Advanced Graph Theory and Combinatorics

To Christelle, Aurore and Maxime

Series Editor
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Foreword

The book content reflects the (good) taste of the author for solid mathematical concepts and results that have exciting practical applications. It is an excellent textbook that should appeal to students and instructors for its very clear presentation of both classical and more recent concepts in graph theory.

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September 2016

Introduction

This book is a result of lecture notes from a graph theory course taught at the University of Liège since 2005. Through the years, this course evolved and lectures were given at different levels ranging from second-year undergraduates in mathematics to students in computer science entering master's studies. It was therefore quite challenging to find a suitable title for this book.

Advanced or not so advanced material?

I hope that the reader will not feel cheated by the title (which is always tricky to choose). In some aspects, the material is rather elementary: we will start from scratch and present basic results on graphs such as connectedness or Eulerian graphs. In the second part of the book, we will analyze in great detail the strongly connected components of a digraph and make use of Perron–Frobenius theory and formal power series to estimate asymptotics on the number of walks of a given length between two vertices. Topics with an algebraic or a combinatorial flavor such as Ramsey numbers, introduction to Robertson–Seymour theorem or PageRank can be considered as more advanced.

In the history of mathematics, we often mention the *seven bridges of Königsberg problem* as the very first problem in graph theory. It was studied by the famous mathematician L. Euler in 1736. It took two centuries to develop and build a complete theory from a few scattered results. Probably the first book on graphs is *Theorie der endlichen und unendlichen Graphen* [KÖN 90] written by the Hungarian mathematician D. König in 1936, a student of J. Kürschák and H. Minkowski. In the middle of the 20th Century, graph theory matured into a well-defined branch of discrete mathematics and combinatorics. We observe many mathematicians turning their attention to graph theory with books by C. Berge, N. Biggs, P. Erdős, C. Kuratowski, W. T. Tutte, K. Wagner, etc. We have seen an important growth during the past decades in combinatorics because of the particular interactions existing with optimization, randomized algorithms, dynamical programming, ergodic or number theory, theoretical computer science, computational geometry, molecular biology, etc. On MathSciNet, if you look for research papers with a Primary Mathematics Subject Classification equal to `05C` (which stands for graph theory and is divided into 38 subfields ranging from planar graphs to connectivity, random walks or hypergraphs), then we find for the period 2011–2015 between $3,300$ and $3,700$ papers published every single year.

In less than a century, many scientists and entrepreneurs have seen the importance of graph theory in real-life applications. In a recent issue of Wired magazine (March 2014), we can read an article entitled *Is graph theory a key to understanding big data?* by R. Marsten. Let us quote his conclusion: “The data that we have today, and often the ways we look at data, are already steeped in the theory of graphs. In the future, our ability to understand data with graphs will take us beyond searching for an answer. Creating and analyzing graphs will bring us to answers automatically”. Later, in the same magazine (May 2014), E. Eifrem wrote: “We’re all well aware of how Facebook and Twitter have used the social graph to dominate their markets, and how Facebook and Google are now using their Graph Search and Knowledge Graph, respectively, to gear up for the next wave of hyper-accurate and hyper-personal recommendations, but graphs are becoming very widely deployed in a host of other industries”.

This book reflects the tastes of the author but also includes some important applications such as Google’s PageRank. It is only assumed that the reader has a working knowledge of linear algebra. Nevertheless, all the definitions and important results are given for the sake of completeness. The aim of the book is to *provide the reader with the necessary theoretical background to tackle new problems or to easily understand new concepts in graph theory*. Algorithms and complexity theory occupy a very small portion of the book (mostly in the first chapters).

This book, others and inspiration

Many other books on graphs do exist and the reader should not limit himself or herself to a single source. The Internet is also a formidable resource. Even if we have to be cautious when looking for information on the Web, Wikipedia contains a wealth of relevant information (but keep a critical eye). The present book starts with some unavoidable general material, then moves on to some particular topics with a combinatorial flavor. Powers of the adjacency matrix and Perron theory have a predominant role. The reader could probably start with this book and then move to [BRU 11] as a good companion to get a deeper knowledge of the links between linear algebra and graphs. See also [BAP 14] or the classical [GOD 01] in algebraic graph theory. Similarly, a comprehensive presentation of PageRank techniques can be found in [LAN 06]. The authors of that book, A. Langville and C. D. Meyer, also specialized in ranking techniques (see [LAN 12]). Another general reference discussing partially the same topics as here – and I do hope, with the same philosophy – is by R. Diestel where much more material and a particular emphasis on infinite graphs may be found. The present book is smaller and is thus well suited for readers who do not want to spend too much time on a specialized topic.

Having given a graph theory course for more than 10 years, I'm probably unconsciously tempted to take ownership of the proofs that I found somewhere else. It is no easy task to cite and give credits to all the sources that inspired me in this process. Let me mention [BIG 93] for algebraic aspects and chromatic polynomial, [TUT 01] for its first chapters and [ORE 90] for planar graphs and his proof of the 5-color theorem. I should also mention [BOL 98] (with an impressive collection of exercises) and [BER 89]. Finally, I remember that projects available on the Web and run by D. Arnold (College of Redwoods) were inspiring.

Practical organization

For a one-semester course, here is the time I usually devote to selected topics with 15 lectures of 90 min. Moreover, sessions for exercises take the same amount of time. The book contains extra material and more than 115 exercises:

– digraphs, paths, connected components (sections 1.1 and 1.2);

– Eulerian graphs, distance and shortest path (sections 1.3 and 1.5);

– introduction to Hamiltonicity, applications (sections 1.4 and 1.6);

– trees (section 4.1);

– homomorphisms, group of automorphisms (sections 7.1 and 7.2);

– Hamiltonian graphs (sections 3.1–3.4);

– topological sort (Chapter 4);

– adjacency matrix, counting walks (sections 8.2 and 8.3);

– primitivity, Perron's theorem and asymptotics (sections 9.1 and 9.4);

– irreducibility and asymptotics (sections 9.2 and 9.4);

– applications of Perron–Frobenius theory (section 9.3);

– Google's PageRank (Chapter 10);

– planar graphs and Euler's formula (sections 6.1–6.3);

– colorings, the five-color theorem (section 6.5);

– Ramsey numbers (section 7.4).

Definitions are emphasized and the most important ones are written in bold face, so that definitions of recurrent notions can be found more easily. Labels of bibliographic entries are based on the first three letters of the last name of the first author and then the year of publication. In the bibliography, entries are sorted in alphabetical order using these labels.

I address special thanks to Émilie Charlier, Aline Parreau and Manon Stipulanti for their great feedback leading to some major improvements in this book. Of course, Valérie Berthé always plays a special role. I am very pleased to blame her (indeed, this adventure produced some pressure from time to time) for what this project finally looks like. She is always supportive and enthusiastic. I also thanks several colleagues: Benoît Donnet, Eric Duchêne, Fabien Durand, Narad Rampersad, Eric Rowland and Élise Vandomme for the valuable time they spent reading drafts of parts of this book. I will not forget Laurent Waxweiler who gave and prepared the very first exercise classes for my course. I also thank my many students along the years. Their questions, queries and enthusiasm allowed me to improve, over the time, the overall presentation and sequencing of this book.

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September, 2016