primitive marked morphisms. We can consider, instead of the marked morphism φ, a suitable power of φ. Thus, without loss of generality we assume that φ is well marked and that $\mathrm{FST}(\varphi_L) = \mathrm{LST}(\varphi_R) = \mathrm{Id}$. For such φ with the conjugacy word w of $\varphi_L \triangleright \varphi_R$ we define the mapping $\Phi : \mathcal{A}^* \to \mathcal{A}^*$ by

$$\Phi(u) = \varphi_R(u)w \quad \text{for all} \quad u \in \mathcal{A}^*.$$

As φ is primitive, each of its powers and also each of its conjugates have the same language. Moreover, if we assume that $\mathbf{u}$ is palindromic, we can deduce using [15, Lemma 15, Lemma 27, Prop. 28] remarkable properties of the mapping Φ.

Lemma 11 ([15]). *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ and $u \in \mathcal{A}^*$. If φ satisfies assumptions of Theorem 10, we have*

(I) *If $u \in \mathcal{L}(\mathbf{u})$, then $\Phi(u) \in \mathcal{L}(\mathbf{u})$.*

(II) $\widetilde{\Phi(u)} = \Phi(\widetilde{u})$.

(III) *The word u is a palindrome if and only if $\Phi(u)$ is a palindrome.*

(IV) *For any $a, b \in \mathcal{A}$, $aub \in \mathcal{L}(\mathbf{u})$ implies $a\Phi(u)b \in \mathcal{L}(\mathbf{u})$.*

(V) *If u is a palindromic (respectively non-palindromic) bispecial factor, then $\Phi(u)$ is a palindromic (respectively non-palindromic) bispecial factor.*

Proof. (I) Let us find v such that $uv \in \mathcal{L}(\mathbf{u})$ with $|\varphi_L(v)| \geq w$. We have

$$\varphi_R(uv)w = \varphi_R(u)w\varphi_L(v).$$

Since $\varphi_R(uv) \in \mathcal{L}(\mathbf{u})$, by erasing a suffix of length greater than or equal to $|w|$ from $\varphi_R(u)w\varphi_L(v)$ we obtain a factor of $\mathcal{L}(\mathbf{u})$, in particular $\varphi_R(u)w \in \mathcal{L}(\mathbf{u})$.

(II) Let $u = u_1 u_2 \cdots u_n$ with $u_i \in \mathcal{A}$. We obtain

$$\Phi(u) = w\varphi_L(u_1) \cdots \varphi_L(u_n) = \varphi_R(u_1) \cdots \varphi_R(u_n)w.$$

Using Theorem 10 we obtain

$$\widetilde{\Phi(u)} = \widetilde{w}\widetilde{\varphi_R(u_n)} \cdots \widetilde{\varphi_R(u_1)} = w\varphi_L(u_n) \cdots \varphi_L(u_1) = \Phi(\widetilde{u}).$$

(III) Let us note that any marked morphism is injective and thus Φ is injective as well. If u is a palindrome, then $\widetilde{\Phi(u)} = \Phi(\widetilde{u}) = \Phi(u)$ from Item (II), therefore $\Phi(u)$ is a palindrome. Conversely, if $\Phi(u)$ is a palindrome, then $\Phi(u) = \widetilde{\Phi(u)} = \Phi(\widetilde{u})$. As φ_L is injective, Φ is injective and the claim follows.

(IV) Let $aub \in \mathcal{L}(\mathbf{u})$. We have $\Phi(aub) \in \mathcal{L}(\mathbf{u})$ and

$$\Phi(aub) = \varphi_R(a)\varphi_R(u)w_{\varphi}\varphi_L(b) = \varphi_R(a)\Phi(u)\varphi_L(b).$$

By our assumption, $\mathrm{LST}(\varphi_R)(c) = \mathrm{FST}(\varphi_L)(c) = \mathrm{Id}(c) = c$ for any $c \in \mathcal{A}$. Thus, $a\Phi(u)b$ is a factor $\Phi(aub) \in \mathcal{L}(\mathbf{u})$.

(V) The statement follows from the previous properties. $\square$

4 Extension graphs of a factor

To study the Zero Defect Conjecture on a multiliteral alphabet, we assign graphs to palindromic and non-palindromic bispecial factors. These graphs were used already in [1] where only words with zero defect are considered. These graphs enable to represent extensions of a bispecial factor and to determine factor complexity, see [8, p.234–235]. They also appear in a more general context in [3]. We use these graphs to demonstrate that the definition of bilateral multiplicity of bispecial factors is related to basic notions of graph theory which we use later in the proofs.

Definition 12 ($\mathbf{\Gamma(w)}$). *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$. We assign to a factor $w \in \mathcal{L}(\mathbf{u})$ the bipartite extension graph $\Gamma(w) = (V, U)$ whose vertices V consist of the disjoint union of $E^-(w)$ and $E^+(w)$*

$$V = \left(E^-(w) \times \{-1\}\right) \cup \left(E^+(w) \times \{+1\}\right)$$

and whose edges U are essentially the elements of $E(w)$:

$$U = \{\{(a, -1), (b, +1)\} : (a, b) \in E(w)\}.$$

Lemma 13. *If $\Gamma(w)$ is connected, then $m(w) \geq 0$ and*

- *$m(w) > 0$ if and only if $\Gamma(w)$ contains a cycle,*

- *$m(w) = 0$ if and only if $\Gamma(w)$ is a tree.*

Proof. Let $G = (V, U)$ be a graph with vertices V and edges U. If G is connected then $\#U - \#V + 1 \geq 0$. A connected graph $G = (V, U)$ is a tree if and only if $\#U - \#V\# + 1 = 0$ and it contains a cycle if and only if $\#U - \#V + 1 > 0$. In the case of the graph $\Gamma(w)$, it turns out that

$$\#U - \#V + 1 = \#E(w) - \#E^-(w) - \#E^+(w) + 1 = m(w). \qquad \square$$

Another graph will be useful in the case when $w = \widetilde{w}$ and when the language $\mathcal{L}(\mathbf{u})$ is closed under reversal. These two hypotheses imply that $E^-(w) = E^+(w)$ and that $E(w)$ is symmetric, i.e. $(a, b) \in E(w)$ if and only if $(b, a) \in E(w)$.

Definition 14 ($\mathbf{\Theta(w)}$). *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ having its language closed under reversal. To a palindromic factor $w \in \mathcal{L}(\mathbf{u})$ we assign a graph $\Theta(w) = (V, U)$ whose vertices $V = E^-(w) = E^+(w)$ are exactly the right (or left) extensions of w and whose edges U are unordered pairs of distinct elements of $E(w)$:*

$$U = \{\{a, b\} : (a, b) \in E(w), a \neq b\}.$$

In particular, $\Theta(w)$ does not contain loops.

The next lemma uses the both-sided symmetric extensions of a factor w which are denoted by

$$E^=(w) = \{a \in \mathcal{A} \colon awa \in \mathcal{L}(\mathbf{u})\}.$$

Lemma 15. *Suppose that the language $\mathcal{L}(\mathbf{u})$ is closed under reversal and $w = \widetilde{w}$. If $\Theta(w)$ is connected, then $m(w) \geq \#E^=(w) - 1$ and*

- *$m(w) > \#E^=(w) - 1$ if and only if $\Theta(w)$ contains a cycle,*

- *$m(w) = \#E^=(w) - 1$ if and only if $\Theta(w)$ is a tree.*

Proof. Using the same argument as for the previous lemma, we compute that

$$\#U = \frac{1}{2}\left(\#E(w) - \#E^=(w)\right) \quad \text{and} \quad \#V = \#E^-(w) = \#E^+(w).$$

Therefore,

$$\#U - \#V + 1 = \frac{1}{2}\left(\#E(w) - \#E^=(w) - \#E^-(w) - \#E^+(w)\right) + 1$$
$$= \frac{1}{2}\left(m(w) - \#E^=(w) + 1\right) \qquad \qquad \square$$

Example 16. Let **u** be the fixed point of the substitution $\eta : a \mapsto aabcacba, b \mapsto aa, c \mapsto a$ used by Bucci and Vaslet. The list of all factors of length 2 is:

$$aa, ab, ac, ba, ca, bc, cb.$$

The list of all factors of length 3 is:

$$aaa, aab, abc, acb, baa, bca, cac, cba.$$

This allows to construct the graphs $\Theta(w)$ and $\Gamma(w)$ for $w \in \{\varepsilon, a, b, c\}$ (see Fig. 1) and the following table of values for the bilateral multiplicity:

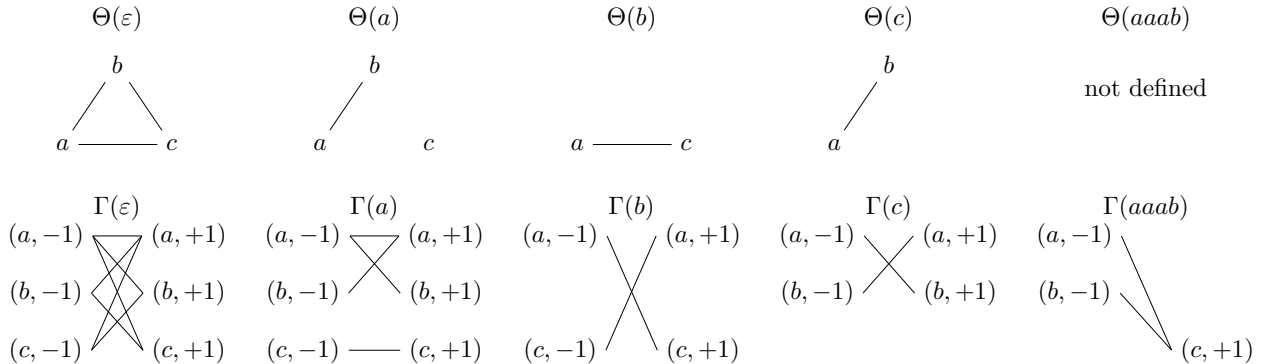| w	ε	a	b	c
$m(w)$	2	-1	-1	-1
$\#E^=(w) - 1$	0	0	-1	-1

.

Figure 1: Example of graphs $\Theta(w)$ and $\Gamma(w)$ for $w \in \{\varepsilon, a, b, c, aaab\}$ in the language of the fixed point of the morphism $a \mapsto aabcacba, b \mapsto aa, c \mapsto a$.

1. The graph $\Theta(\varepsilon)$ has vertices $V = \{a, b, c\}$ and edges $U = \left\{\{a, b\}, \{a, c\}, \{b, c\}\right\}$. The graph $\Theta(\varepsilon)$ contains a cycle. The bilateral multiplicity equals $m(\varepsilon) = 2 > 0 = \#E^=(\varepsilon) - 1$.

2. The graph $\Theta(a)$ has vertices $V = \{a, b, c\}$ and edges $U = \left\{\{a, b\}\right\}$. The graph $\Theta(a)$ is not connected. The bilateral multiplicity equals $m(a) = -1 < 0 = \#E^=(a) - 1$.

3. The graph $\Theta(b)$ has vertices $V = \{a, c\}$ and edges $U = \left\{\{a, c\}\right\}$. The graph $\Theta(b)$ is a tree. The bilateral multiplicity equals $m(b) = -1 = \#E^=(b) - 1$.

It is easy to see that the graph $\Theta(c)$ is isomorphic to $\Theta(b)$. The construction of the graphs $\Gamma(w)$ is analogous. From the extension set $E(aaab) = \{(a, c), (b, c)\}$ of the non-palindromic left special word $w = aaab$, the graph $\Gamma(aaab)$ can be constructed (see Fig. 1). Notice that it is a tree.

5 Words with finite palindromic defect

The graphs introduced in the previous section allow to interpret the palindromic defect in terms of graph theory. In this section we focus on properties of graphs of a factor under the assumption of finite palindromic defect (Theorem 21 and Corollary 22). In Section 7, we study properties of a graph of a factor under the assumption of positive palindromic defect (Theorem 26).

The proof of the main results of this section, namely Theorem 21, can be excerpted from [1, proof of Theorem 3.10]. However, the mentioned theorem has a stronger assumption (the palindromic defect of $\mathbf{u}$ is zero) and it is not stated in terms of graphs as done below in Corollary 22. Therefore, Theorem 21 is accompanied here with an independent proof. The proof requires the next two lemmas, which explain the link between complete mirror return word to a factor w and the connectedness of its associated graphs, and a proposition on the relation of factor and palindromic complexity in a word having finite palindromic defect.

Lemma 17. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. Suppose that v is a palindromic complete mirror return word to $w \in \mathcal{L}(\mathbf{u})$ such that $b\widetilde{w}$ is a suffix of v and $av \in \mathcal{L}(\mathbf{u})$ for some letters $a, b \in \mathcal{A}$. Then $\{(a, -1), (b, +1)\}$ is an edge of the graph $\Gamma(w)$. If w is a palindrome and $a \neq b$, then $\{a, b\}$ is an edge of the graph $\Theta(w)$.*

Proof. Let $s \in \mathcal{A}^*$ such that $v = sb\widetilde{w}$. Since v is a palindrome, we get $v = wb\widetilde{s}$. Therefore, $awb \in \mathcal{L}(\mathbf{u})$ being a prefix of av and $(a, b) \in E(w)$. We conclude that $\{(a, -1), (b, +1)\}$ is an edge of the graph $\Gamma(w)$. Also if $w = \widetilde{w}$ and $a \neq b$, we conclude that $\{a, b\}$ is an edge of the graph $\Theta(w)$. $\square$

Lemma 18. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal, $w \in \mathcal{L}(\mathbf{u})$ and suppose that occurrences of w and $\widetilde{w}$ alternate in $\mathbf{u}$. Suppose that all complete mirror return words to w are palindromes. Then $\Gamma(w)$ is connected. If w is a palindrome, then $\Theta(w)$ is connected.*

Proof. It suffices to show that there is a path from any vertex $(a, -1)$ to any vertex $(b, +1)$ in $\Gamma(w)$. Let $(a, -1)$ and $(b, +1)$ be two vertices of $\Gamma(w)$. Then $aw, wb \in \mathcal{L}(\mathbf{u})$. Let $v \in \mathcal{L}(\mathbf{u})$ be such that aw is a prefix of av and $b\widetilde{w}$ is a suffix of av. If there are no other occurrences of factors of $\mathcal{A}w \cup \mathcal{A}\widetilde{w}$ in v, then $\{(a, -1), (b, +1)\}$ is an edge of the graph $\Gamma(w)$ from Lemma 17. Suppose that

$$a_1 w, \; b_1 \widetilde{w}, \; a_2 w, \; b_2 \widetilde{w}, \; \ldots, \; a_n w, \; b_n \widetilde{w}$$

are consecutive occurrences of factors of $\mathcal{A}w \cup \mathcal{A}\widetilde{w}$ in v where $a = a_1$, $b = b_n$ and $n \geq 2$. From Lemma 17, $\{(a_i, -1), (b_i, +1)\}$ is an edge of the graph $\Gamma(w)$ for all i with $1 \leq i \leq n$. Also, $\{(a_{i+1}, -1), (b_i, +1)\}$ is an edge of the graph $\Gamma(w)$ for all i with $1 \leq i \leq n-1$. Therefore, we conclude that there exists a path from $(a, -1)$ to $(b, +1)$.

Assume $w = \widetilde{w}$. Let $a, b \in E^-(w) = E^+(w)$ be two distinct vertices of $\Theta(w)$. Then $aw, bw \in \mathcal{L}(\mathbf{u})$. We want to show that there exists a path from a to b in $\Theta(w)$. Among the occurrences of factors in $\mathcal{A}w$, if there exist two consecutive occurrences of aw and bw, then $\{a, b\}$ is an edge of $\Theta(w)$ from Lemma 17. Otherwise, we conclude that there exists a path from a to b by transitivity. $\square$

Corollary 19. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. If $D(\mathbf{u}) < +\infty$, then there exists an integer K such that for each bispecial factor $w \in \mathcal{L}(\mathbf{u})$ with $|w| \geq K$ the graph $\Gamma(w)$ is connected. If w is moreover a palindrome, then also the graph $\Theta(w)$ is connected.*

Proof. If $\mathbf{u}$ is not aperiodic, then the claim is trivially satisfied as there is only a finite number of bispecial factors.

If $\mathbf{u}$ is aperiodic, Theorem 6 implies that there exists a positive integer K such that for every factor $w \in \mathcal{L}(\mathbf{u})$ longer than K, the occurrences of w and $\widetilde{w}$ alternate and every complete mirror return to w in $\mathbf{u}$ is a palindrome. We conclude from Lemma 18 that the graph $\Gamma(w)$ is connected. Also if w is a palindrome, then $\Theta(w)$ is connected. □

The following claim may be deduced from [2].

Proposition 20 ([2, Th. 2, Prop. 6]). *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. If $D(\mathbf{u}) < +\infty$, then there exists an integer M such that for all $n \geq M$ we have*

$$\Delta^2 \mathcal{C}(n) = \mathcal{P}(n+2) - \mathcal{P}(n).$$

Proof. Since $\mathcal{L}(\mathbf{u})$ is closed under reversal, Proposition 6 from [2] says that there exists an integer M such that for all $n \geq M$ we have

$$\Delta \mathcal{C}(n) + 2 \geq \mathcal{P}(n+1) + \mathcal{P}(n).$$

Since $D(\mathbf{u}) < +\infty$, Theorem 2 from [2] together with the above inequality implies that there exists an integer M such that for all $n \geq M$ we have

$$\Delta \mathcal{C}(n) + 2 = \mathcal{P}(n+1) + \mathcal{P}(n).$$

From this we conclude:

$$\Delta^2 \mathcal{C}(n) = \Delta \mathcal{C}(n+1) - \Delta \mathcal{C}(n) = \mathcal{P}(n+2) + \mathcal{P}(n+1) - \mathcal{P}(n+1) - \mathcal{P}(n) = \mathcal{P}(n+2) - \mathcal{P}(n). \quad \square$$

Theorem 21. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. If $D(\mathbf{u}) < +\infty$, then there exists an integer K such that each bispecial factor $w \in \mathcal{L}(\mathbf{u})$ with $|w| \geq K$ satisfies*

$$m(w) = \begin{cases} 0 & \text{if } w \neq \widetilde{w}, \\ \#E^=(w) - 1 & \text{if } w = \widetilde{w}. \end{cases}$$

Proof. Let K_1 be the constant given by Corollary 19. If w is a bispecial factor with $|w| > K_1$, we conclude from Lemma 18 that the graph $\Gamma(w)$ is connected. Also if w is a palindrome, then $\Theta(w)$ is connected. It follows from Lemma 13 that $m(w) \geq 0$. If w is a palindrome, Lemma 15 implies $m(w) \geq \#E^=(w) - 1$.

If w is not a bispecial factor, then $m(w) = 0$ and, moreover, if w is not a bispecial factor and $w = \widetilde{w}$, then by closedness under reversal we have $\#E^=(w) = 1$, and thus $m(w) = 0 = \#E^=(w) - 1$.

Suppose by contradiction that for every integer N there exists a non-palindromic factor v of length $|v| > N$ such that $m(v) > 0$ or there exists a palindromic factor v of length $|v| > N$ such that $m(v) > \#E^=(v) - 1$. As closedness under reversal implies recurrence, using (1) we obtain that for every integer N there exists $n = |v| > N$ such that

$$\Delta^2 \mathcal{C}(n) = \sum_{\substack{w \in \mathcal{L}(\mathbf{u}) \\ |w|=n \\ w \neq \widetilde{w}}} m(w) + \sum_{\substack{w \in \mathcal{L}(\mathbf{u}) \\ |w|=n \\ w = \widetilde{w}}} m(w) > 0 + \sum_{\substack{w \in \mathcal{L}(\mathbf{u}) \\ |w|=n \\ w = \widetilde{w}}} (\#E^=(w) - 1) = \mathcal{P}(n+2) - \mathcal{P}(n). \quad (3)$$

This contradicts Proposition 20 and ends the proof of the theorem with $K = \max\{K_1, M\}$. □

The following result is a direct consequence of Lemma 13 and Lemma 15. It allows to interpret the previous theorem in terms of graph theory.

Corollary 22. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be an infinite word with its language closed under reversal and $D(\mathbf{u}) < +\infty$. There exists a positive integer K such that for every $w \in \mathcal{L}(\mathbf{u})$ of length at least K*

- *if w is not a palindrome, then the graph $\Gamma(w)$ is a tree,*

- *if w is a palindrome, then the graph $\Theta(w)$ is a tree.*

Proof. Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. From Corollary 19, there exists an integer K_1 such that for each bispecial factor $w \in \mathcal{L}(\mathbf{u})$ with $|w| \geq K_1$ the graph $\Gamma(w)$ is connected. If w is moreover a palindrome, then also the graph $\Theta(w)$ is connected.

From Theorem 21, it follows that there exists a constant K_2 such that every factor w longer than K_2 satisfies

$$m(w) = \begin{cases} 0 & \text{if } w \neq \widetilde{w}, \\ \#E^{=}(w) - 1 & \text{if } w = \widetilde{w}. \end{cases}$$

Let $K = \max\{K_1, K_2\}$ and w be a factor of $\mathcal{L}(\mathbf{u})$ such that $|w| > K$. If $w \neq \widetilde{w}$, Lemma 13 implies that $\Gamma(w)$ is a tree. If $w = \widetilde{w}$, Lemma 15 implies that $\Theta(w)$ is a tree. $\qquad\square$

6 Proof of Zero Defect Conjecture for binary alphabet

The binary alphabet offers less variability for the construction of a strange phenomenon. The recent counterexamples to two conjectures concerning palindromes in fixed points of primitive morphisms — namely the Bucci-Vaslet counterexample to the Zero Defect Conjecture and the Labbé counterexample to the Hof-Knill-Simon (HKS) conjecture — use ternary alphabet. That conjecture [12] asks whether all palindromic fixed points of primitive substitutions are fixed by some conjugate of a morphism of the form $\alpha \mapsto p_\alpha p$ where p_α and p are palindromes. On a binary alphabet, Tan demonstrated the validity of the HKS conjecture, see [20]. Here we prove the Zero Defect Conjecture on a binary alphabet.

Lemma 23. *Let $\mathcal{A} = \{0, 1\}$ and $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$. If $\mathcal{L}(\mathbf{u})$ is closed under reversal and $D(\mathbf{u}) > 0$, then there exists a non-palindromic factor $q \in \mathcal{L}(\mathbf{u})$ such that $0q0, 0q1, 1q0, 1q1 \in \mathcal{L}(\mathbf{u})$.*

Proof. By Proposition 5, as $D(\mathbf{u}) > 0$, there exist factors v and w in $\mathcal{L}(\mathbf{u})$ such that v is a complete mirror return word to w and v is not a palindrome. Let us consider the shortest v with this property. For this fixed v we find the longest w such that v is a complete mirror return word to w. It means that v has a prefix wa and a suffix $b\widetilde{w}$ where $a, b \in \mathcal{A}$ and $a \neq b$. Since on a binary alphabet every complete mirror return word to a letter is always a palindrome, we have $|w| > 1$. Without loss of generality we can write $w = 0q$ with $q \neq \varepsilon$. Consequently $v = 0u0$. Clearly u has a prefix q, the word u has a suffix $\widetilde{q}$ and u is not a palindrome. Our choice of v (to be the shortest non-palindromic mirror return to a factor) implies that u is not a complete mirror return word to q and thus q or $\widetilde{q}$ has another occurrence inside u. Since v is a complete mirror return word to $w = 0q$,

$$0q \text{ and } \widetilde{q}0 \text{ do not occur in } u. \tag{4}$$

Let us suppose that $q = \widetilde{q}$. Consider the shortest prefix of u which has exactly two occurrences of q. It is palindrome. Since v has a prefix $wa = 0qa$ the second occurrence of q is extended to the left as aq. Analogously, consider the shortest suffix of u which contains exactly two occurrences of q. It is a palindrome and thus the penultimate occurrence of q is extended to the right as qb. This contradicts (4) as $a \neq b$. We conclude that q is not a palindrome.

Now we show that occurrences of q and $\widetilde{q}$ in u alternate. Assume that there exists a factor of u, denoted by u', such that q is a prefix and a suffix of u' and u' does not contain $\widetilde{q}$. It follows that the longest palindromic suffix of u' is not unioccurrent in u'. Therefore $D(u') \geq 1$ (see Section 2.2), which contradicts the minimality of $|v|$.

The minimality of $|v|$ implies that all mirror return words to q in u are palindromes. Therefore, the leftmost occurrence of $\widetilde{q}$ in u is extended to the left as $a\widetilde{q}$ and the rightmost occurrence of q in u is extended to the right as qb. From (4) we deduce that $0qa$, $a\widetilde{q}1$, $1qb$, and $b\widetilde{q}0$ belong to $\mathcal{L}(\mathbf{u})$. The assumption that $\mathcal{L}(\mathbf{u})$ is closed under reversal and the fact that $a \neq b$ finish the proof. $\qquad\square$

Theorem 24. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ be a fixed point of a primitive morphism φ over a binary alphabet $\mathcal{A}$. If $D(\mathbf{u}) < +\infty$, then $D(\mathbf{u}) = 0$ or $\mathbf{u}$ is periodic.*

Proof. Assume the contrary, i.e., $\mathbf{u}$ is not periodic and $D(\mathbf{u}) > 0$ and let $\mathcal{A} = \{0, 1\}$.

Since $D(\mathbf{u})$ is finite, $\mathbf{u}$ is palindromic. As φ is primitive, $\mathcal{L}(\mathbf{u})$ is uniformly recurrent. Any uniformly recurrent word which is palindromic has its language closed under reversal. Due to Lemma 23 there exists a strong bispecial non-palindromic factor q with $m(q) = 1$.

Since $\mathbf{u}$ is not periodic, the morphism φ is acyclic. On the binary alphabet, it means that φ is well-marked. Applying repeatedly Lemma 11 (IV) and (V), we can construct an infinite sequence of strong bispecial factors $q, \Phi(q), \Phi^2(q), \Phi^3(q), \ldots$, each with bilateral multiplicity 1. By Lemma 11 (III), all these bispecial factors are non-palindromic. This contradicts Theorem 21. $\qquad\square$

7 Proof of Zero Defect Conjecture for marked morphisms

At first we have to stress that unlike the binary version, the statement of Theorem 1 does not speak about periodic fixed points. The following result from [15] allows to deduce that on a larger alphabet there is no ultimately periodic infinite word $\mathbf{u}$ fixed point of a primitive marked morphism such that $0 < D(\mathbf{u}) < \infty$.

Proposition 25. [15, Cor. 30, Cor. 32] *Let $\mathbf{u}$ be an eventually periodic fixed point of a primitive marked morphism φ over an alphabet $\mathcal{A}$. If $\mathbf{u}$ is palindromic, then $\mathcal{A} = \{0, 1\}$ is a binary alphabet and $\mathbf{u}$ equals $(01)^\omega$ or $(10)^\omega$.*

Due to the previous proposition, a fixed point of a marked morphisms on binary alphabet is either not eventually periodic or equal to $(01)^\omega$ or $(10)^\omega$. Since both words $(01)^\omega$ and $(10)^\omega$ have defect zero and the Zero Defect Conjecture for binary alphabet is proven by Theorem 24, we may restrict ourselves to alphabets with cardinality at least three.

First, we prove a multiliteral analogue of Lemma 23 for words with its language closed under reversal and with positive palindromic defect.

Theorem 26. *Let $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ have its language closed under reversal. If $D(\mathbf{u}) > 0$, then either*

1. *there exists a non-palindrome $q \in \mathcal{L}(\mathbf{u})$ such that $\Gamma(q)$ contains a cycle or*

2. *there exists a palindrome $q \in \mathcal{L}(\mathbf{u})$ such that $\Theta(q)$ contains a cycle.*

Moreover, if the empty word is the unique factor q with the above property, then there exists a letter with a non-palindromic complete return word.

Proof. Since $D(\mathbf{u}) > 0$, there exists a word $v = v_0 v_1 \cdots v_n$ such that w is a prefix of v, $\widetilde{w}$ is a suffix of v, v does not contain other occurrences of w or $\widetilde{w}$, v is not a palindrome and $|w| \geq 1$. Suppose that v is a word of minimal length with this property and suppose that w is the longest prefix of v such that $\widetilde{w}$ is a suffix of v. Then there exist letters $\alpha \neq \beta$ such that $w\alpha$ is a prefix and $\beta\widetilde{w}$ is a suffix of v. Let us define $t \in \mathcal{A}$ and $q \in \mathcal{A}^*$ to satisfy $w = tq$ (see Figure 2).
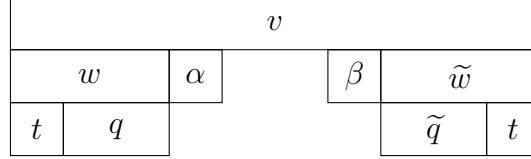
v					
w	α		β	$\widetilde{w}$	
t	q			$\widetilde{q}$	t

Figure 2: The complete mirror return word v to the factor w.

We discuss three cases:

1. Let us suppose $q = \widetilde{q} \neq \varepsilon$. Due to the minimality of $v = v_0 v_1 \ldots v_n = tq\alpha \cdots \beta qt$, the non-palindromic factor $v_1 v_2 \ldots v_{n-1} = q\alpha \cdots \beta q$ cannot be a complete return word to q and thus contains at least 3 occurrences of q. Let k be the number of occurrences q in v. For $i = 1, 2, \ldots, k$, denote by γ_i the letter which precedes the ith occurrence of q and by δ_i the letter which succeeds the ith occurrence of q.

   • Obviously, $\gamma_1 = t$, $\delta_1 = \alpha$, and $\gamma_k = \beta$ and $\delta_k = t$.

   • Since v is a complete mirror return word to the factor $w = tq$, necessarily $t \neq \gamma_i$ for $i = 2, \ldots, k$ and $t \neq \delta_i$ for $i = 1, \ldots, k-1$. In particular, $\alpha \neq t$ and $\beta \neq t$.

   • Since each complete return word to q in v is a palindrome, $\delta_i = \gamma_{i+1}$ for $i = 1, 2, \ldots, k-1$. We artificially put $\gamma_{k+1} = \delta_k = t$.

   According to the definition of $\Theta(q)$, if $\gamma_i \neq \gamma_{i+1} = \delta_i$, then the pair $\{\gamma_i, \gamma_{i+i}\}$ forms an edge. We want to find a cycle in $\Theta(q)$. For this purpose we modify the sequence of letters $\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma_{k+1}$ as follows: If $\gamma_{j+1} = \gamma_j$ for some index $j = 1, \ldots, k$, then we erase from the sequence the $(j+1)^{th}$ entry γ_{j+1}. Then the modified sequence is a path in $\Theta(q)$ which starts and ends at t. The second vertex on the path is α, the penultimate vertex is β. As $\alpha \neq \beta$, the graph $\Theta(q)$ contains a cycle.

2. Let us suppose that $q = \varepsilon$. Now $v = v_0 v_1 \ldots v_n = t\alpha v_2 v_3 \cdots \beta t$. It means that v is a complete return to the letter t which is non-palindromic. If $v_i \neq v_{i+1}$, the pair of consecutive letters $\{v_i, v_{i+1}\}$ is an edge in the graph $\Theta(\varepsilon)$ connecting vertices v_i and v_{i+1}. If we erase from the sequence $v_0, v_1, \ldots, v_n$ each vertex v_{j+1} which coincides with its predecessor v_j, we get a path starting and ending in the vertex t. The first edge on this path is $\{t, \alpha\}$, the last one is $\{t, \beta\}$. As $\alpha \neq \beta$, the graph $\Theta(\varepsilon)$ contains a cycle.

3. Now we assume that $q \neq \widetilde{q}$. Note that occurrences of q and $\widetilde{q}$ alternate inside v. Indeed, suppose the contrary, that is there exists a complete return word z of q that has no occurrences of $\widetilde{q}$ and z is a factor of v. The longest palindrome suffix of z must be shorter than q. Therefore the longest palindromic suffix of z is not unioccurrent in z. This contradicts the minimality of v. Note also that v must contain other occurrences of q or $\widetilde{q}$ inside or otherwise we get

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a contradiction on minimality of v. Let us denote k the number of occurrences of q in v. Clearly k equals to the number of occurrences of $\widetilde{q}$ as well.

Again we denote by γ_i the letter which precedes the ith occurrence of q and by δ_i the letter which succeeds the ith occurrence of q. In particular, $\gamma_1 = t$ and $\delta_1 = \alpha$. Analogously, we denote by $\tilde{\gamma}_i$ the letter which precedes the ith occurrence of $\widetilde{q}$ and by $\tilde{\delta}_i$ the letter which succeeds the ith occurrence of $\widetilde{q}$. In particular, $\tilde{\gamma}_k = \beta$ and $\tilde{\delta}_k = t$. Point out three important facts:

- $\gamma_i q \delta_i \in \mathcal{L}(\mathbf{u})$ implies $\{(\gamma_i, -1), (\delta_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$.

- As the language $\mathcal{L}(\mathbf{u})$ is closed under reversal, $\tilde{\gamma}_i \widetilde{q} \tilde{\delta}_i \in \mathcal{L}(\mathbf{u})$ implies $\{(\tilde{\delta}_i, -1), (\tilde{\gamma}_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$.

- Due to minimality of v, any mirror return to q in v is a palindrome. Thus $\delta_i = \tilde{\gamma}_i$ for $i = 1, 2, \ldots, k$ and $\tilde{\delta}_i = \gamma_{i+1}$ for $i = 1, 2, \ldots, k-1$.

Therefore, $\{(\gamma_i, -1), (\tilde{\gamma}_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$, $\{(\tilde{\gamma}_i, +1), (\gamma_{i+1}, -1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k-1$ and $\{(\tilde{\gamma}_k, +1), (\tilde{\delta}_k, -1)\}$ is an edge in $\Gamma(q)$. We can summarize that the sequence of vertices
$$(\gamma_1, -1), (\tilde{\gamma}_1, +1), (\gamma_2, -1), (\tilde{\gamma}_2, +1), \ldots, (\gamma_k, -1), (\tilde{\gamma}_k, +1), (\tilde{\delta}_k, -1)$$
forms a path in the bipartite graph $\Gamma(q)$ with $\gamma_1 = \tilde{\delta}_k = t$ and $\tilde{\gamma}_1 = \alpha \neq \beta = \tilde{\gamma}_k = t$. In this path the first and the last vertices coincide and the second and the penultimate vertices are distinct. Thus the graph $\Gamma(q)$ contains a cycle. $\square$

As we have seen in Example 16 for the fixed point $\mathbf{u}$ of the morphism $\eta : a \mapsto aabcacba, b \mapsto aa, c \mapsto a$ for which the defect is known to be positive, the graph $\Theta(\varepsilon)$ contains a cycle. Since the defect of $\mathbf{u}$ is finite, Corollary 22 also applies. Thus there are no arbitrarily large palindromic factors w containing a cycle in their graph $\Theta(w)$ nor non-palindromic factors w containing a cycle in their graph $\Gamma(w)$. This is readily seen on the conjugacy word of $\eta_L \triangleright \eta_R$ which is aaa (see Fig. 3).

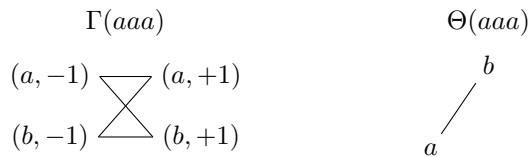$$\Gamma(aaa) \qquad\qquad \Theta(aaa)$$

Figure 3: $\Gamma(aaa)$ contains a cycle but $\Theta(aaa)$ is a tree in the language of the fixed point of the morphism $a \mapsto aabcacba, b \mapsto aa, c \mapsto a$.

We are now ready to finish the proof for the multiliteral case.

Proof of Theorem 1. As the languages of the fixed points of φ and φ^k coincide, we may assume without loss of generality that the marked morphism φ has already the property $\mathrm{LST}(\varphi_R) = \mathrm{FST}(\varphi_L) = \mathrm{Id}$.

Proving that the Zero Defect Conjecture holds in the case of marked morphisms amounts to prove that the defect is either zero or $+\infty$. Let us assume on the contrary that $0 < D(\mathbf{u}) < +\infty$. It follows that $\mathbf{u}$ is palindromic. The primitivity of φ implies that $\mathcal{L}(\mathbf{u})$ is closed under reversal.

Theorem 26 implies that there exists a factor q such that if $q \neq \widetilde{q}$ the graph $\Gamma(q)$ contains a cycle, or if $q = \widetilde{q}$, the graph $\Theta(q)$ contains a cycle. Lemma 11, property (IV), implies that for all n, there is a cycle in the graph of $\Phi^n(q)$.

If $q \neq \varepsilon$, then the primitivity of φ implies that $\lim_{n \to +\infty} |\Phi^n(q)| = +\infty$. If $q = \varepsilon$, then, again by Theorem 26, there exists a letter having non-palindromic complete return word. By the assumption of the theorem, there must exist a conjugate of φ distinct from φ itself. It implies that the conjugacy word of $\varphi_L \triangleright \varphi_R$ is nonempty, i.e., $\Phi(\varepsilon) \neq \varepsilon$. Moreover, $\lim_{n \to +\infty} |\Phi^n(q)| = +\infty$.

To conclude, we have that $\lim_{n \to +\infty} |\Phi^n(q)| = +\infty$ and there is a cycle in the graph of $\Phi^n(q)$ for all n. This is a contradiction with Corollary 22. $\qquad\square$

8 Comments

Let us comment two conjectures concerning palindromes in languages of fixed points of primitive morphisms.

- The counterexample to the Zero Defect Conjecture in full generality was already mentioned in the Introduction. It is taken from [7]. The fixed point of

$$\varphi : a \mapsto aabcacba, b \mapsto aa, c \mapsto a$$

  has finite positive palindromic defect and is not periodic. There is a remarkable property of the fixed point $\mathbf{u} = \varphi(\mathbf{u})$.

  Let $\mu : a \mapsto ap, p \mapsto apaaaapaaaap$ be a morphism over the binary alphabet $\{a, p\}$. Let us denote $\mathbf{v}$ the fixed point of μ. Then one can easily verify that $\mathbf{u} = \pi(\mathbf{v})$, where $\pi : a \mapsto a, p \mapsto abcacba$. Moreover, $\mathbf{v}$ has zero defect.

  In other words, the counterexample word is just an image under π of a purely morphic binary word with zero defect.

- The counterexample to the question of Hof, Knill and Simon (recalled in Section 6) given in [14] by the first author is
$$\psi : a \mapsto aca, b \mapsto cab, c \mapsto b.$$

  As mentioned in [18], the fixed point $\mathbf{u} = \psi(\mathbf{u})$ is again an image of a Sturmian word $\mathbf{v}$ under a morphism $\pi : \{0, 1\} \mapsto \{a, b, c\}$ and the Sturmian word $\mathbf{v}$ itself is a fixed point of a morphism over binary alphabet $\{0, 1\}$. Since $\mathbf{v}$ is Sturmian, its defect is zero.

Both counterexamples are in some sense degenerate. Both words are on ternary alphabet, but the binary alphabet is hidden in their structure. For further research in this area, it would be instructive to find another kind of counterexamples to both mentioned conjectures. In this context we mention that the second and third authors showed in [19] that any uniformly recurrent infinite word $\mathbf{u}$ with a finite defect is a morphic image of a word $\mathbf{v}$ with defect 0.

Acknowledgements

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