Time-varying modal parameters identification in the modal domain

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The research focuses on the identification of time-varying systems

$$\boldsymbol{M}(t)\,\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}(t)\,\dot{\boldsymbol{y}}(t) + \boldsymbol{K}(t)\,\boldsymbol{y}(t) = \boldsymbol{f}(t)$$

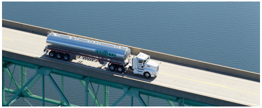
Dynamics of such systems is characterized by :

- Non-stationary time series
- Instantaneous modal properties
  - Frequencies : $\omega_r(t)$
  - Damping ratio's : $\xi_r(t)$
  - Modal deformations : $\boldsymbol{q}_r(t)$

Why time-varying behaviour can occur ?

Several possible origins :

- Structural changes



- Operating conditions

- Damage appearance

Outline of the presentation

Presentation of the proposed method

Presentation of the experimental test setup

Linear time invariant and time-varying identifications of the system

Presentation of the results

The parameterization is done by modal state-space modeling

First in the LTI case

Starting with the following ***innovation state-space model***:

$$\begin{cases} \boldsymbol{x}[t+1] &= \boldsymbol{F}\,\boldsymbol{x}[t] + \boldsymbol{K}\,\boldsymbol{e}[t] \\ \boldsymbol{y}[t] &= \boldsymbol{C}\,\boldsymbol{x}[t] + \boldsymbol{e}[t] \end{cases}$$

we can transform it into a ***modal form***

$$\begin{cases} \boldsymbol{\eta}[t+1] &= \boldsymbol{A}\,\boldsymbol{\eta}[t] + \boldsymbol{\Psi}\,\boldsymbol{e}[t] \\ \boldsymbol{y}[t] &= \boldsymbol{\Phi}\,\boldsymbol{\eta}[t] + \boldsymbol{e}[t] \end{cases}$$

with

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{V}^{-1}\,\boldsymbol{F}\,\boldsymbol{V}, \\ \boldsymbol{\eta} &= \boldsymbol{V}^{-1}\,\boldsymbol{x}, \\ \boldsymbol{\Phi} &= \boldsymbol{C}\,\boldsymbol{V}, \\ \boldsymbol{\Psi} &= \boldsymbol{V}^{-1}\,\boldsymbol{K}. \end{aligned}$$

The parameterization is done by modal state-space modeling

First in the LTI case

To avoid treating such complex values, all the parameters are separated into their real and imaginary parts.

The modal decoupling is still valid.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & & & \\ & \boldsymbol{A}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_n \end{bmatrix} \qquad \begin{aligned} \boldsymbol{\Phi} &= \begin{bmatrix} \boldsymbol{\Phi}_1^{\mathcal{R}} & \boldsymbol{\Phi}_1^{\mathcal{I}} & \boldsymbol{\Phi}_2^{\mathcal{R}} & \boldsymbol{\Phi}_2^{\mathcal{I}} & \cdots \end{bmatrix} \\ \boldsymbol{\Psi}^T &= \begin{bmatrix} \boldsymbol{\Psi}_1^{\mathcal{R}} & \boldsymbol{\Psi}_1^{\mathcal{I}} & \boldsymbol{\Psi}_2^{\mathcal{R}} & \boldsymbol{\Psi}_2^{\mathcal{I}} & \cdots \end{bmatrix} \end{aligned}$$

with

$$\boldsymbol{A}_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}$$

All the coefficients of these matrices are stacked in a parameter vector $\boldsymbol{\theta}$.

Identification of the parameters

The identification is performed by the minimization of the ***prediction error*** $\boldsymbol{e}[t, \boldsymbol{\theta}] = \boldsymbol{y}[t] - \hat{\boldsymbol{y}}[t, \boldsymbol{\theta}]$.

If a hat ($\hat{}$) denotes estimated quantities, we have

$$\begin{cases} \hat{\boldsymbol{\eta}}[t+1, \boldsymbol{\theta}] & = & \boldsymbol{A}\,\hat{\boldsymbol{\eta}}[t, \boldsymbol{\theta}] + \boldsymbol{\Psi}\,\left(\boldsymbol{y}[t] - \hat{\boldsymbol{y}}[t, \boldsymbol{\theta}]\right) \\ \hat{\boldsymbol{y}}[t, \boldsymbol{\theta}] & = & \boldsymbol{\Phi}\,\hat{\boldsymbol{\eta}}[t, \boldsymbol{\theta}] \end{cases},$$

or equivalently

$$\begin{cases} \hat{\boldsymbol{\eta}}[t+1, \boldsymbol{\theta}] & = & \left(\boldsymbol{A} - \boldsymbol{\Psi}\boldsymbol{\Phi}\right)\,\hat{\boldsymbol{\eta}}[t, \boldsymbol{\theta}] + \boldsymbol{\Psi}\,\boldsymbol{y}[t] \\ \hat{\boldsymbol{y}}[t, \boldsymbol{\theta}] & = & \boldsymbol{\Phi}\,\hat{\boldsymbol{\eta}}[t, \boldsymbol{\theta}] \end{cases}.$$

Identification of the parameters

The minimization is performed in a least squares sense, i.e. minimizing the following cost function

$$\begin{aligned} V(\boldsymbol{\theta}) &= \frac{1}{N} \sum_{t=1}^{N} \boldsymbol{e}[t,\boldsymbol{\theta}]^T \boldsymbol{e}[t,\boldsymbol{\theta}] \\ &= \frac{1}{N} \boldsymbol{E}(\boldsymbol{\theta})^T \boldsymbol{E}(\boldsymbol{\theta}). \end{aligned}$$

with $\boldsymbol{E}(\boldsymbol{\theta}) = \left[\boldsymbol{e}[1,\boldsymbol{\theta}]^T \; \boldsymbol{e}[2,\boldsymbol{\theta}]^T \; \cdots \; \boldsymbol{e}[N,\boldsymbol{\theta}]^T \right]^T$ gathering all the residual terms in a single vector.

Identification of the parameters

A classical ***Levenberg-Marquardt*** optimization scheme is used. The set of parameters is iteratively updated

$$\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \boldsymbol{d}$$

where $\boldsymbol{d}$ is the update to the vector of parameters. This update is the solution of the following equation:

$$\left(\boldsymbol{J}^T\boldsymbol{J} + \lambda\boldsymbol{I}\right)\,\boldsymbol{d} = \boldsymbol{J}^T\boldsymbol{E}$$

in which $\boldsymbol{J}$ is the *Jacobian* of the residual vector and λ is the *Marquardt parameter* regularizing the equation.

Identification of the parameters

The calculation of the Jacobian matrix represents the major task of the algorithm.

Taking the derivative of $\boldsymbol{E}(\boldsymbol{\theta})$ with respect to the $\boldsymbol{\theta}$ vector, we have

$$\boldsymbol{J} = \frac{\partial \boldsymbol{E}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} = -\frac{\partial \hat{\boldsymbol{Y}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T}$$

if $\hat{\boldsymbol{Y}}(\boldsymbol{\theta}) = \left[\hat{\boldsymbol{y}}[1]^T \; \hat{\boldsymbol{y}}[2]^T \; \cdots \; \hat{\boldsymbol{y}}[N]^T\right]^T$.

The Jacobian matrix is computed by calculating the derivative of the output estimate

- If θ_i belongs to the $\boldsymbol{A}$ matrix :

$$\begin{cases} \hat{\boldsymbol{z}}[t+1] & = & (\boldsymbol{A} - \boldsymbol{\Psi}\boldsymbol{\Phi})\ \hat{\boldsymbol{z}}[t] + \boldsymbol{E}_{k,l}\,\hat{\boldsymbol{\eta}}[t,\boldsymbol{\theta}] \\ \dfrac{\partial \hat{\boldsymbol{y}}[t,\boldsymbol{\theta}]}{\partial \theta_i} & = & \boldsymbol{\Phi}\,\hat{\boldsymbol{z}}[t] \end{cases},$$

- If θ_i belongs to the $\boldsymbol{\Phi}$ matrix :

$$\begin{cases} \hat{\boldsymbol{z}}[t+1] & = & (\boldsymbol{A} - \boldsymbol{\Psi}\boldsymbol{\Phi})\ \hat{\boldsymbol{z}}[t] - \boldsymbol{\Psi}\boldsymbol{E}_{k,l}\,\hat{\boldsymbol{\eta}}[t,\boldsymbol{\theta}] \\ \dfrac{\partial \hat{\boldsymbol{y}}[t,\boldsymbol{\theta}]}{\partial \theta_i} & = & \boldsymbol{\Phi}\,\hat{\boldsymbol{z}}[t] + \boldsymbol{E}_{k,l}\,\hat{\boldsymbol{\eta}}[t,\boldsymbol{\theta}] \end{cases},$$

- If θ_i belongs to the $\boldsymbol{\Psi}$ matrix :

$$\begin{cases} \hat{\boldsymbol{z}}[t+1] & = & (\boldsymbol{A} - \boldsymbol{\Psi}\boldsymbol{\Phi})\ \hat{\boldsymbol{z}}[t] - \boldsymbol{E}_{k,l}\boldsymbol{\Phi}\,\hat{\boldsymbol{\eta}}[t,\boldsymbol{\theta}] + \boldsymbol{E}_{k,l}\,\boldsymbol{y}[t] \\ \dfrac{\partial \hat{\boldsymbol{y}}[t,\boldsymbol{\theta}]}{\partial \theta_i} & = & \boldsymbol{\Phi}\,\hat{\boldsymbol{z}}[t] \end{cases},$$

where $\hat{\boldsymbol{z}}[t] = \frac{\partial \hat{\boldsymbol{\eta}}[t,\boldsymbol{\theta}]}{\partial \theta_i}$ and $\boldsymbol{E}_{k,l}$ is an appropriate single entry matrix corresponding to θ_i.

Time-varying expansion of the model

Let us introduce the time variation into the model:

- All the elements in the $\boldsymbol{A}$, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ matrices may vary with time

$$\left\{ \begin{array}{rcl} \boldsymbol{\eta}[t+1] & = & \boldsymbol{A}[t]\,\boldsymbol{\eta}[t] + \boldsymbol{\Psi}[t]\,\boldsymbol{e}[t] \\ \boldsymbol{y}[t] & = & \boldsymbol{\Phi}[t]\,\boldsymbol{\eta}[t] + \boldsymbol{e}[t] \end{array} \right.$$

We assume that each of the time-varying parameters $\boldsymbol{\theta}[t]$ is expanded in a ***basis of time functions***

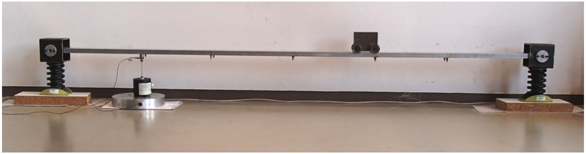
$$\theta_i[t] = \sum_{j=1}^{n_f} \theta_{i,\,j}\, f_j[t]$$

The identification is now performed on the time invariant $\theta_{i,\,j}$ coefficients.

The experimental setup

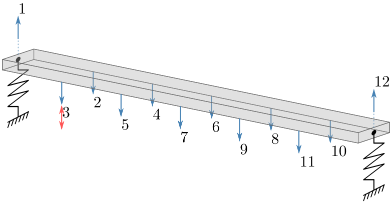
The experimental setup is an aluminum beam with a moving mass.

The whole system is supported by springs and excited by a shaker.

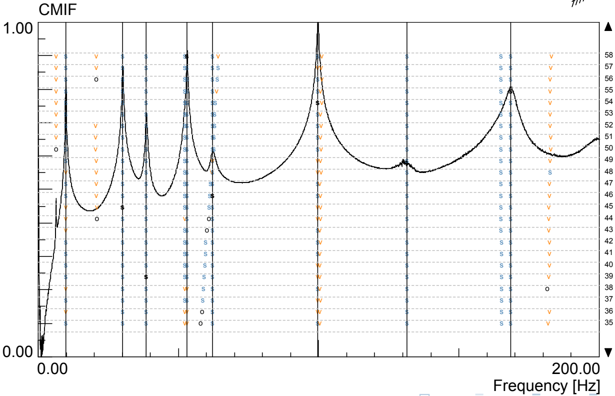


- 2.1 meter long and 8×2 cm for the cross section
- 9 kg for the beam and 3.475 kg for the moving mass (ratio of 38.6 %)

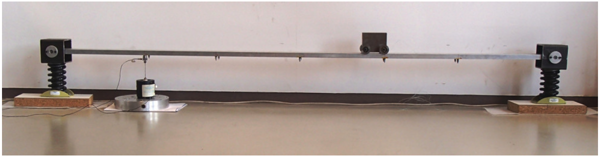
Let us do first a LTI modal analysis of the beam subsystem

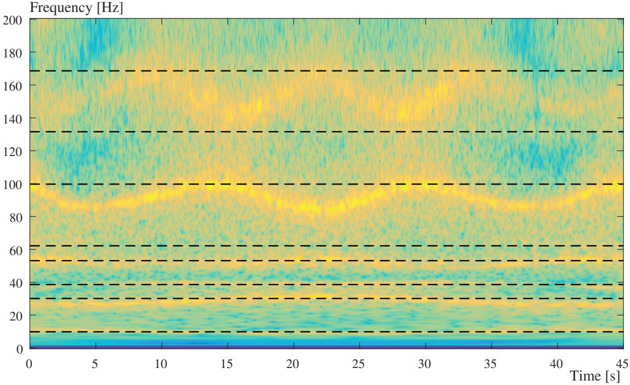


	f_r [Hz]	ζ_r [%]
1	9.86	0.32
2	30.12	0.52
3	38.6	0.65
4	53.14	0.28
5	62.17	1.57
6	99.70	0.28
7	131.57	2.039
8	168.60	0.99



Identification of this system with the time-varying modal model





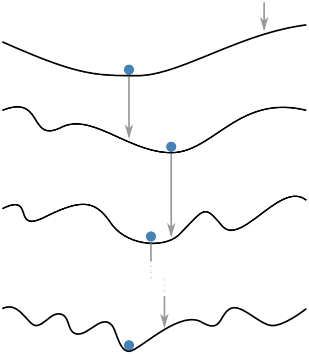
A note on the computational aspect

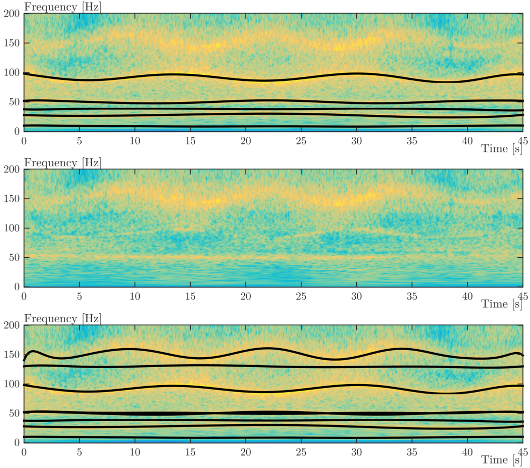
The computational efforts and memory management are not negligible with this method.

- The number of parameters may be large (risk of local minima).
- Therefore the number of state-space realizations for the Jacobian computation may be large too.
- The size of the Jacobian matrix may be huge.

One solution is to perform the optimization by steps

It is possible to benefit from the modal structure of the model for graduated optimization.





Thanks to the multivariate modeling, the mode shapes may be obtained too

To conclude

The basis functions approach is suitable to manage the time variation of the dynamic properties.

The The modal structure of the proposed model offer some advantages.

The varying modal parameters are well tracked with the proposed model. The time-varying mode shapes are identified too.

Thank you for your attention