Introducing a reasoning system based on ternary projective relations

Roland Billen[1] and Eliseo Clementini[2]

[1] Center for Geosciences, Department of Geography and Geomatics, University of Glasgow, Glasgow G12 8QQ, Scotland (UK), rbillen@geog.gla.ac.uk
[2] Department of Electrical Engineering, University of L'Aquila, I-67040 Poggio di Roio (AQ), Italy, eliseo@ing.univaq.it

Abstract. This paper introduces a reasoning system based on ternary projective relations between spatial objects. The model applies to spatial objects of the kind point and region, is based on basic projective invariants and takes into account the size and shape of the three objects that are involved in a relation. The reasoning system uses permutation and composition properties, which allow the inference of unknown relations from given ones.

Introduction

The field of Qualitative Spatial Reasoning (QSR) has experienced a great interest in the spatial data handling community due to its potential applications [1]. An important topic in QSR is the definition of reasoning systems on qualitative spatial relations. For example, regarding topological relations, the 9-intersection model [2] provides formal definitions for the relations and a reasoning system based on composition tables [3] establishes a mechanism to find new relations from a set of given ones.

Topological relations take into account an important part of geometric knowledge and can be used to formulate qualitative queries about the connection properties of close spatial objects, like "retrieve the lakes that are *inside* Scotland". Other qualitative queries that involve disjoint objects cannot be formulated in topological terms, for example: "the cities that are *between* Glasgow and Edinburgh", "the lakes that are *surrounded* by the mountains", "the shops that are on the *right* of the road", "the building that is *before* the crossroad". All these examples can be seen as semantic interpretations of underlying projective properties of spatial objects. As discussed in [4], geometric properties can be subdivided in three groups: topological, projective and metric. Most qualitative relations between spatial objects can be defined in terms of topological or projective properties [5], with the exception of qualitative distance and direction relations (such as *close*, *far*, *east*, *north*) that are a qualitative interpretation of metric distances and angles [6]. The use of projective properties for the definition of spatial relations is rather new. A model for ternary projective relations has been introduced for points and regions in [7]. The model is based on a basic geometric invariant in projective space, the collinearity of three points, and takes into account the size and shape of the three objects involved in a relation.

In first approximation, this work can be compared to research on qualitative relations dealing with relative positioning or cardinal directions [8-13]. Most approaches consider binary relations to which is associated a frame of reference, never avoiding the use of metric properties (minimum bounding rectangles, angles, etc.). To this respect, the main difference in our approach is that we only deal with projective invariants, disregarding distances and angles. Most work on projective relations deals with point abstractions of spatial features. In [9], the authors develop a model for cardinal directions between extended objects. Composition tables for the latter model have been developed in [14]. Freksa's double-cross calculus [15] is similar to our approach in the case of points. Such a calculus, as it has been further discussed in [16, 17], is based on ternary directional relations between points. However, in Freksa's model, an intrinsic frame of reference centred in a given point partitions the plane in four quadrants that are given by the front-back and right-left dichotomies. This leads to a greater number of qualitative distinctions with different algebraic properties and composition tables.

In this paper, we establish a reasoning system based on the ternary projective relations that were introduced in [7]. From a basic set of rules about the permutation and composition of relations, we will show how it is possible to infer unknown relations using the algebraic properties of projective relations. The paper is organized as follows. We start in Section 2 with introducing the general aspects of a reasoning system with ternary relations. In Section 3 we summarize the model for ternary projective relations between points and we present the associated reasoning systems. In section 4, recall the model in the case of regions and we introduce the reasoning system for this case too. In Section 5, we draw short conclusions and discuss some future developments.

2. Reasoning systems on ternary relations

In this section, we present the basis of a reasoning system on ternary projective relations. Usually, reasoning systems apply to binary spatial relations, for example, to topological relations [3] and to directional relations [18]. For binary relations, given three objects *a,b,c* and two relations *r(a,b)* and *r(b,c)*, the reasoning system allows to find the relation *r(a,c)*. This is done by giving an exhaustive list of results for all possible input relations, in the form of a composition table. The inverse relations complete the reasoning system, by finding, given the relation *r(a,b)*, the relation *r(b,a)*.

Reasoning with ternary relations is slightly more complex and it is not been applied a lot to spatial relations till now, with few exceptions [16, 17, 19]. The notation we use for ternary relations is of the kind $r(PO, RO_1, RO_2)$, where the first object *PO* represents the primary object, the second object RO_1 represents the first reference object and the third object RO_2 represents the second reference object. The primary object is the one that holds the relation *r* with the two reference objects, i.e., *PO* holds the relation *r* with RO_1 and RO_2. A reasoning system with ternary relations is based on two different sets of rules:

- a set of rules for permutations. Given three objects *a,b,c*, and a relation *r(a,b,c)*, these rules allow to find which are the other relations with permutations of the three arguments. There are 6 (=3!) potential arrangements of the arguments. The permutation rules correspond to the inverse relation of binary systems.
- a set of rules for composition. Given four objects *a,b,c,d*, and the two relations *r(a,b,c)* and *r(b,c,d)*, these rules allow to find the relation *r(a,c,d)*. The composition of relations r_1 and r_2 is indicated $r_1 \oplus r_2$.

Considering a set of relations P, it is possible to prove that the four following rules, three permutations and one composition, are sufficient to derive all the possible ternary relations out of a set of four arguments.

(1) $r(a,b,c) \rightarrow r'(a,c,b)$

(2) $r(a,b,c) \rightarrow r''(b,a,c)$

(3) $r(a,b,c) \rightarrow r'''(c,a,b)$

(4) $r_1(a,b,c) \oplus r_2(b,c,d) \rightarrow r_3(a,c,d)$

In the next sections, we will see how to apply such a ternary reasoning system in the case of projective ternary relations between points and between regions.

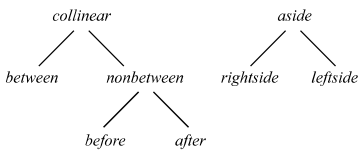
3. Reasoning system on ternary projective relations between points

The projective ternary relations between points have been introduced in a previous paper [7]. They have a straightforward definition because they are related to common concepts of projective geometry [20]. In section 3.1, we will only present the definitions and the concepts necessary for a good understanding of the reasoning system. In section 3.2, we show how to apply the reasoning system on these ternary projective relations.

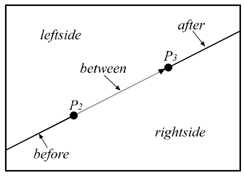
3.1. Ternary projective relations between points

Our basic set of projective relations is based on the most important geometric invariant in a projective space: the collinearity of three points. Therefore, the nature of projective relations is intrinsically ternary.

Given a relation $r(P_1, P_2, P_3)$, the points that act as reference objects must be distinct, in such a way they define a unique line passing through them, indicated with $\overline{P_2 P_3}$. When the relation needs an orientation on this line, the orientation is assumed to be from the first reference object to the second one: the oriented line is indicated with $\overrightarrow{P_2 P_3}$. The most general projective relations between three points are the *collinear* relation and its complement, the *aside* relation. The former one can be refined into *between* and *nonbetween* relations, and the latter one into *rightside* or *leftside* relations. In turn, the *nonbetween* relation can be subdivided into *before* and *after* relations, completing the hierarchical model of the projective relations between three points of the plane (see Figure 1.a). Out of this hierarchical model, five basic projective relations (*before*, *between*, *after*, *rightside*, *leftside*) are extracted. They correspond to the finest projective partition of the plane (see Figure 1.b).



a. Hierarchical model of the relations



b. Projective partition of the plane

Fig. 1. Projective relations between points

Definitions are given only for the *collinear* relation and the five basic relations.

Definition 1. A point P_1 is *collinear* to two given points P_2 and P_3, with $P_2 \neq P_3$, $collinear(P_1, P_2, P_3)$, if $P_1 \in \overline{P_2 P_3}$.

Definition 2. A point P_1 is *before* points P_2 and P_3, with $P_2 \neq P_3$, $before(P_1, P_2, P_3)$, if $collinear(P_1, P_2, P_3)$ and $P_1 \in (-\infty, P_2)$, where the last interval is part of the oriented line $\overrightarrow{P_2 P_3}$.

Definition 3. A point P_1 is *between* two given points P_2 and P_3, with $P_2 \neq P_3$, $between(P_1, P_2, P_3)$, if $P_1 \in [P_2 P_3]$.

Definition 4. A point P_1 is *after* points P_2 and P_3, with $P_2 \neq P_3$, $after(P_1, P_2, P_3)$, if $collinear(P_1, P_2, P_3)$ and $P_1 \in (P_3, +\infty)$, where the last interval is part of the oriented line $\overrightarrow{P_2 P_3}$.

Considering the two halfplanes determined by the oriented line $\overrightarrow{P_2 P_3}$, respectively the halfplane to the right of the line, which we indicate with $HP^+(\overrightarrow{P_2 P_3})$, and the halfplane to the left of the line, which we indicate with $HP^-(\overrightarrow{P_2 P_3})$, we may define the relations rightside and leftside.

Definition 5. A point P_1 is *rightside* of two given points P_2 and P_3, $rightside(P_1, P_2, P_3)$, if $P_1 \in HP^+(\overrightarrow{P_2 P_3})$.

Definition 6. A point P_1 is *leftside* of two given points P_2 and P_3, $leftside(P_1, P_2, P_3)$, if $P_1 \in HP^-(\overrightarrow{P_2 P_3})$.

3.2. Reasoning system

Using this model for ternary relations between points, it is possible to build a reasoning system, which allows the prediction of ternary relations between specific points. Such a reasoning system is an application of the reasoning system on ternary relations previously introduced. The four rules become:

(1) $r(P_1, P_2, P_3) \rightarrow r'(P_1, P_3, P_2)$

(2) $r(P_1, P_2, P_3) \rightarrow r''(P_2, P_1, P_3)$

(3) $r(P_1, P_2, P_3) \rightarrow r'''(P_3, P_1, P_2)$

(4) $r_1(P_1, P_2, P_3) \oplus r_2(P_2, P_3, P_4) \rightarrow r_3(P_1, P_3, P_4)$

For any ternary relations (P_1,P_2,P_3), Table 1 gives the corresponding relations resulting from permutation rules (1), (2) and (3). The following abbreviations are used: *bf* for *before*, *bt* for *between*, *af* for *after*, *rs* for *rightside* and *ls* for *leftside*. For example, knowing $bf(P_1,P_2,P_3)$, one can derive the relationships corresponding to the permutation of the three points, which are this case $af(P_1,P_3,P_2)$, $bt(P_2,P_1,P_3)$ and $af(P_3,P_1,P_2)$.

Table 1. Permutation table of ternary projective relations between points

$r(P_1, P_2, P_3)$	$r(P_1, P_3, P_2)$	$r(P_2, P_1, P_3)$	$r(P_3, P_1, P_2)$
bf	*af*	*bt*	*af*
bt	*bt*	*bf*	*bf*
af	*bf*	*af*	*bt*
rs	*ls*	*ls*	*rs*
ls	*rs*	*rs*	*ls*

Table 2 gives relations resulting from the composition rule (4). The first column of the table contains the basic ternary relations for (P_1, P_2, P_3) and the first row contains the basic ternary relations (P_2, P_3, P_4). The other cells give the deduced transitive relations for (P_1, P_3, P_4). For some entries, several cases may occur and all the possibilities are presented in the table.

Table 2. Composition table of ternary projective relations between points

	bf	***bt***	***af***	***rs***	***ls***
bf	*bf*	*af, bt*	*af*	*rs*	*ls*
bt	*bf*	*bt*	*af, bt*	*rs*	*ls*
af	*af, bt*	*bf*	*bf*	*ls*	*rs*
rs	*rs*	*ls*	*ls*	*af, rs, ls, bt*	*bf, rs, ls*
ls	*ls*	*rs*	*rs*	*bf, rs, ls*	*af, rs, ls, bt*

Using this reasoning system and knowing any two ternary relations between three different points out of a set of four, it is possible to predict the ternary relations between all the other possible combinations of three points out of the same set.

4. Reasoning system on ternary projective relations between regions

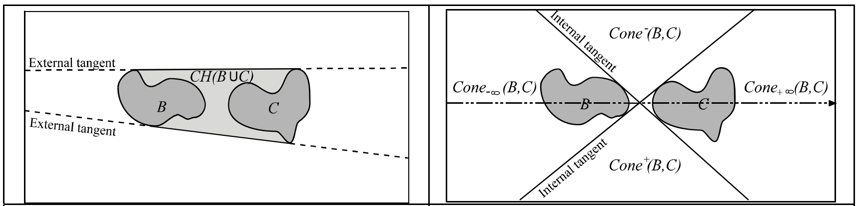
Likewise the section on relations between points, we first recall some concepts about the ternary projective relations between regions (section 4.1). Afterwards, we introduce the associated reasoning system including an example of application of such a system (section 4.2).

4.1. Ternary projective relations between regions

We will assume that a region is a regular closed point set possibly with holes and separate components. We will only present briefly the basic projective relations and the related partition of the space, while we refer to [7] for a more extended treatment. In the following, we indicate the convex hull of a region with a unary function *CH*(). As in the case of points, we use the notation $r(A, B, C)$ for projective relations between regions, where the first argument *A* is a region that acts as the primary object, while the second and third arguments *B* and *C* are regions that act as reference objects. The latter two regions must satisfy the condition $CH(B) \cap CH(C) = \varnothing$, that is, the intersection of their convex hulls must be empty. This condition allows to build a reference frame based on *B* and *C*, as it will be defined in this section. We also use the concept of orientation, which is represented by an oriented line connecting any point in *B* with any point in *C*.

Definition 7. Given two regions *B* and *C*, with $CH(B) \cap CH(C) = \varnothing$, a region *A* is *collinear* to regions *B* and *C*, $collinear(A, B, C)$, if for every point $P \in A$, there exists a line *l* intersecting *B* and *C* that also intersects *P*, that is:
$\forall P \in A, \exists l, (l \cap B \neq \varnothing) \wedge (l \cap C \neq \varnothing) \mid l \cap P \neq \varnothing$.

The projective partition of the space into five regions corresponding to the five basic projective relations is based, as it was for the points, on the definition of the general *collinear* relation between three regions. The portion of the space where this relation is true is delimited by four lines that are the common external tangents and the common internal tangents. Common external tangents of *B* and *C* are defined by the fact that they also are tangent to the convex hull of the union of *B* and *C* (figure 2.a). Common internal tangents intersect inside the convex hull of the union of regions B and C and divide the plane in four cones (figure 2.b). In order to distinguish the four cones, we consider an oriented line from region *B* to region *C* and we call $Cone_{-\infty}(B,C)$ the cone that contains region *B*, $Cone_{+\infty}(B,C)$ the cone that contains region *C*, $Cone^{+}(B,C)$ the cone that is to the right of the oriented line, $Cone^{-}(B,C)$ the cone that is to the left of the oriented line. We obtain a partition of the space into five regions, which correspond to the five basic projective relations *before*, *between*, *after*, *rightside*, and *leftside* (figure 3.a).



a. Common external tangents	b. Common internal tangents

Fig. 2. Internal and external tangents

Definition 8. A region *A* is *before* two regions *B* and *C*, $before(A,B,C)$, with $CH(B) \cap CH(C) = \varnothing$, if $A \subset Cone_{-\infty}(B,C) - CH(B \cup C)$.

Definition 9. A region *A* is *between* two regions *B* and *C*, $between(A,B,C)$, with $CH(B) \cap CH(C) = \varnothing$, if $A \subseteq CH(B \cup C)$.

Definition 10. A region *A* is *after* two regions *B* and *C*, $after(A,B,C)$, with $CH(B) \cap CH(C) = \varnothing$, if $A \subset Cone_{+\infty}(B,C) - CH(B \cup C)$.

Definition 11. A region *A* is *rightside* of two regions *B* and *C*, $rightside(A,B,C)$, with $CH(B) \cap CH(C) = \varnothing$, if *A* is contained inside $Cone^{+}(B,C)$ minus the convex hull of the union of regions *B* and *C*, that is, if $A \subset (Cone^{+}(B,C) - CH(B \cup C))$.

Definition 12. A region *A* is *leftside* of two regions *B* and *C*, $leftside(A,B,C)$, with $CH(B) \cap CH(C) = \varnothing$, if *A* is contained inside $Cone^{-}(B,C)$ minus the convex hull of the union of regions *B* and *C*, that is, if $A \subset (Cone^{-}(B,C) - CH(B \cup C))$.

The set of five projective relations *before*, *between*, *after*, *rightside*, and *leftside* can be used as a set of basic relations to build a model for all projective relations between three regions of the plane. The model, that we call the *5-intersection*, is synthetically expressed by a matrix of five values that are the empty/non-empty intersections of a region *A* with the five regions defined in Figure 3.b. In the matrix, a value 0 indicates an empty intersection, while a value 1 indicates a non-empty intersection. The five basic relations correspond to values of the matrix with only one non-empty value (Figure 4). In total, the 5-intersection matrix can have 2^5 different values that correspond to the same theoretical number of projective relations. Excluding the configuration with all zero values, which cannot exist, we are left with 31 different projective relations between the three regions *A*, *B* and *C*.

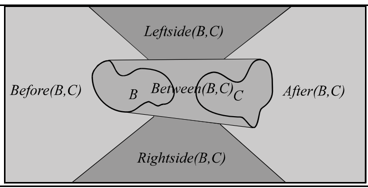
		$A \cap$ *leftside*(*B*,*C*)	
	$A \cap$ *Before*(*B*,*C*)	$A \cap$ *between*(*B*,*C*)	$A \cap$ *after*(*B*,*C*)
		$A \cap$ *rightside*(*B*,*C*)	
a. The partition of the plane in five regions	b. The 5-intersection model		

Fig. 3. Projective relations between regions

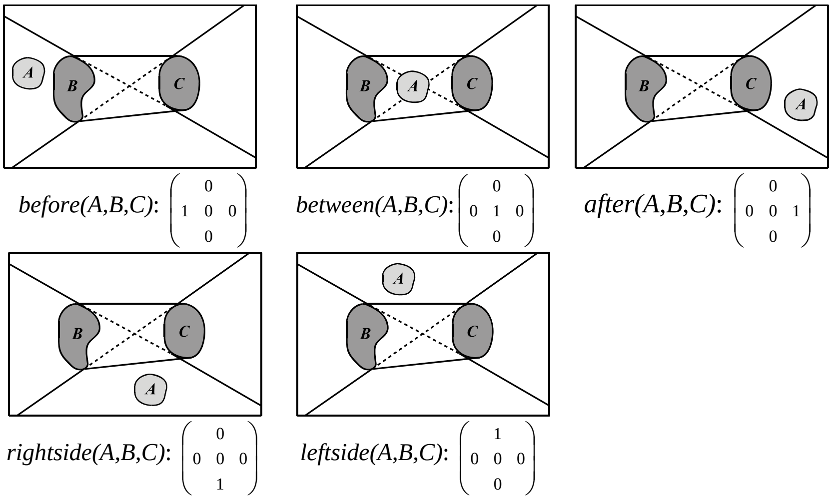


Fig. 4. The projective relations with object *A* intersecting only one of the regions of the plane.

4.2. Reasoning system

The reasoning system for regions is fully defined on the basis of the following relations:

(1) $r(A,B,C) \rightarrow r'(A,C,B)$

(2) $r(A,B,C) \rightarrow r''(B,A,C)$

(3) $r(A,B,C) \rightarrow r'''(C,A,B)$

(4) $r_1(A,B,C) \oplus r_2(B,C,D) \rightarrow r_3(A,C,D)$

Currently, the reasoning system has been established for the five basic projective relations only. We are working on its extension to the whole set of projective relations by developing a system which will combine the results of permutations and composition of the basic cases.

Table 3. Permutation table of ternary projective relations between regions

$r(A,B,C)$	$r'(A,C,B)$	$r''(B,A,C)$	$r'''(C,A,B)$
bf	*af*	bt , $(bt \wedge rs)$, $(bt \wedge ls)$, $(bt \wedge rs \wedge ls)$	af , $(af \wedge rs)$, $(af \wedge ls)$, $(af \wedge rs \wedge ls)$
bt	*bt*	bf , $(bf \wedge rs)$, $(bf \wedge ls)$, $(bf \wedge rs \wedge ls)$	bf , $(bf \wedge rs)$, $(bf \wedge ls)$, $(bf \wedge rs \wedge ls)$
af	*bf*	af , $(af \wedge rs)$, $(af \wedge ls)$; $(af \wedge rs \wedge ls)$	bt , $(bt \wedge rs)$, $(bt \wedge ls)$, $(bt \wedge rs \wedge ls)$
rs	*ls*	*ls*	*rs*
ls	*rs*	*rs*	*ls*

For any basic ternary relation *r(A,B,C)*, Table 3 gives the corresponding relations resulting from permutation rules (1), (2) and (3). The similarity with the permutation table for three points is clear. Only for some cases, there are exceptions to the basic permutations for points. In those cases, the "strong" relation (which is the one that holds also for points) can be combined with one or both of *leftside* and *rightside* relations.

The results of the composition rule (4) of the reasoning system are presented in Table 4. The first column of the table contains the basic ternary relations for r_1 (*A*,*B*,*C*) and the first row contains the basic ternary relations for r_2 (*B*,*C*,*D*). The other cells give the deduced $r_3(A,C,D)$ relations. In this table, we present only the single relations as results. The full composition relations can be obtained by combinations of these single relations. For example, the result of the composition *before(A,B,C)* $\oplus$ *before(B,C,D)* is: bf , rs , ls , $bf \wedge rs$, $bf \wedge ls$, $ls \wedge rs$, $bf \wedge rs \wedge ls$.

Table 4. Composition table of ternary projective relations between regions

	bf	***bt***	***af***	***rs***	***ls***
bf	*bf, rs, ls*	*bt, af, rs, ls*	*af*	*af, rs*	*af, ls*
bt	*bf*	*bt*	*bt, af, rs, ls*	*bf, bt, rs*	*bf, bt, ls*
af	*bf, bt, af, rs, ls*	*bf, bt, rs, ls*	*bt, af*	*bf, bt, ls*	*bf, bt, rs*
rs	*bf, bt, af, rs*	*bf, bt, af, ls*	*bf, bt, af, ls*	*bt, af, rs, ls*	*bf, rs, ls*
ls	*bf, bt, af, ls*	*bf, bt, af, rs*	*bf, bt, af, rs*	*bf, rs, ls*	*bt, af, rs, ls*

We will end this section by an example of application of the reasoning system. Given the relations *before(A,B,C)* and *rightside(B,C,D)*, we find out the potential relations for *r(A,B,D)*.

- Step 1: apply (1) to the first term of the transitive relations, and (2) to the second term (fig. 5a, b and c):

  $before(A, B, C) \rightarrow after(A, C, B)$;

  $rightside(B, C, D) \rightarrow leftside(C, B, D)$.

- Step 2: apply (4) to the following composition:

  $after(A, C, B) \oplus leftside(C, B, D) \rightarrow$

  $before(A, B, D)$ (fig 5.d)

  $\vee between(A, B, D)$ (fig 5.e)

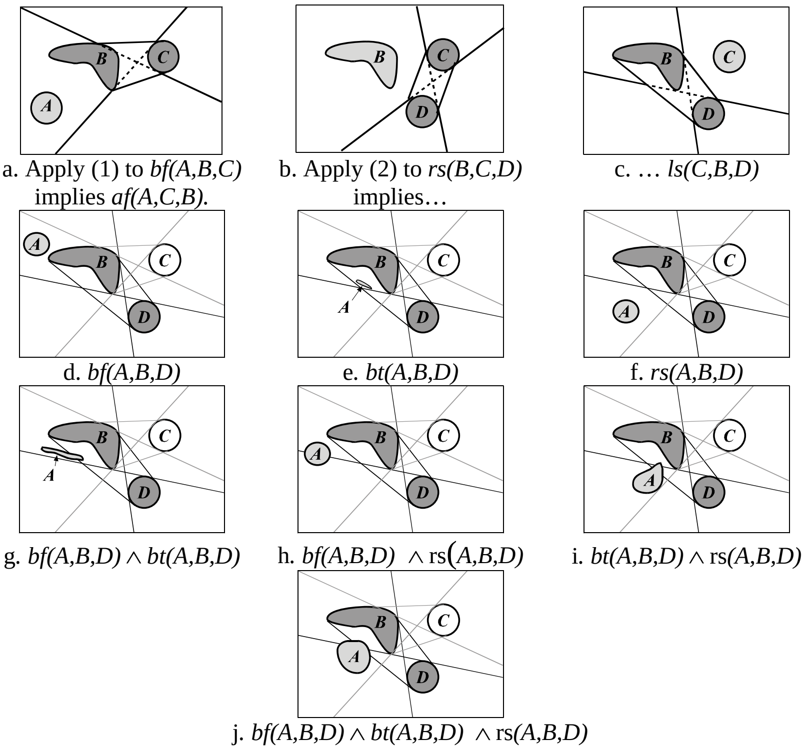
  $\vee rightside(A, B, D)$ (fig 5.f)

  $\vee (before(A, B, D) \wedge between(A, B, D))$ (fig 5.g)

  $\vee (before(A, B, D) \wedge rightside(A, B, D))$ (fig 5.h)

  $\vee (between(A, B, D) \wedge rightside(A, B, D))$ (fig 5.i)

  $\vee (before(A, B, D) \wedge between(A, B, D) \wedge rightside(A, B, D))$. (fig 5.j)



a. Apply (1) to *bf(A,B,C)* implies *af(A,C,B).*

b. Apply (2) to *rs(B,C,D)* implies…

c. … *ls(C,B,D)*

d. *bf(A,B,D)*

e. *bt(A,B,D)*

f. *rs(A,B,D)*

g. *bf(A,B,D)* ∧ *bt(A,B,D)*

h. *bf(A,B,D)* ∧ rs*(A,B,D)*

i. *bt(A,B,D)* ∧ rs*(A,B,D)*

j. *bf(A,B,D)* ∧ *bt(A,B,D)* ∧ rs*(A,B,D)*

Fig. 5. Example of application of the reasoning system between regions

5. Conclusion and future work

In this paper, we have introduced a reasoning system based on ternary projective relations between points and between regions. These sets of qualitative spatial relations, invariant under projective transformations, provide a new classification of configurations between three objects based on a segmentation of the space in five regions. The associated reasoning system allows inferring relations between three objects using permutations and compositions rules. It is the first step of the establishment of a whole qualitative reasoning based on projective properties of space.

In the future, the reasoning system has to be more formally defined; in particular the relations contained in permutation and composition tables have to be proved. Another issue that should be explored is the realisation of a complete qualitative spatial calculus for reasoning about ternary projective relations.

Acknowledgements

This work was supported by M.I.U.R. under project "Representation and management of spatial and geographic data on the Web".

References

1. Cohn, A.G. and S.M. Hazarika, *Qualitative Spatial Representation and Reasoning: An Overview.* Fundamenta Informaticae, 2001. **46**(1-2): p. 1-29.
2. Egenhofer, M.J. and J.R. Herring, *Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases*. 1991, Department of Surveying Engineering, University of Maine, Orono, ME.
3. Egenhofer, M.J., *Deriving the composition of binary topological relations.* Journal of Visual Languages and Computing, 1994. **5**(1): p. 133-149.
4. Clementini, E. and P. Di Felice, *Spatial Operators.* ACM SIGMOD Record, 2000. **29**(3): p. 31-38.
5. Waller, D., et al., *Place learning in humans: The role of distance and direction information.* Spatial Cognition and Computation, 2000. **2**: p. 333-354.
6. Clementini, E., P. Di Felice, and D. Hernández, *Qualitative representation of positional information.* Artificial Intelligence, 1997. **95**: p. 317-356.
7. Billen, R. and E. Clementini, *A model for ternary projective relations between regions*, in *EDBT2004 - 9th International Conference on Extending DataBase Technology,* E. Bertino, Editor. 2004, Springer-Verlag: Heraklion - Crete, Greece. p. 310-328.
8. Gapp, K.-P. *Angle, Distance, Shape, and their Relationship to Projective Relations*. in *Proceedings of the 17th Conference of the Cognitive Science Society*. 1995. Pittsburgh, PA.

9. Goyal, R. and M.J. Egenhofer, *Cardinal directions between extended spatial objects.* IEEE Transactions on Knowledge and Data Engineering, 2003. **(in press)**.
10. Kulik, L. and A. Klippel, *Reasoning about Cardinal Directions Using Grids as Qualitative Geographic Coordinates*, in *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science: International Conference COSIT'99*, C. Freksa and D.M. Mark, Editors. 1999, Springer. p. 205-220.
11. Schmidtke, H.R., *The house is north of the river: Relative localization of extended objects*, in *Spatial Information Theory. Foundations of Geographic Information Science: International Conference, COSIT 2001*, D.R. Montello, Editor. 2001, Springer. p. 415-430.
12. Moratz, R. and K. Fischer. *Cognitively Adequate Modelling of Spatial Reference in Human-Robot Interaction*. in *Proc. of the 12th IEEE International Conference on Tools with Artificial Intelligence, ICTAI 2000*. 2000. Vancouver, BC, Canada.
13. Schlieder, C., *Reasoning about ordering*, in *Spatial Information Theory: A Theoretical Basis for GIS - International Conference, COSIT'95*, A.U. Frank and W. Kuhn, Editors. 1995, Springer-Verlag: Berlin. p. 341-349.
14. Skiadopoulos, S. and M. Koubarakis, *Composing cardinal direction relations.* Artificial Intelligence, 2004. **152**(2): p. 143-171.
15. Freksa, C., *Using Orientation Information for Qualitative Spatial Reasoning*, in *Theories and Models of Spatio-Temporal Reasoning in Geographic Space*, A.U. Frank, I. Campari, and U. Formentini, Editors. 1992, Springer-Verlag: Berlin. p. 162-178.
16. Scivos, A. and B. Nebel, *Double-Crossing: Decidability and Computational Complexity of a Qualitative Calculus for Navigation*, in *Spatial Information Theory. Foundations of Geographic Information Science: International Conference, COSIT 2001*, D.R. Montello, Editor. 2001, Springer. p. 431-446.
17. Isli, A. *Combining Cardinal Direction Relations and other Orientation Relations in QSR*. in *AI&M 14-2004, Eighth International Symposium on Artificial Intelligence and Mathematics*. 2004. January 4-6, 2004, Fort Lauderdale, Florida.
18. Frank, A.U., *Qualitative Reasoning about Distances and Directions in Geographic Space.* Journal of Visual Languages and Computing, 1992. **3**(4): p. 343-371.
19. Isli, A. and A.G. Cohn, *A new approach to cyclic ordering of 2D orientations using ternary relation algebras.* Artificial Intelligence, 2000. **122**: p. 137-187.
20. Struik, D.J., *Projective Geometry*. 1953, London: Addison-Wesley.