Balanced words and related concepts: applications and complexity issues

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comex
combinatorial optimization:
metaheuristics & exact methods

Introduction

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Outline

Apportionment

- A House of representatives has 50 seats
- 37000 citizens elect representatives
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- Well-studied problem (for a couple of centuries)
- Apportionment algorithms are mostly sequential allocation methods: first seat, then second one,...
- E.g., Webster's method for above example:
  A B A B A A B C A B A C...

Just-In-Time production scheduling

- p product types (A, B, C,...; red, blue, green,...)
- n_i items of type $i = 1, \ldots, p$
- $n = \sum_i n_i$ = total number of items
- unit production times
- $r_i = \frac{n_i}{n}$ = proportion of items of type i
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- Determine a production schedule of all items such that, at every instant k, the number of items of type i that have been produced is as close as possible to kr_i.

JIT scheduling: Example

$n_1 = 3$ $n_2 = 3$ $n_3 = 1$

$r_1 = 3/7$ $r_2 = 3/7$ $r_3 = 1/7$

kr_1	3/7	6/7	9/7	12/7	15/7	18/7	21/7
x_{1k}	1	1	2	2	2	3	3
dev	4/7	1/7	5/7	2/7	1/7	3/7	0

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  A B A C A D A E A F A G A H A

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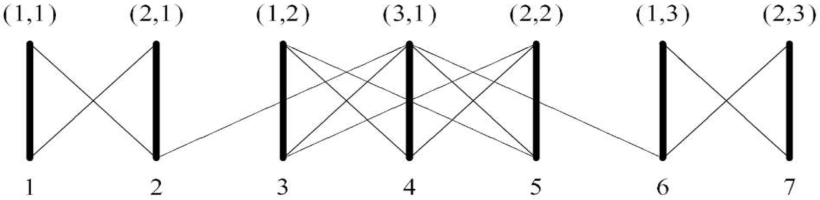
- Bipartite matching model:
  - precompute the time-slots k to which j-th item of type i can be assigned so that $|j - kr_i| \leq B$
  - put an edge between (i, j) and k
  - determine whether the graph has a perfect matching

JIT scheduling: Bipartite graph

- 3 part types
- $n_1 = 3$, $n_2 = 3$, $n_3 = 1$
- $n = 7$ time slots
- $B = 5/7$



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- Can we do better?

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- Pseudo-polynomial (input length is: $\log n_1 + \log n_2 + \ldots \log n_p$)
- Can we do better?
- Is the MDJIT problem in NP?

Algebraic characterization

Theorem (Brauner and Crama *DAM* 2004)

MDJIT has a solution with maximum deviation at most B if and only if the following hold for all $x_1, x_2 \in \{1, 2, \ldots, n\}$ with $x_1 \leq x_2$:

$\sum_i \max(0, \lfloor x_2 r_i + B \rfloor - \lceil (x_1 - 1) r_i - B \rceil) \geq x_2 - x_1 + 1$

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MDJIT is in co-NP.

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MDJIT is in co-NP.

Corollary 2

For fixed p, MDJIT can be solved in polynomial time.

- **Proof.** Express the NSC as linear inequalities in integer variables; use Lenstra's algorithm.
- Easy when $p = 2$. We know nothing smarter when $p \geq 3$

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Corollary 4 (Jost 2006)

There always exists a 3-*balanced schedule* of items, i.e., a schedule such that the difference between the number of occurrences of parts of a same type in any two (time) intervals of the same length is at most 3.

- Recall: for $(n_1, n_2, \dots, n_8) = (8, 1, 1, 1, 1, 1, 1, 1)$,
- Webster's method yields A A A A B C D E F G H A A A A
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- JIT scheduling yields A B A C A D A E A F A G A H A
- The latter JIT schedule is *balanced* (difference is at most 1).

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- interval $[s, t]$: number of A's is in
  $\big((t-s+1)\, r_A - 2, (t-s+1)\, r_A + 2\big)$
- same holds for any other interval of length $(t - s)$
- so, the number of A's differs by 3 units, at most.

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If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then the word W^* is *balanced* and all numbers n_i are pairwise distinct.

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- Proved by Kubiak (2003), Brauner and Jost (2008)
- Nice connections with Fraenkel's conjecture on balanced words.

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Definitions

Sequences and words

A *sequence* is a subset of $\mathbb{Z}$.
A *word* on p letters (or colors) is a partition of $\mathbb{Z}$ into p sequences $S_1, \ldots, S_p$ or, equivalently, a mapping $\mathbb{Z} \to \{1, 2, \ldots, p\}$.

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A *balanced sequence* is a sequence S such that, for every two intervals I_1 and I_2 of the same length, the difference between the number of elements of the sequence in the two intervals is at most 1: that is, if $I_1 = \{i_1, \ldots, i_1 + t\}$ and $I_2 = \{i_2, \ldots, i_2 + t\}$, then $-1 \leq |I_1 \cap S| - |I_2 \cap S| \leq 1$.

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A word W is *balanced* if all its associated sequences S_i, $i \in \{1, \ldots, p\}$ are balanced.

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- abacababababacaaba...: not balanced

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Every balanced word has a *density vector* δ, where δ_i, the *density* of letter i, is the limit, when $t \to \infty$, of the proportion of occurrences of letter i in the interval $\{1, \ldots, t\}$.

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Example:

- abacaba abacaba ... $\delta = (\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$

Fraenkel's conjecture

Fraenkel's conjecture (1973)

For all $p \geq 3$, W^p is a balanced word on p letters such that all components of its density vector δ^p are pairwise distinct if and only if

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- F^p has length $2^p - 1$, and the letter frequencies are $(2^{p-1}, 2^{p-2}, \dots, 1)$.

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- Theorem (C&B (2004)): If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then W^* is balanced and all numbers n_i are pairwise distinct.

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If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then W is *symmetric*.

Links with MDJIT

- Theorem (C&B (2004)): If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then W^* is balanced and all numbers n_i are pairwise distinct.
- Conjecture (C&B (2004)): If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then $n_i = 2^{p-i}$ for all $i = 1, \ldots, p$.
- Conjecture would follow from Fraenkel's conjecture.

Theorem (Brauner and Jost (2008))

If W is a schedule with $B(W) < \frac{1}{2}$ for the instance $(n_1, n_2, \ldots, n_p)$, then W is *symmetric*.

Theorem (Symmetric case of Fraenkel's conjecture; B&J (2008))

For all $p \geq 3$, W^p is a symmetric and balanced word on p letters such that all components of its density vector δ^p are pairwise distinct if and only if $\delta_i^p = \frac{2^{p-i}}{2^p-1}$ for all $i = 1, \ldots, p$.

Summary

- JIT scheduling asks for "regular" scheduling of item types with given densities.
- MDJIT asks for a schedule minimizing the maximum deviation from "ideal frequencies" kr_i.
- MDJIT is in co-NP.
- MDJIT can be solved in pseudo-polynomial time, and even in polynomial time when p is fixed.
- Complexity is unknown in general.

- Optimal schedules are almost balanced (3-balanced).
- When $B^* < \frac{1}{2}$, the optimal schedule is balanced. But this is a rare instance.
- What about 2-balanced schedules?

Extensions

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More generally:

- Given a vector δ in $\mathbb{R}^p$, can we decide (efficiently) whether there exists a “nicely regular” word with density δ?
- Can we better understand the structure of such “nicely regular” words?
- How do we define a “nicely regular” word??

Questions, questions,...

Outline

What words are balanced?

Balanceable vectors

Vector $\delta \in \mathbb{R}^p$ is *balanceable* if there exists a balanced word on p letters with density vector δ.

- $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$ is a balanceable vector.

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Question:

Can we characterize *all* balanceable vectors?

- Probably very ambitious, so let's start slowly...
- What is known already?

Balanceable vectors

On 2 letters:

$(\alpha, 1 - \alpha)$ is balanceable for all $0 < \alpha < 1$.

Balanceable vectors

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$(\alpha, 1-\alpha)$ is balanceable for all $0 < \alpha < 1$.

On more than 2 letters:

Much more complex

A class of balanced words

Congruence sequence

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A word consisting of congruence sequences $S_1, S_2, \ldots, S_p$.
The density of letter i is $1/a_i$.

Example: $W = (abacabad)^*$

- Positions of a : $1, 3, 5, 7, ... = 2n + 1$
- Positions of b : $2, 6, 10... = 4n + 2$
- Positions of c : $4, 12, 20, ... = 8n + 4$

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Property

Every congruence word is balanced.

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Congruence substitution $W_{A,j}$

given a word W, a *congruence word* A, and a letter j of W, replace the k-th occurrence of j in W by k-th letter of A, cyclically.

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A *congruence expansion* of a word W is the result of iterative applications of congruence substitutions on W.

Property.

Every congruence expansion of a balanced word is balanced.

- How general is this construction?

Irrational densities

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Proposition (Hubert 2000)

A word with *irrational densities* is balanced if and only if it is a congruence expansion of a balanced word on two letters. These words are non-periodic.

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Experiment:

for small p and D, generate all balanceable vectors with rational densities of the form $\left(\frac{d_1}{D}, \frac{d_2}{D}, \ldots, \frac{d_p}{D}\right)$.

Experimental observations

In all cases, the balanceable vectors on p letters fall into one of the following classes:

- density vectors of congruence expansions of balanced words on fewer letters
- Fraenkel density $\left(\frac{2^{p-1}}{2^p-1}, \frac{2^{p-2}}{2^p-1}, \ldots, \frac{1}{2^p-1}\right)$
- and not much more...

Experimental observations

- $N = 3$: $(\alpha/2, \alpha/2, 1-\alpha)$, for all $0 < \alpha < 1$, and $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$ (this is the complete list)

Experimental observations

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- $N = 4$, results for $D \leq 200$; sporadic cases:

  $\left(\frac{6}{11}, \frac{3}{11}, \frac{1}{11}, \frac{1}{11}\right)$ $\left(\frac{6}{11}, \frac{2}{11}, \frac{2}{11}, \frac{1}{11}\right)$ $\left(\frac{4}{11}, \frac{4}{11}, \frac{2}{11}, \frac{1}{11}\right)$ $\left(\frac{8}{14}, \frac{4}{14}, \frac{1}{14}, \frac{1}{14}\right)$
  $\left(\mathbf{\frac{8}{15}}, \mathbf{\frac{4}{15}}, \mathbf{\frac{2}{15}}, \mathbf{\frac{1}{15}}\right)$

  (proved to be complete)

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  $\left(\mathbf{\frac{8}{15}}, \mathbf{\frac{4}{15}}, \mathbf{\frac{2}{15}}, \mathbf{\frac{1}{15}}\right)$

  (proved to be complete)
- $N = 5$, results for $D \leq 130$; sporadic cases:

  $\left(\frac{8}{13}, \frac{2}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}\right)$ $\left(\frac{6}{17}, \frac{6}{17}, \frac{2}{17}, \frac{2}{17}, \frac{1}{17}\right)$ $\left(\frac{12}{23}, \frac{6}{23}, \frac{3}{23}, \frac{1}{23}, \frac{1}{23}\right)$
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Experimental observations

- $N = 6$, test exhaustif pour $D \leq 80$; sporadic cases:

$(\frac{5}{13}, \frac{3}{13}, \frac{2}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13})$ $(\frac{10}{19}, \frac{5}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19})$ $(\frac{8}{21}, \frac{8}{21}, \frac{2}{21}, \frac{1}{21}, \frac{1}{21}, \frac{1}{21})$

$(\frac{12}{35}, \frac{12}{35}, \frac{6}{35}, \frac{2}{35}, \frac{2}{35}, \frac{1}{35})$ $(\frac{9}{16}, \frac{3}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$ $(\frac{10}{19}, \frac{3}{19}, \frac{2}{19}, \frac{2}{19}, \frac{1}{19}, \frac{1}{19})$

$(\frac{12}{25}, \frac{6}{25}, \frac{3}{25}, \frac{2}{25}, \frac{1}{25}, \frac{1}{25})$ $(\frac{12}{35}, \frac{12}{35}, \frac{4}{35}, \frac{4}{35}, \frac{2}{35}, \frac{1}{35})$ $(\frac{10}{17}, \frac{2}{17}, \frac{2}{17}, \frac{1}{17}, \frac{1}{17}, \frac{1}{17})$

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$(\frac{6}{19}, \frac{4}{19}, \frac{4}{19}, \frac{2}{19}, \frac{2}{19}, \frac{1}{19})$ $(\frac{18}{35}, \frac{6}{35}, \frac{6}{35}, \frac{2}{35}, \frac{2}{35}, \frac{1}{35})$ $(\frac{24}{47}, \frac{8}{47}, \frac{8}{47}, \frac{4}{47}, \frac{2}{47}, \frac{1}{47})$

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Which leads us to...

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Besides well-identified infinite families, there only exists a finite number of balanceable vectors for each p.

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Besides well-identified infinite families, there only exists a finite number of balanceable vectors for each p.

More precisely:

Conjecture

If a word W on p letters is balanced, then

(1) W is a *congruence expansion* of a balanced word on two letters, or

(2) W is D-periodical for some $D \leq 2^p - 1$.

- For irrational densities, Condition (1) holds.
- Condition (2) implies that the number of "sporadic cases" is finite for each p.
- Proof for $p \geq 5$??

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- How difficult is it to recognize whether a vector δ is the density vector of a congruence word?
- Example: $\delta = \frac{1}{15}(5, 5, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$ is the density of the congruence word

  $$132465172xx31425xx162374152xxx.$$

  But when applied to δ, classical methods for building "regular schedules" (e.g., MDJIT algorithms, or Webster's method of divisors) do not produce a congruence word.
- Given p congruence sequences $S(a_i, b_i)$, how difficult is it to recognize whether they form a congruence word, i.e, whether they partition $\mathbb{Z}$?

  Polynomial for fixed p. In NP otherwise. Hard?

Outline

m-balanced words

Various related notions have been considered in the OR literature.

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- **Definition.** A word W is m-balanced if, for all i and all t, every subword of W of length t contains the same number of occurrences of letter i, up to m units.
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  (Follows from $B^* < 1$ for the MDJIT problem.)
- **Question:** What about 2-balance??

Tree words

- **Definition.** A *tree word* (or tree schedule) W is recursively built as follows:
  - start with the constant word $W = (a)^*$
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- Tree words are a subclass of congruence words.
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Conclusions

- Many interesting (and hard) questions relating to the structure of “almost regular” words and of their density vectors.
- More fundamentally: what is the “right” notion of regularity? *m*-balance (different versions), congruence words, weighted measure of deviation, etc.
- Other untouched connections: apportionment problems, queueing, Beatty sequences, billiard words, etc.
- Recognition problems: given a vector δ, decide whether δ is the density of a “regular” word.
- Optimization problems: given a vector δ, find a “regular” word whose density is as close as possible to δ.

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