9ème cours de Mécanique Analytique (24 Novembre 2016)

Chapitre 2 : Mécanique hamiltonienne

- 2.0 Transformée de Legendre : notions

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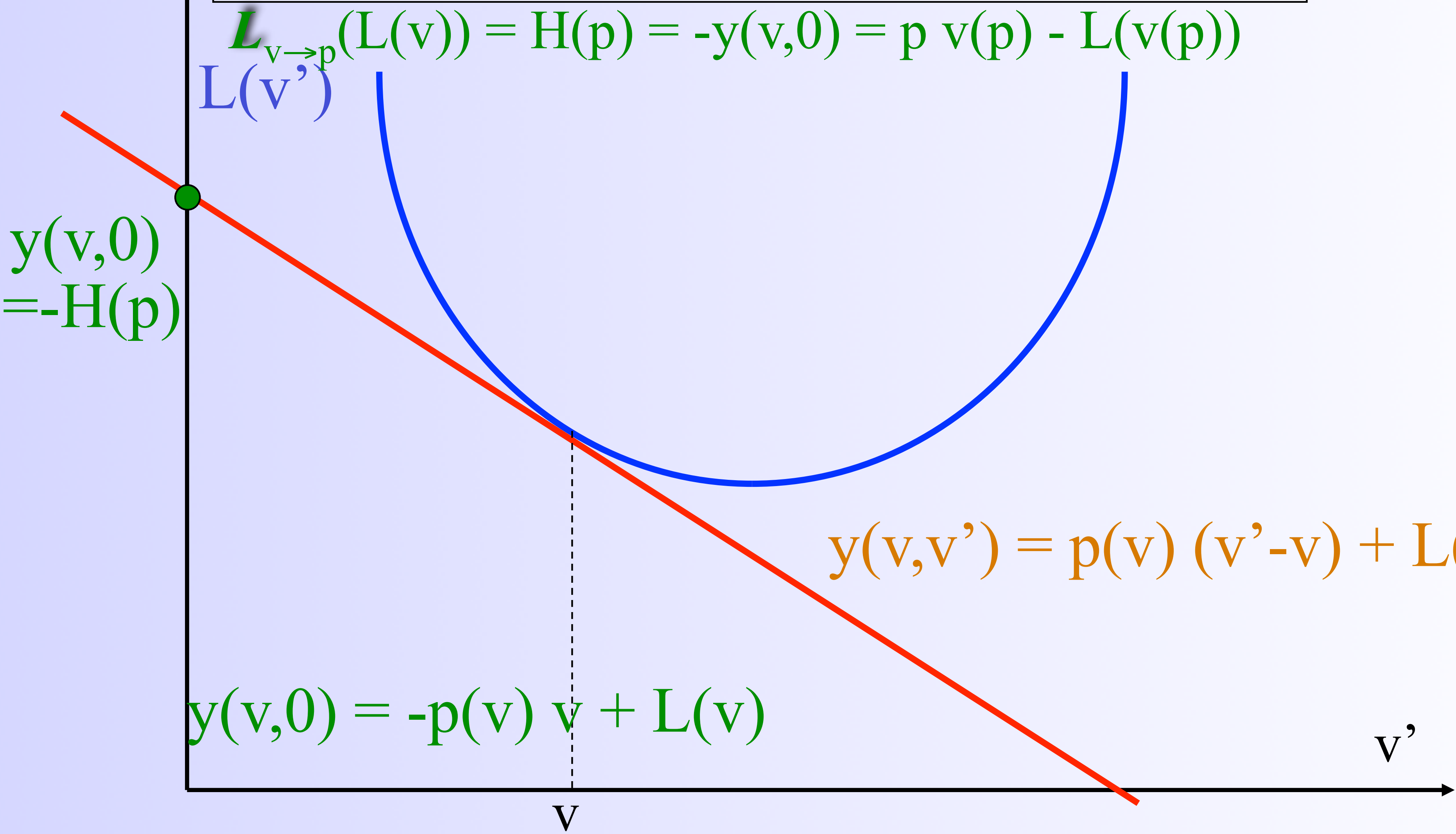
Si $p = p(v) = (dL/dv')_v$

Si $dp(v)/dv \neq 0$; $d^2L/d^2v \neq 0 \Rightarrow v = v(p)$

$\mathcal{L}_{v \to p}(L(v)) = H(p) = -y(v,0) = p\, v(p) - L(v(p))$



- 2.0 Transformée de Legendre : notions

$$H(p) \Rightarrow L(v)$$

$$u(p) = dH/dp$$

$$u = \frac{dH}{dp} = v(p) + p\frac{dv}{dp} - \frac{dL}{dv}\frac{dv}{dp} = v(p)$$

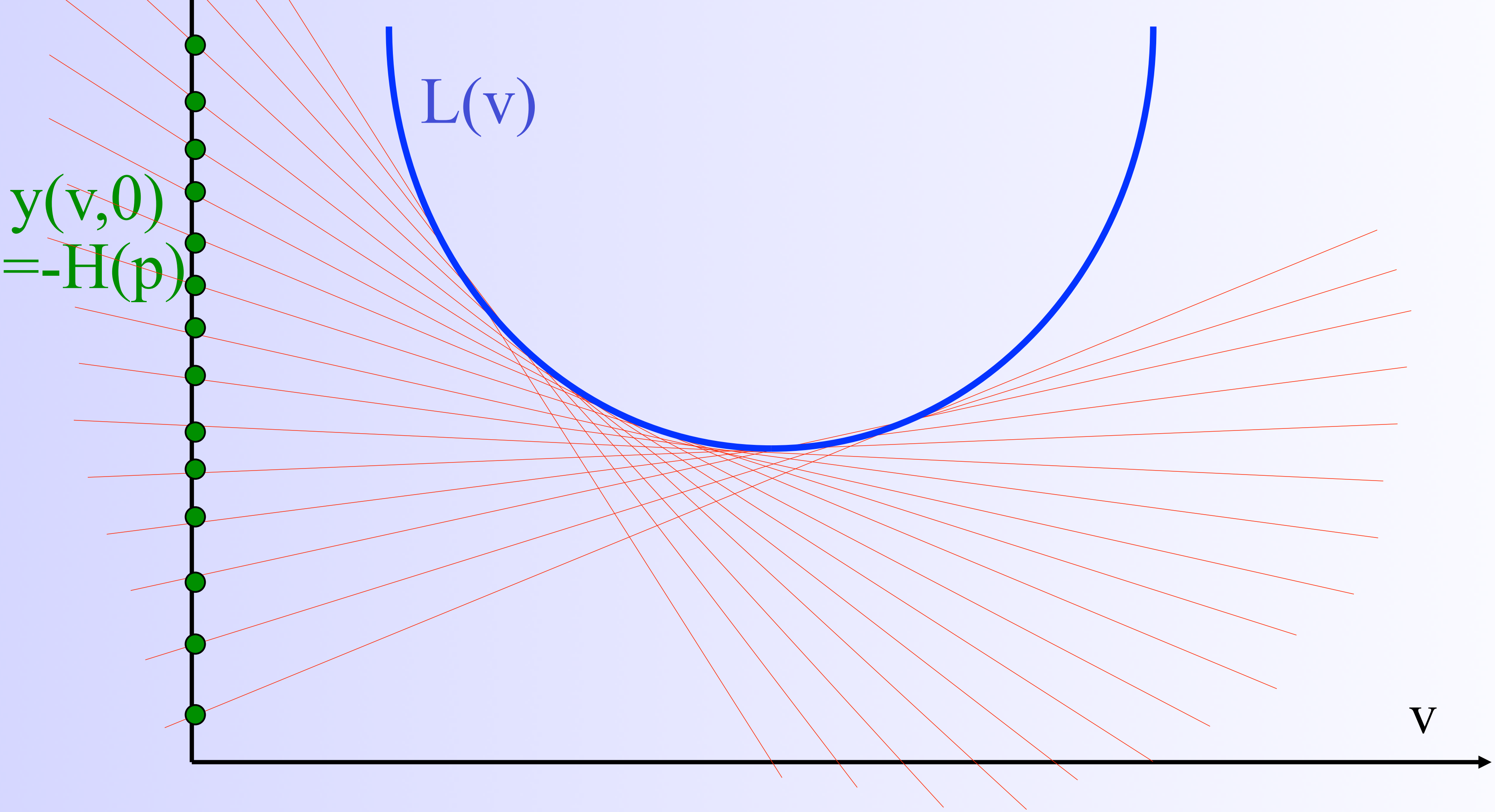
$$\frac{du}{dp} = \frac{dv}{dp} = \left(\frac{dp}{dv}\right)^{-1} \neq 0$$

$$\mathcal{L}_{p\to v}(H(p)) = up - H = vp(v) - p(v)v + L = L$$

• 2.0 Transformée de Legendre : notions

$$y(v,0) = -p(v)\, v + L(v)$$

$$\mathcal{L}_{v \to p}(L(v)) = H(p) = -y(v,0) = p\, v(p) - L(v(p))$$



• 2.0 Transformée de Legendre : notions

la transformée de Legendre de $L(v) = (1/2)mv^2$.

$$p = dL/dv = mv$$

$$dp/dv = m \neq 0$$

$$H(p) = p^2/(2m)$$

$$\mathcal{L}_{p\to v}(H(p)) = L(v)$$

- 2.0 Transformée de Legendre : notions

si la fonction L dépend de f variables v_i

$$p_i = \partial L / \partial v_i$$

$$\frac{\partial p_i(v_k)}{\partial v_j} = \frac{\partial^2 L}{\partial v_i \partial v_j}$$

$$\mathcal{L}_{v_i \to p_j}(L(v_i)) = H(p_j) = p_k v_k(p_j) - L(v_i(p_j))$$

• 2.1 Equations de Hamilton et systèmes canoniques

Lagrangien : f; $q_i(t)$, $\dot{q}_i$ ⇒ Hamiltonien : 2f; $q_i(t)$, $p_i(t)$, t

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (i = 1, 2, ..., f) \quad (2.1)$$

$$dtm\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) = dtm\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0 \quad (2.3)$$

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 (T - V)}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \quad (2.2)$$

$$\dot{q}_i = \dot{q}_i(q, p, t) \quad (2.4)$$

$$H(q, p, t) = \mathcal{L}_{(q)_i \to p_j}(L(q, \dot{q}, t)) = p_i \dot{q}_i(q, p, t) - L(q, \dot{q}(q, p, t), t) \quad (2.5)$$

• 2.1 Equations de Hamilton et systèmes canoniques

$$\boxed{H(q,p,t) = \mathcal{L}L(q,p,t) = p_i\dot{q}_i(q,p,t) - L(q,\dot{q}(q,p,t),t)} \qquad (2.6)$$

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i - \frac{\partial L}{\partial t}dt \qquad (2.7)$$

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt = -\dot{p}_i dq_i + \dot{q}_i dp_i - \frac{\partial L}{\partial t}dt \qquad (2.8)$$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, ..., f)} \qquad (2.9)$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}} \qquad (2.10)$$

$p_i \rightarrow q_i \qquad H \rightarrow t$

• 2.1 Equations de Hamilton et systèmes canoniques

$$\boxed{H(q,p,t) = \mathcal{L}L(q,p,t) = p_i\dot{q}_i(q,p,t) - L(q,\dot{q}(q,p,t),t)} \quad (2.6)$$

Si $L(q_1, q_2, .. q_{j-1}, \cancel{q_j}, q_{j+1}, \dots q_f, \dot{q}, t)$

$\Rightarrow H(q_1, q_2, .. q_{j-1}, \cancel{q_j}, q_{j+1}, \dots q_f, p, t)$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, ..., f)} \quad (2.9)$$

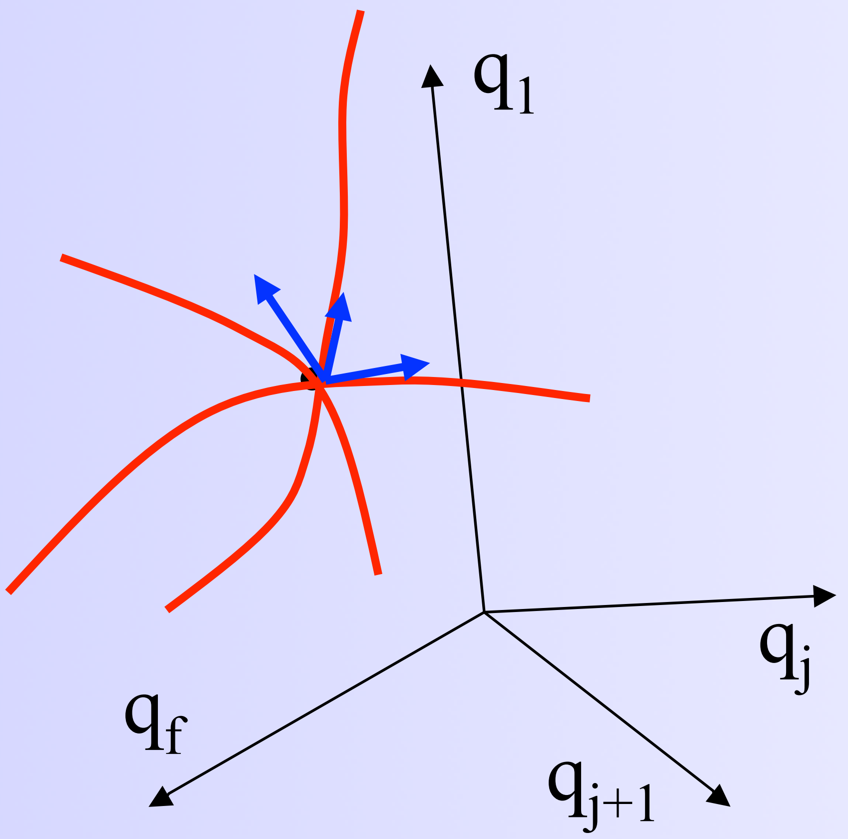
Si $L(q, \dot{q}, \cancel{t}) \Rightarrow H(q, p, \cancel{t})$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{dt} = -\frac{\partial L}{\partial t}} \quad (2.10)$$

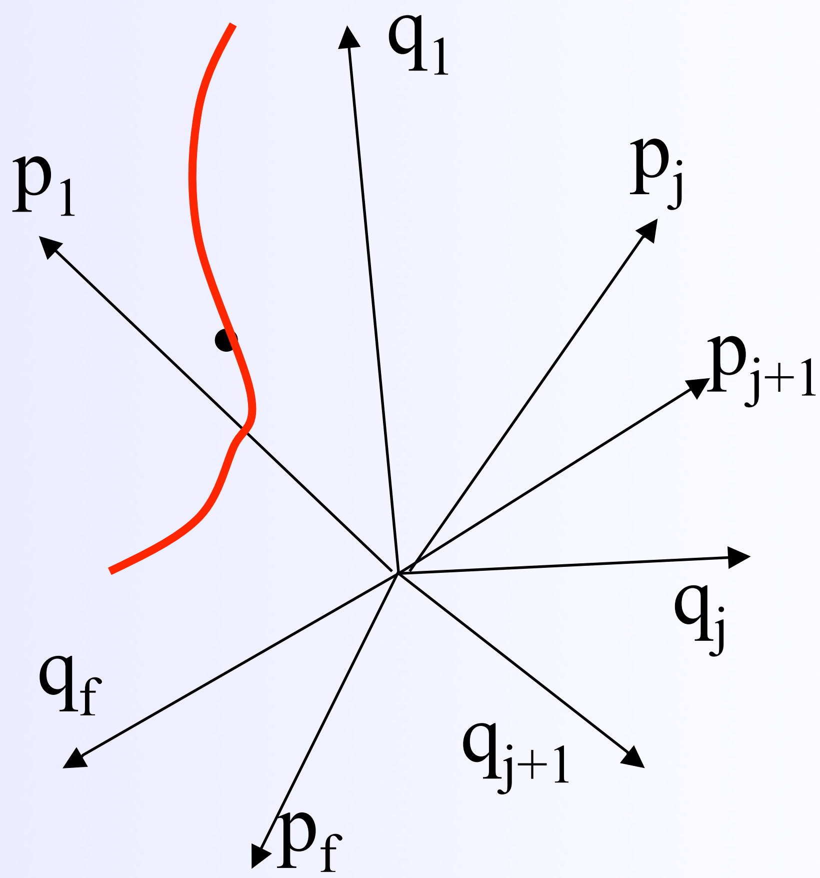
• 2.1 Equations de Hamilton et systèmes canoniques

Suivant Lagrange ...

Suivant Hamilton ...



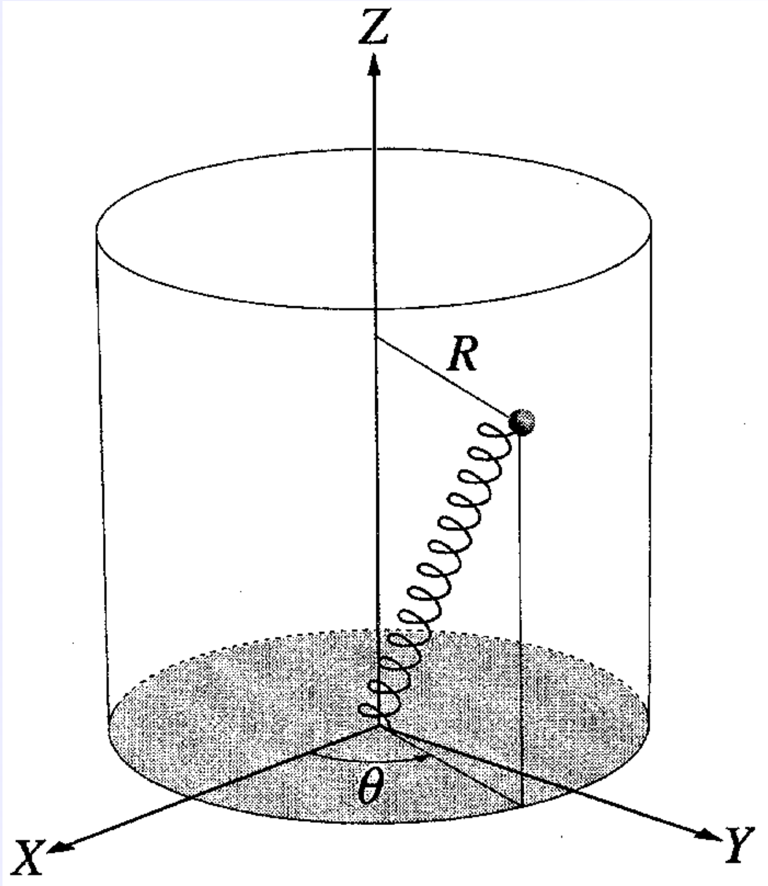
Espace de configuration



Espace de phase

• 2.2 Exemples de systèmes canoniques

Exemple 1 :



$$x^2 + y^2 = R^2$$

$$\vec{F} = -k\vec{r}$$

coordonnées généralisées θ et z

$$V = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2) = \frac{1}{2}k(R^2 + z^2)$$

$x = R\cos\theta$ $\quad$ $y = R\sin\theta$ $\quad$ $R =$ constante

• 2.2 Exemples de systèmes canoniques

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2$$

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right)$$

$$L = T - V = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$$

$$\begin{cases} p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \\ p_z = \dfrac{\partial L}{\partial \dot{z}} = m\dot{z} \end{cases}$$

$$dtm\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = dtm\left(\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right) = dtm\begin{pmatrix} mR^2 & 0 \\ 0 & m \end{pmatrix} = m^2R^2 \neq 0$$

• 2.2 Exemples de systèmes canoniques

$$\dot{\theta} = \frac{p_\theta}{mR^2}\,,\ \dot{z} = \frac{p_z}{m}$$

$$\begin{aligned} H &= p_\theta \dot{\theta} + p_z \dot{z} - L = mR^2\dot{\theta}^2 + m\dot{z}^2 - \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) + \frac{1}{2}k(R^2 + z^2) \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2 + \frac{1}{2}kR^2 = T + V \end{aligned}$$

$$H(z, p_\theta, p_z) = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2$$

- 2.2 Exemples de systèmes canoniques

$$H(z, p_\theta, p_z) = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2$$

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \\
\frac{dz}{dt} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \\
\frac{dp_\theta}{dt} &= -\frac{\partial H}{\partial \theta} = 0 \\
\frac{dp_z}{dt} &= -\frac{\partial H}{\partial z} = -kz
\end{aligned}$$

$$p_\theta = mR^2\dot{\theta} = \text{ constante}$$

$$\ddot{z} + \omega_0^2 z = 0 \quad \text{avec} \quad \omega_0^2 = \frac{k}{m}$$

• 2.2 Exemples de systèmes canoniques

Exemple 2 : La pendule plan

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell \cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta}$$

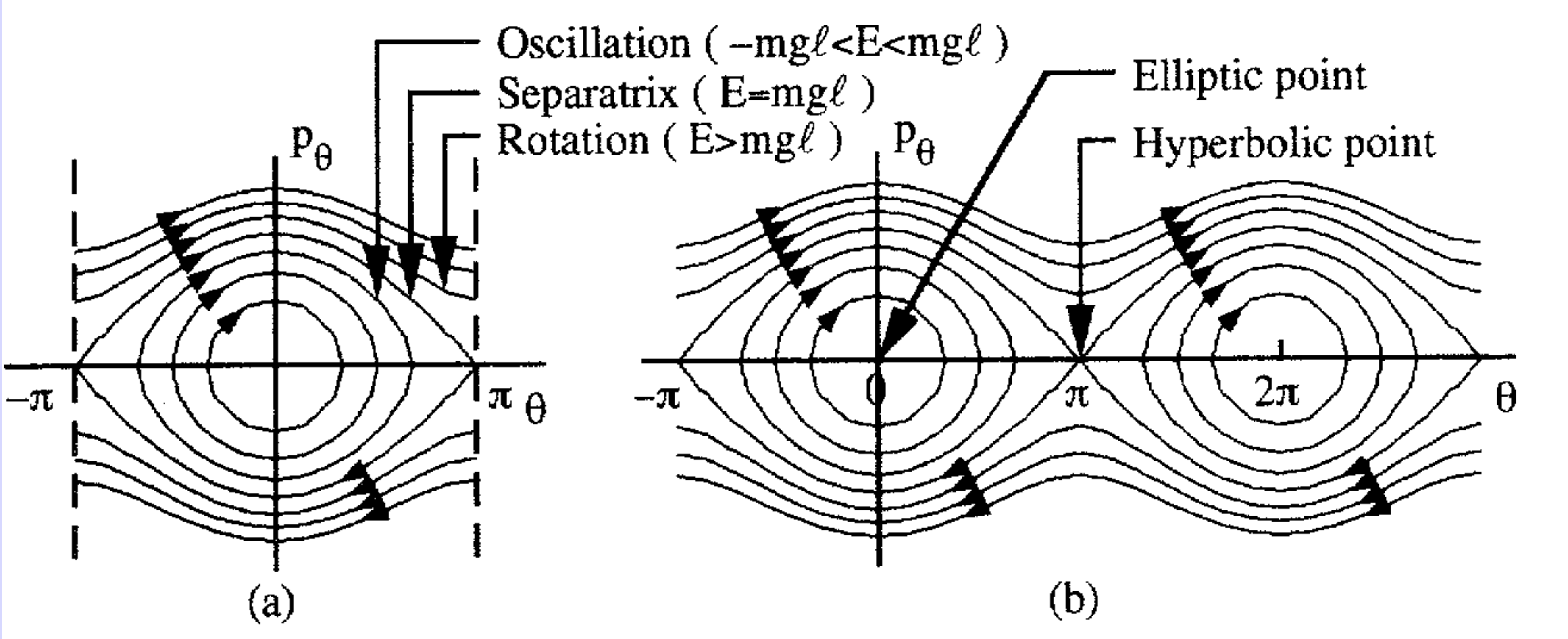
$$H = p_\theta \dot{\theta} - L = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell \cos\theta = \frac{p_\theta^2}{2m\ell^2} - mg\ell \cos\theta$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \\ \frac{dp_\theta}{dt} &= -\frac{\partial H}{\partial \theta} = -mg\ell \sin\theta \end{aligned}$$

• 2.2 Exemples de systèmes canoniques

(θ, p_θ)

$$\frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta = E$$

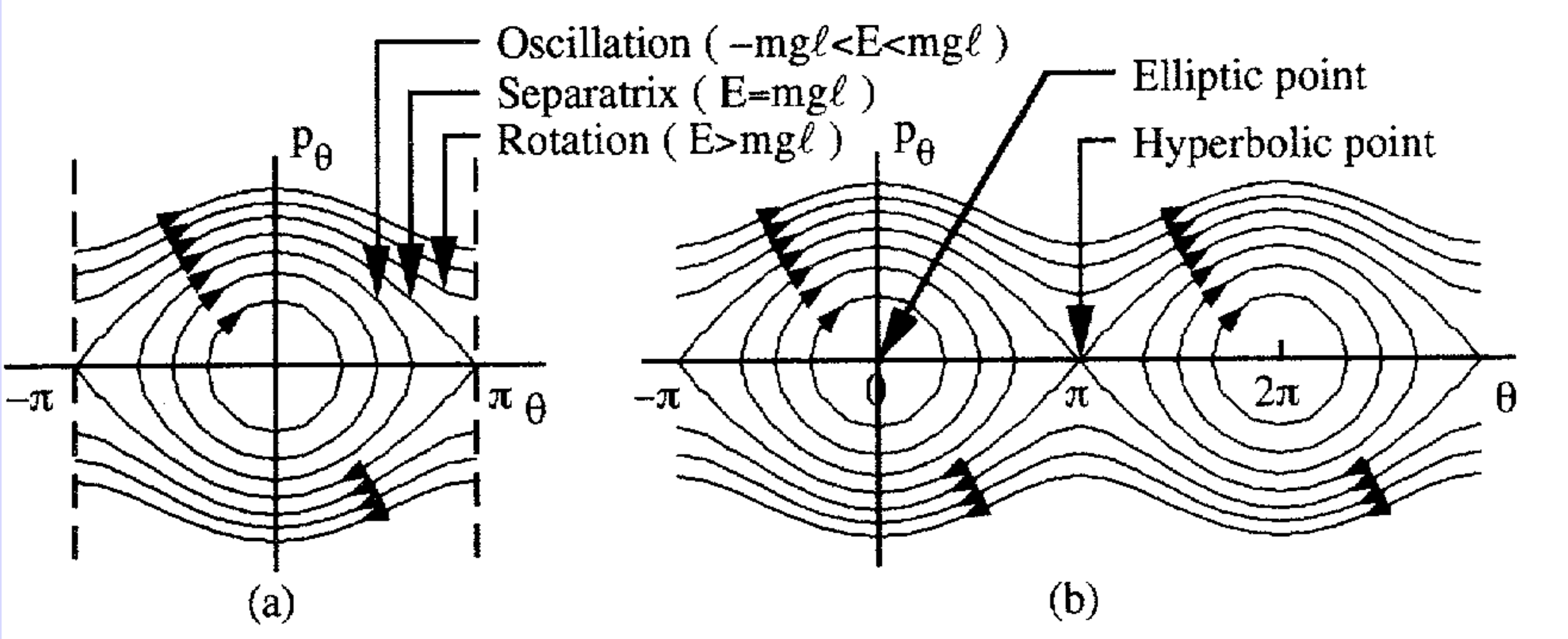


$\theta = \pi$ et $\theta = -\pi$

$(\theta = \pm\pi, p_\theta)$

$(\theta = \mp\pi, p_\theta)$

• 2.2 Exemples de systèmes canoniques



$$0 < E + mgl \ll mgl,$$

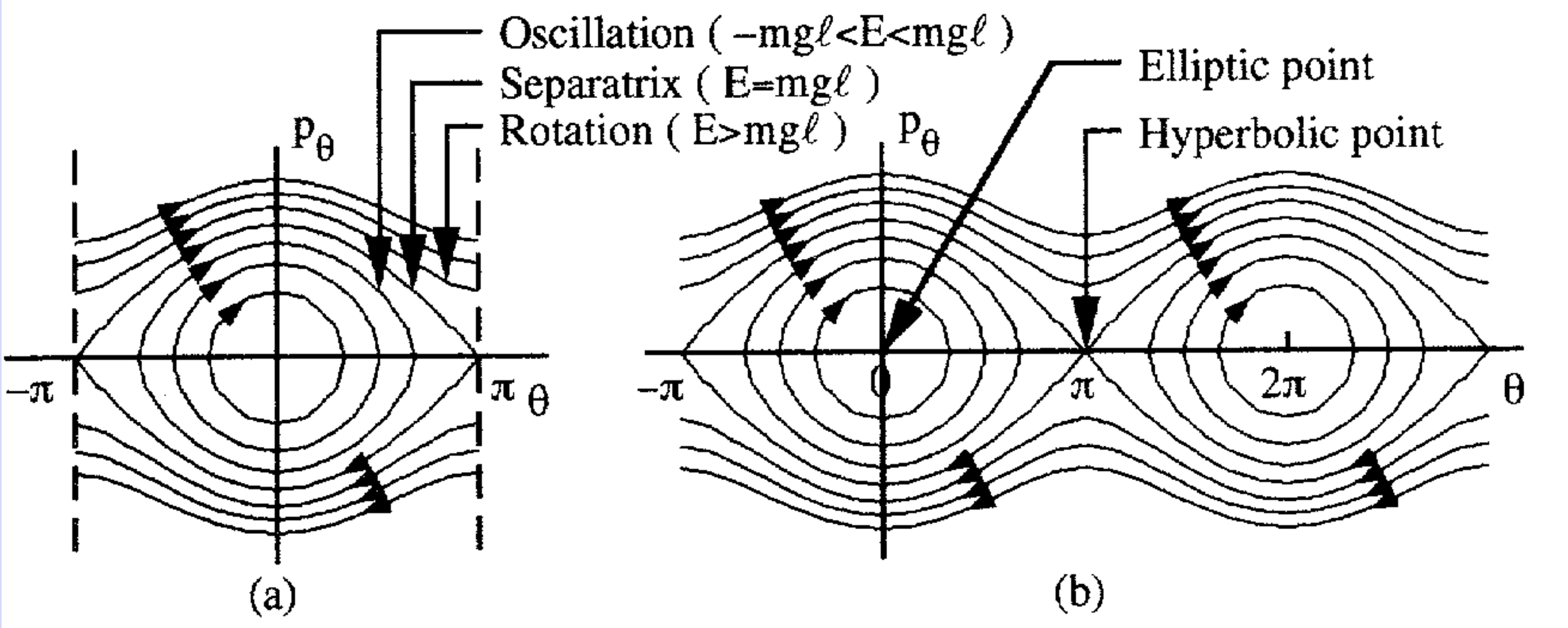
$$\frac{p_\theta^2}{2m\ell^2} + \frac{1}{2} mg\ell\theta^2 \simeq E + mg\ell$$

• 2.2 Exemples de systèmes canoniques

$$E = mg\ell$$

$$\theta = \pi \text{ (avec } p_\theta = 0)$$

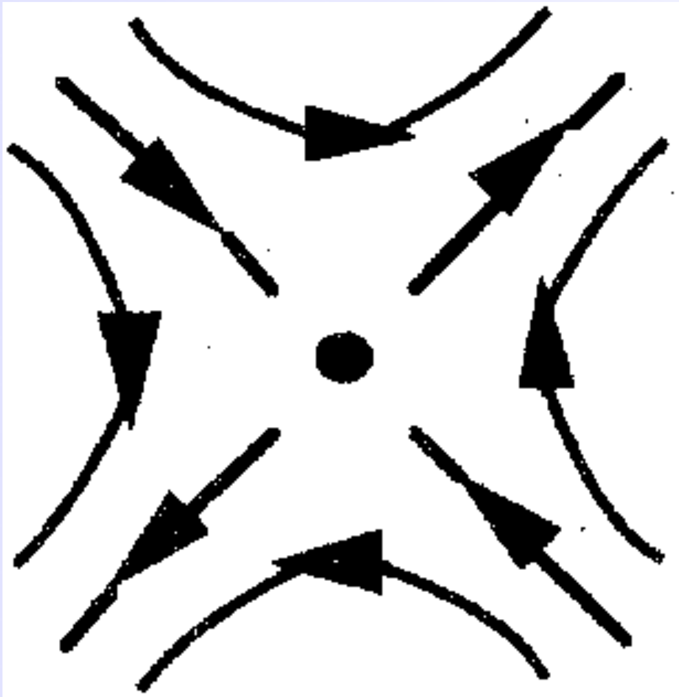
$$p_\theta = \pm 2m\ell\sqrt{g\ell}\cos(\theta/2)$$



• 2.2 Exemples de systèmes canoniques

$(\pm\pi, 0)$

$$\frac{p_\theta^2}{2m\ell^2} - \frac{1}{2}mg\ell(\pm\pi - \theta)^2 \simeq E - mg\ell$$



signe de $dp_\theta / d\theta$

• 2.3 Le principe variationnel d'Hamilton modifié

$$L(q, \dot{q}(q,p,t), t) = p_i \dot{q}_i(q,p,t) - H(q,p,t) \qquad (2.11)$$

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q,p,t)\right) dt = 0 \qquad (2.12)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i \dot{q}_i - H)}{\partial \dot{q}_k}\right) = \frac{\partial(p_i \dot{q}_i - H)}{\partial q_k} \Rightarrow \boxed{\dot{p}_k = -\frac{\partial H}{\partial q_k}} \qquad (2.13)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i \dot{q}_i - H)}{\partial \dot{p}_k}\right) = \frac{\partial(p_i \dot{q}_i - H)}{\partial p_k} \Rightarrow \boxed{0 = \dot{q}_k - \frac{\partial H}{\partial p_k}} \qquad (2.14)$$