Numerical properties of a Discontinuous Galerkin formulation for Electro-Thermal coupled problems

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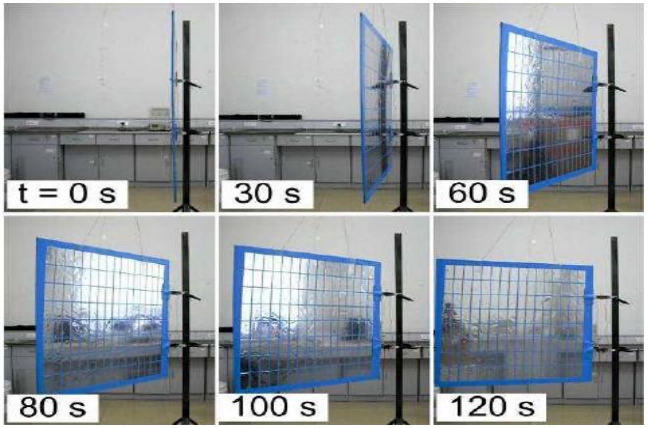
Applications of Electro-Thermal materials

Electro-thermal materials convert electricity into heat and vice versa

Applications:

- Thermo-Electric Generator (TEG)
  - e.g. production of electricity from waste heat on an automobile
  - Cooling applications
  - e.g. Electronic, medical…

- Heat applications
  - e.g. Activation of fiber reinforced shape memory polymer composite [1].



Shape recovery process of a prototype of solar array actuated by SMPC hinge

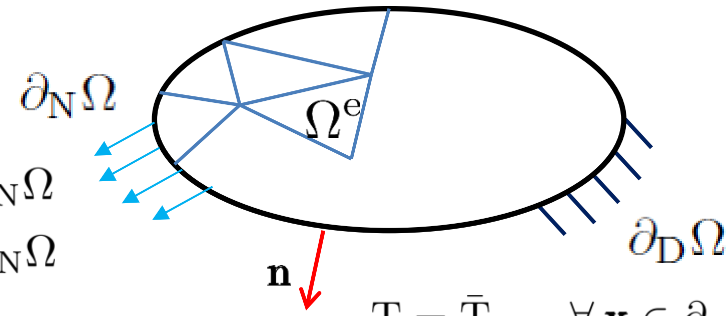
[1] L. Xin and Z. Ruirui. Smart materials and structures, 2011

Outline

- Introduction
  - Constitutive equations
  - Main concept and equation of Discontinuous Galerkin (DG)
- DG Formulation for Electro-Thermal coupled problem
  - Weak form of equations
  - Numerical properties i.e. solution uniqueness, convergence rate…
- Numerical examples
- Conclusions & Perspectives

Governing equations for Electro-Thermal coupling

$$\forall \mathrm{V}, \mathrm{T} \in \mathrm{H}^2(\Omega) \times \mathrm{H}^{2^+}(\Omega)$$



$$\mathbf{j}_{\mathrm{e}} \cdot \mathbf{n} = \bar{\mathrm{j}}_{\mathrm{e}} \quad \forall\, \mathbf{x} \in \partial_{\mathrm{N}}\Omega$$
$$\mathbf{j}_{\mathrm{y}} \cdot \mathbf{n} = \bar{\mathrm{j}}_{\mathrm{y}} \quad \forall\, \mathbf{x} \in \partial_{\mathrm{N}}\Omega$$

$$\mathrm{T} = \bar{\mathrm{T}} \qquad \forall\, \mathbf{x} \in \partial_{\mathrm{D}}\Omega$$
$$\mathrm{V} = \bar{\mathrm{V}} \qquad \forall\, \mathbf{x} \in \partial_{\mathrm{D}}\Omega$$

Conservation of Electric charge

$$\nabla \cdot \mathbf{j}_{\mathrm{e}} = 0 \qquad \forall\, \mathrm{x} \in \Omega$$

Conservation of Energy

$$\nabla \cdot \mathbf{j}_{\mathrm{y}} = -\partial_{\mathrm{t}}\mathrm{y} \qquad \forall\, \mathrm{x} \in \Omega$$

Electric current density flow

$$\mathbf{j}_{\mathrm{e}} = \mathbf{l} \cdot (-\nabla \mathrm{V}) + \alpha \mathbf{l} \cdot (-\nabla \mathrm{T})$$

$$\mathbf{j}_{\mathrm{y}} = \mathbf{q} + \mathrm{V}\,\mathbf{j}_{\mathrm{e}}$$ Energy flux

$$\mathbf{q} = \mathbf{k} \cdot (-\nabla \mathrm{T}) + \alpha \mathrm{T}\,\mathbf{j}_{\mathrm{e}}$$ Thermal flux

$$\mathrm{y} = \mathrm{y}_0 + \mathrm{c}_{\mathrm{v}}\,\mathrm{T}$$ Internal energy

Electro-thermal constitutive relations

- Vector of the unknown fields: $\mathbf{M} = \begin{pmatrix} f_V \\ f_T \end{pmatrix} = \begin{pmatrix} -\frac{V}{T} \\ \frac{1}{T} \end{pmatrix}$
- Matrix form of fluxes:

$$\mathbf{j} = \begin{pmatrix} \mathbf{j}_e \\ \mathbf{j}_y \end{pmatrix} = \begin{pmatrix} \frac{1}{f_T}\mathbf{l} & -\frac{f_V}{f_T^2}\mathbf{l} + \alpha\frac{1}{f_T^2}\mathbf{l} \\ -\frac{f_V}{f_T^2}\mathbf{l} + \alpha\frac{1}{f_T^2}\mathbf{l} & \frac{\mathbf{k}}{f_T^2} - 2\alpha\frac{f_V}{f_T^3}\mathbf{l} + \alpha^2\frac{1}{f_T^3}\mathbf{l} + \frac{f_V^2}{f_T^3}\mathbf{l} \end{pmatrix} \begin{pmatrix} \nabla f_V \\ \nabla f_T \end{pmatrix}$$

Fluxes **Coefficients matrix** **Field gradients**

$$\mathbf{j} = \mathbf{Z} \quad \nabla\mathbf{M}$$

- Strong form:

$$\mathbf{M} \in H^2(\Omega) \times H^{2^+}(\Omega) \qquad \begin{cases} \operatorname{div}(\mathbf{j}) & = \mathbf{i} & \forall\, x \in \Omega \\ \mathbf{M} & = \bar{\mathbf{M}} & \forall\, x \in \partial_D\Omega \\ \bar{\mathbf{n}}\,\mathbf{j} & = \bar{\mathbf{j}} & \forall\, x \in \partial_N\Omega \end{cases}$$

With $\mathbf{i} = \begin{pmatrix} 0 \\ -\partial_t y \end{pmatrix}, \quad \bar{\mathbf{j}} = \begin{pmatrix} \bar{j}_e \\ \bar{j}_y \end{pmatrix}$

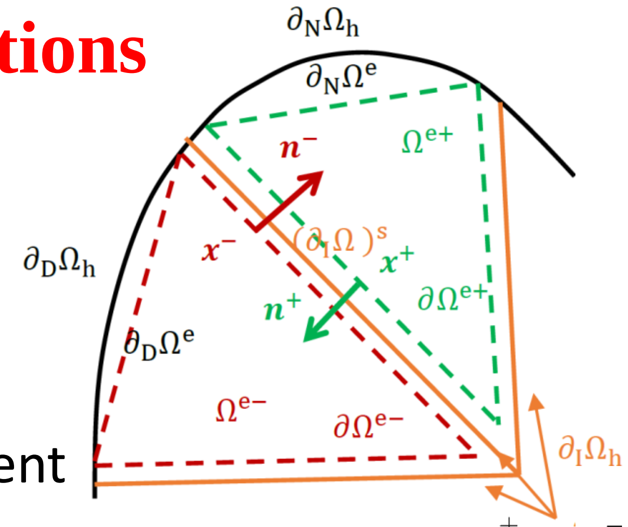
Discontinuous Galerkin (DG) introduction

- Similarity to FEM, to solve PDE's
  - Geometry approximated by polyhedral elements
  - Continuity ensured inside elements
    - Polynomial solution of finite degree
- Main difference with FEM:
  - Compatibilty weakly ensured
    - Inter-element continuity weakly constrained
    - Support of nodal shape function restrained to one element
  - Allows / eases (with high scalability and high accuracy order):
    - Discontinuous polynomial spaces of high degree

$$\mathrm{X}^{\mathrm{k}^{(+)}} = \left\{ \mathbf{M}_{\mathrm{h}} \in \mathrm{L}^2(\Omega_{\mathrm{h}}) \times \mathrm{L}^{2^{(+)}}(\Omega_{\mathrm{h}}) \,|_{\mathbf{M}_{\mathrm{h}}|_{\Omega^{\mathrm{e}}} \in \mathbb{P}^{\mathrm{k}}(\Omega^{\mathrm{e}}) \times \mathbb{P}^{\mathrm{k}^{(+)}}(\Omega^{\mathrm{e}})\ \forall \Omega^{\mathrm{e}} \in \Omega_{\mathrm{h}}} \right\}$$

    - Irregular and non-conforming meshes
    - hp-adaptivity

DG main concepts and equations



Interface between two elements Ω^{e+} and Ω^{e-}

- Let us recall the strong form

$$\nabla(\mathbf{j}) = \nabla(\mathbf{Z}\nabla\mathbf{M}) \quad = \mathbf{i} \;\; (+\mathrm{BC's})$$

- Weak form for DG scheme: multiply it by test functions $\delta\mathbf{M}$ + $\int$ by parts element by element
- Defining:

  Jump operator $[\![\mathbf{M}]\!] = \mathbf{M}^+ - \mathbf{M}^-$, Average operator $\langle\mathbf{M}\rangle = \dfrac{\mathbf{M}^+ + \mathbf{M}^-}{2}$

$$\int_{\Omega_h} \nabla\delta\mathbf{M}\,\mathbf{j}\mathrm{dV} + \int_{\partial_I\Omega_h} [\![\delta\mathbf{M}]\!]\,\langle\mathbf{j}\rangle\,\mathbf{n}^-\mathrm{dS} = -\int_{\Omega_h} \mathbf{i}\,\delta\mathbf{M}\mathrm{dV} \; + \mathrm{BC's\,terms}$$

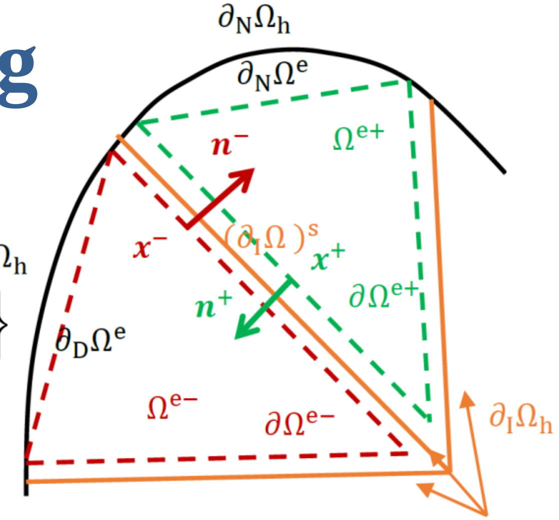
  - Supplementary terms:
    - **Consistency** term (appears naturally above)
    - **Symmetrisation** term (optimal convergence rate)

$$\int_{\partial_I\Omega_h} [\![\mathbf{M}]\!]\,\langle \mathrm{Z}\nabla\delta\mathbf{M}\rangle\,\mathbf{n}^-\mathrm{dS}$$

    - Quadratic **stabilization** term ($\mathcal{B}$ = stabilisation parameter)

$$\int_{\partial_I\Omega_h} [\![\delta\mathbf{M}]\!]\,\mathbf{n}^-\left\langle\frac{\mathbf{Z}\mathcal{B}}{\mathrm{h_s}}\right\rangle\mathbf{n}^-\,[\![\mathbf{M}]\!]\,\mathrm{dS}$$

Discontinuous Galerkin formulation for Electro-Thermal Coupling



Find $\mathbf{M} \in X^+$

$$X^{(+)} = \left\{\mathbf{M} \in L^2(\Omega_h) \times L^{2^{(+)}}(\Omega_h) \,|_{\mathbf{M}_{|\Omega^e} \in H^2(\Omega^e)\times H^{2^{(+)}}(\Omega^e) \quad \forall \Omega^e \in \Omega_h}\right\}$$

$$a(\mathbf{M}, \delta\mathbf{M}) = b(\bar{\mathbf{M}}, \delta\mathbf{M}) - \int_{\Omega_h} \delta\mathbf{M}^T \mathbf{i} d\Omega \quad \forall \delta\mathbf{M} \in X$$

Structural term + DG terms **= Boundary terms - Time derivative term**

$$a(\mathbf{M}, \delta\mathbf{M}) = \int_{\Omega_h} \nabla\delta\mathbf{M}^T \mathbf{j}(\mathbf{M}, \nabla\mathbf{M}) d\Omega + \int_{\partial_I\Omega_h \cup \partial_D\Omega_h} [\![\delta\mathbf{M}_{\mathbf{n}}^T]\!] \langle \mathbf{j}(\mathbf{M}, \nabla\mathbf{M}) \rangle dS$$
$$+ \int_{\partial_I\Omega_h \cup \partial_D\Omega_h} [\![\mathbf{M}_{\mathbf{n}}^T]\!] \langle \mathbf{Z}(\mathbf{M})\nabla\delta\mathbf{M} \rangle dS + \int_{\partial_I\Omega_h \cup \partial_D\Omega_h} [\![\delta\mathbf{M}_{\mathbf{n}}^T]\!] \left\langle \frac{\mathcal{B}}{h_s} \mathbf{Z}(\mathbf{M}) \right\rangle [\![\mathbf{M}_{\mathbf{n}}]\!] dS \quad \forall \delta\mathbf{M} \in X$$

Consistency (second term)

Compatibility (third term)

Stability (fourth term)

$$b(\bar{\mathbf{M}}; \delta\mathbf{M}) = \int_{\partial_N\Omega_h} \delta\mathbf{M}^T \bar{\mathbf{j}} dS - \int_{\partial_D\Omega_h} \bar{\mathbf{M}}_{\mathbf{n}}^T (\mathbf{Z}(\bar{\mathbf{M}})\nabla\delta\mathbf{M}) dS + \int_{\partial_D\Omega_h} \delta\mathbf{M}_{\mathbf{n}}^T \left(\frac{\mathcal{B}}{h_s}\mathbf{Z}(\bar{\mathbf{M}})\right) \bar{\mathbf{M}}_{\mathbf{n}} dS$$

Solution uniqueness

➢ The mesh dependent norm

$$|||\,\mathbf{M}\,|||_1^2 = \sum_e \|\mathbf{M}\|^2_{H^1(\Omega^e)} + \sum_s h_s \,\|\,\mathbf{M}\,\|^2_{H^1(\partial\Omega^e)} + \sum_s h_s^{-1} \|\, [\![\mathbf{M_n}]\!] \,\|^2_{L^2(\partial\Omega^e)}$$

Where $\partial\Omega^e = \partial_I\Omega^e \cup \partial_D\Omega^e$

➢ Consistency form

$\mathbf{M}^e \in H^2(\Omega) \times H^{2^+}(\Omega)$ the solution of the strong form, with $\mathbf{i} = 0$

Thus as $[\![\mathbf{M}^e]\!] = 0$ on $\partial_I\Omega^e$ and $[\![\mathbf{M}^e]\!] = -\mathbf{M}^e = -\bar{\mathbf{M}}$ on $\partial_D\Omega^e$

$$a(\mathbf{M}^e, \delta\mathbf{M}^e) = b(\bar{\mathbf{M}}, \delta\mathbf{M}^e) \quad \forall \delta\mathbf{M}^e \in X, \qquad (1)$$

➢ Weak form

The weak form, with $\mathbf{i} = 0$, reads as finding $\mathbf{M}_h \in X^k$, such that

$$a(\mathbf{M}_h, \delta\mathbf{M}_h) = b(\bar{\mathbf{M}}; \delta\mathbf{M}_h) \quad \forall \delta\,\mathbf{M}_h \in X^k \subset X \qquad (2)$$

Replacing $\delta\mathbf{M}^e = \delta\mathbf{M}_h$, then subtracting (2) from (1)

$$a(\mathbf{M}^e, \delta\mathbf{M}_h) - a(\mathbf{M}_h, \delta\mathbf{M}_h) = b(\bar{\mathbf{M}}, \delta\mathbf{M}_h) - b(\bar{\mathbf{M}}, \delta\mathbf{M}_h) = 0 \quad \forall \delta\mathbf{M}_h \in X^k$$

Solution uniqueness

$$\mathcal{A}(\underline{\mathbf{M}}^{\rm e};\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\delta\mathbf{M}_{\rm h})+\mathcal{B}(\underline{\mathbf{M}}^{\rm e};\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\delta\mathbf{M}_{\rm h}) = \mathcal{N}(\mathbf{M}^{\rm e},\mathbf{M}_{\rm h};\delta\mathbf{M}_{\rm h})$$

Fixed **Fixed**

$\mathcal{A}$, $\mathcal{B}$ **Bilinear**

$$\begin{aligned}\mathcal{A}(\mathbf{M}^{\rm e};\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\delta\mathbf{M}_{\rm h}) &= \int_{\Omega_{\rm h}}\nabla\delta\mathbf{M}_{\rm h}^{\rm T}\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})(\nabla\mathbf{M}^{\rm e}-\nabla\mathbf{M}_{\rm h})d\Omega\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\delta\mathbf{M}_{{\rm h}_{\mathbf{n}}^{\rm T}}\right]\!\right]\langle\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})(\nabla\mathbf{M}^{\rm e}-\nabla\mathbf{M}_{\rm h})\rangle dS\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\mathbf{M}_{\mathbf{n}}^{\rm e^T}-\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\langle\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})\nabla\delta\mathbf{M}_{\rm h}\rangle dS\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\mathbf{M}_{\mathbf{n}}^{\rm e^T}-\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\left\langle\frac{\mathcal{B}}{h_s}\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})\right\rangle[\![\delta\mathbf{M}_{{\rm h}_{\mathbf{n}}}]\!]dS,\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\mathbf{M}^{\rm e};\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\delta\mathbf{M}_{\rm h}) &= \int_{\Omega_{\rm h}}\nabla\delta\mathbf{M}_{\rm h}^{\rm T}(\mathbf{j}_{\mathbf{M}}(\mathbf{M}^{\rm e},\nabla\mathbf{M}^{\rm e})(\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h}))d\Omega\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\delta\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\langle\mathbf{j}_{\mathbf{M}}(\mathbf{M}^{\rm e},\nabla\mathbf{M}^{\rm e})(\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h})\rangle dS.\end{aligned}$$

Nonlinear

$$\begin{aligned}\mathcal{N}(\mathbf{M}^{\rm e},\mathbf{M}_{\rm h};\delta\mathbf{M}_{\rm h}) &= \int_{\Omega_{\rm h}}\nabla\delta\mathbf{M}_{\rm h}^{\rm T}(\bar{\mathbf{R}}_{\mathbf{j}}(\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\nabla\mathbf{M}^{\rm e}-\nabla\mathbf{M}_{\rm h}))d\Omega\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\delta\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\langle\bar{\mathbf{R}}_{\mathbf{j}}(\mathbf{M}^{\rm e}-\mathbf{M}_{\rm h},\nabla\mathbf{M}^{\rm e}-\nabla\mathbf{M}_{\rm h})\rangle dS\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\mathbf{M}_{\mathbf{n}}^{\rm e^T}-\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\langle(\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})-\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}_{\rm h}))\nabla\delta\mathbf{M}_{\rm h}\rangle dS\\ &+\int_{\partial_{\rm I}\Omega_{\rm h}\cup\partial_{\rm D}\Omega_{\rm h}}\left[\!\left[\mathbf{M}_{\mathbf{n}}^{\rm e^T}-\mathbf{M}_{{\rm h}_{\mathbf{n}}}^{\rm T}\right]\!\right]\left\langle\frac{\mathcal{B}}{h_s}(\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}^{\rm e})-\mathbf{j}_{\nabla\mathbf{M}}(\mathbf{M}_{\rm h}))\right\rangle[\![\delta\mathbf{M}_{{\rm h}_{\mathbf{n}}}]\!]dS\end{aligned}$$

Solution uniqueness

Spliting ζ into its components $\Rightarrow$

$$\zeta = \mathbf{M}^{\mathrm{e}} - \mathbf{M}_{\mathrm{h}} - \mathrm{I}_{\mathrm{h}}\mathbf{M} + \mathrm{I}_{\mathrm{h}}\mathbf{M}$$
$$= \boldsymbol{\eta} + \boldsymbol{\xi}$$

→ An interpolant of $\mathbf{M}^{\mathrm{e}}$ in X^{k}

$$\boldsymbol{\eta} = \mathbf{M}^{\mathrm{e}} - \mathrm{I}_{\mathrm{h}}\mathbf{M} \in \mathrm{X} \qquad \boldsymbol{\xi} = \mathrm{I}_{\mathrm{h}}\mathbf{M} - \mathbf{M}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}}$$

$$\mathcal{A}(\mathbf{M}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{M} - \mathbf{M}_{\mathrm{h}}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{B}(\mathbf{M}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{M} - \mathbf{M}_{\mathrm{h}}, \delta\mathbf{M}_{\mathrm{h}})$$
$$= \mathcal{A}(\mathbf{M}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{B}(\mathbf{M}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{N}(\mathbf{M}^{\mathrm{e}}, \mathbf{M}_{\mathrm{h}}; \delta\mathbf{M}_{\mathrm{h}})$$

Fixed point formulation

Map $\mathrm{S}_{\mathrm{h}} : \mathrm{X}^{\mathrm{k}} \to \mathrm{X}^{\mathrm{k}}$ as follows

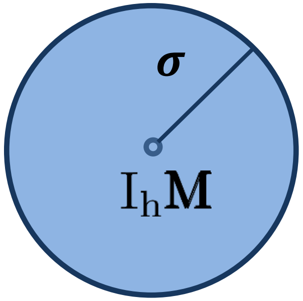
$$\forall \mathbf{y} \in \mathrm{X}^{\mathrm{k}}, \text{ Find } \mathrm{S}_{\mathrm{h}}(\mathbf{y}) = \mathbf{M}_{\mathbf{y}} \in \mathrm{X}^{\mathrm{k}}$$

$$\mathcal{A}(\mathbf{M}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{M} - \mathbf{M}_{\mathbf{y}}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{B}(\mathbf{M}^{\mathrm{e}}; \mathrm{I}_{\mathrm{h}}\mathbf{M} - \mathbf{M}_{\mathbf{y}}, \delta\mathbf{M}_{\mathrm{h}})$$
$$= \mathcal{A}(\mathbf{M}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{B}(\mathbf{M}^{\mathrm{e}}; \boldsymbol{\eta}, \delta\mathbf{M}_{\mathrm{h}}) + \mathcal{N}(\mathbf{M}^{\mathrm{e}}, \mathbf{y}; \delta\mathbf{M}_{\mathrm{h}}) \qquad (*)$$

Solution uniqueness

Definition of the ball O_σ

- Radius: σ
- Center: $\mathrm{I_h}\mathbf{M}$ the interpolant of $\mathbf{M}^{\mathrm{e}}$



$$\mathrm{O}_\sigma(\mathrm{I_h}\mathbf{M}) = \left\{ \mathbf{y} \in \mathrm{X}^{\mathrm{k}} \text{ such that } ||| \, \mathrm{I_h}\mathbf{M} - \mathbf{y} \, |||_1 \leq \sigma \right\}$$

$$\text{with} \quad \sigma = \frac{||| \, \mathrm{I_h}\mathbf{M} - \mathbf{M}^{\mathrm{e}} \, |||_1}{\mathrm{h_s^\varepsilon}}, \quad 0 < \varepsilon < \frac{1}{4}$$

Solution uniqueness

Brouwer fixed point theorem

The existence of the discrete solution M_h

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The existence of a fixed point $S_h(y) = y$ in the map S_h

S_h map to itself

S_h is continuous map

T. Gudi, N. Nataraj, A. K. Pani, Numer. Math. 2008

Solution uniqueness

1. Assumption C_α, C_y, C^k and Lemmas (e.g. trace inequality, inverse inequality . .)
2. Bound the bilinear terms $\mathcal{A}$, $\mathcal{B}$
3. Bound the nonlinear term $\mathcal{N}$

} for stabilization parameter β >Const (C_α ,C_y ,C^k..)

S_h maps $O_\sigma(I_h\mathbf{M})$ into itself

Continuity of S_h in the ball $O_\sigma(I_h\mathbf{M})$

$h_s \longrightarrow 0 \Rightarrow I_h\mathbf{M} - \mathbf{M}_\mathbf{y} \longrightarrow 0$

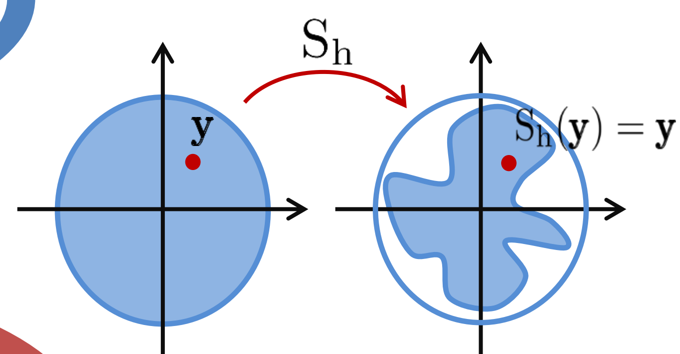
$||| \mathbf{M}_{\mathbf{y}_1} - \mathbf{M}_{\mathbf{y}_2} ||| \le C^k h_s^{\mu-2-\varepsilon} ||| \mathbf{y}_1 - \mathbf{y}_2 |||$

$\mathbf{y} \in O_\sigma(I_h\mathbf{M})$

$S_h(\mathbf{y}) = \mathbf{y}$

Brouwer fixed point

$S_h(\mathbf{y})$ **has a fixed point** $\mathbf{M}_h$



The existence of unique solution of the nonlinear elliptic problem for k ≥2

A prior error estimate

H^1-norm

$$|||\, \mathbf{M}^e - \mathbf{M}_h \,|||_1 \quad \le C^k h_s^{\mu-1} \,\|\, \mathbf{M}^e \,\|_{H^s(\Omega_h)}$$

$$\mu = \min\{s, k+1\}$$

L^2-norm

$$\|\, \mathbf{M}^e - \mathbf{M}_h \,\|_{L^2(\Omega_h)} \le C^k h_s^{\mu} \,\|\, \mathbf{M}^e \,\|_{H^s(\Omega_h)}$$

H^1, L^2-norms are optimal in the mesh size for <u>linear</u> elliptic problem

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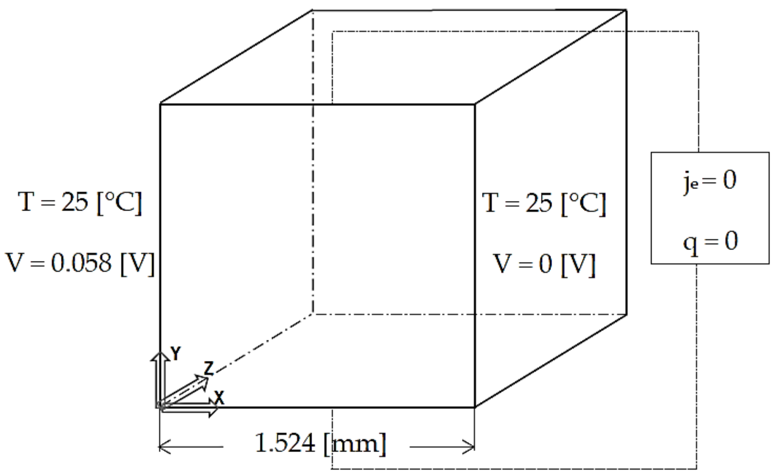
H^1, L^2-norms are optimal in the mesh size for <u>nonlinear</u> elliptic problem

Numerical result

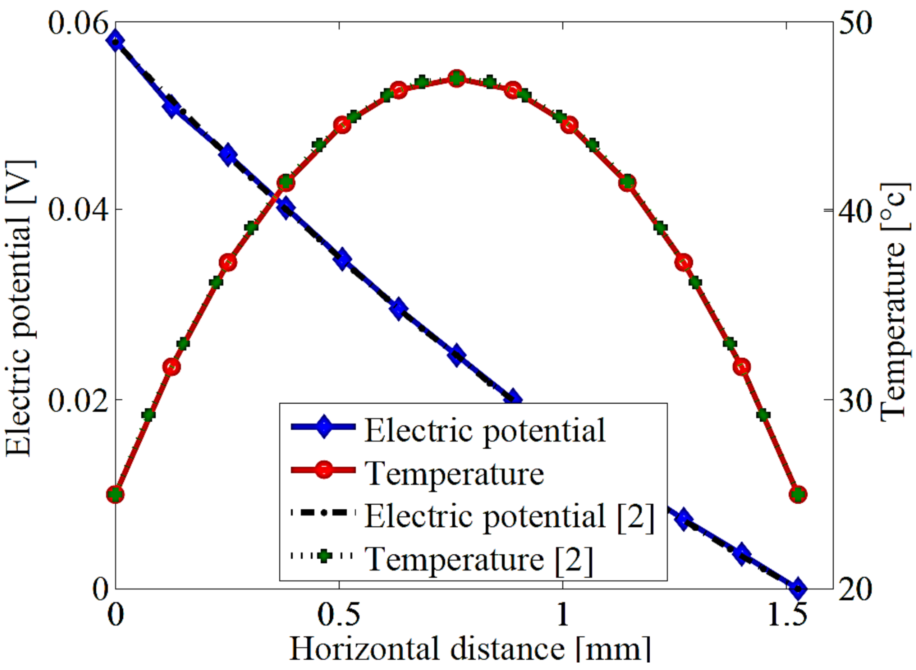
1-D example with one material

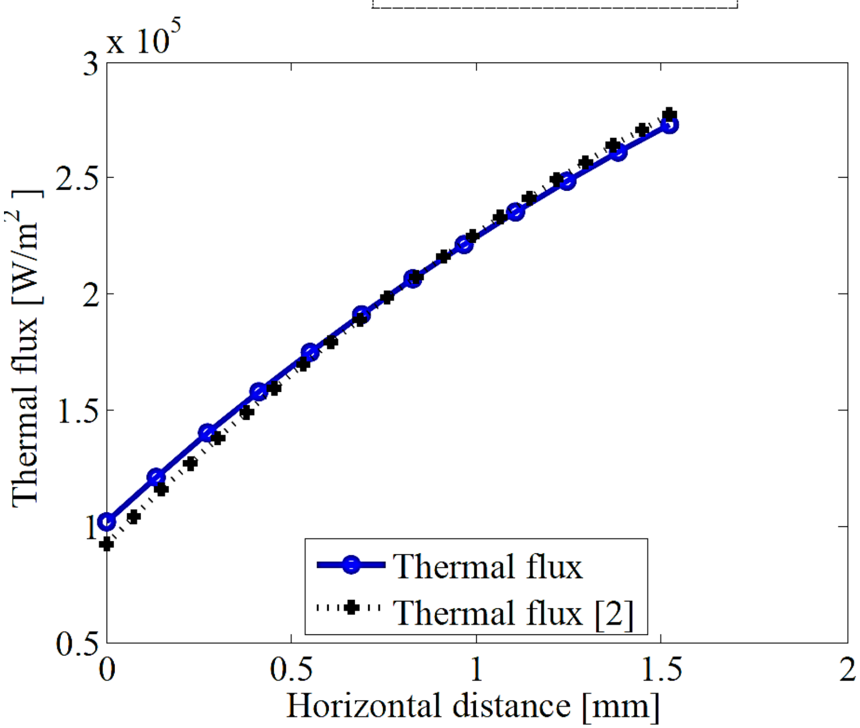


Material parameters of bismuth telluride

$\mathbf{l}$ [S/m]	$\mathbf{k}$ [W/(K · m)]	α [V/K]
diag(8.422×10^4)	diag(1.612)	1.941×10^{-4}

[2]. L. Liu. International Journal of Engineering Science, 2012

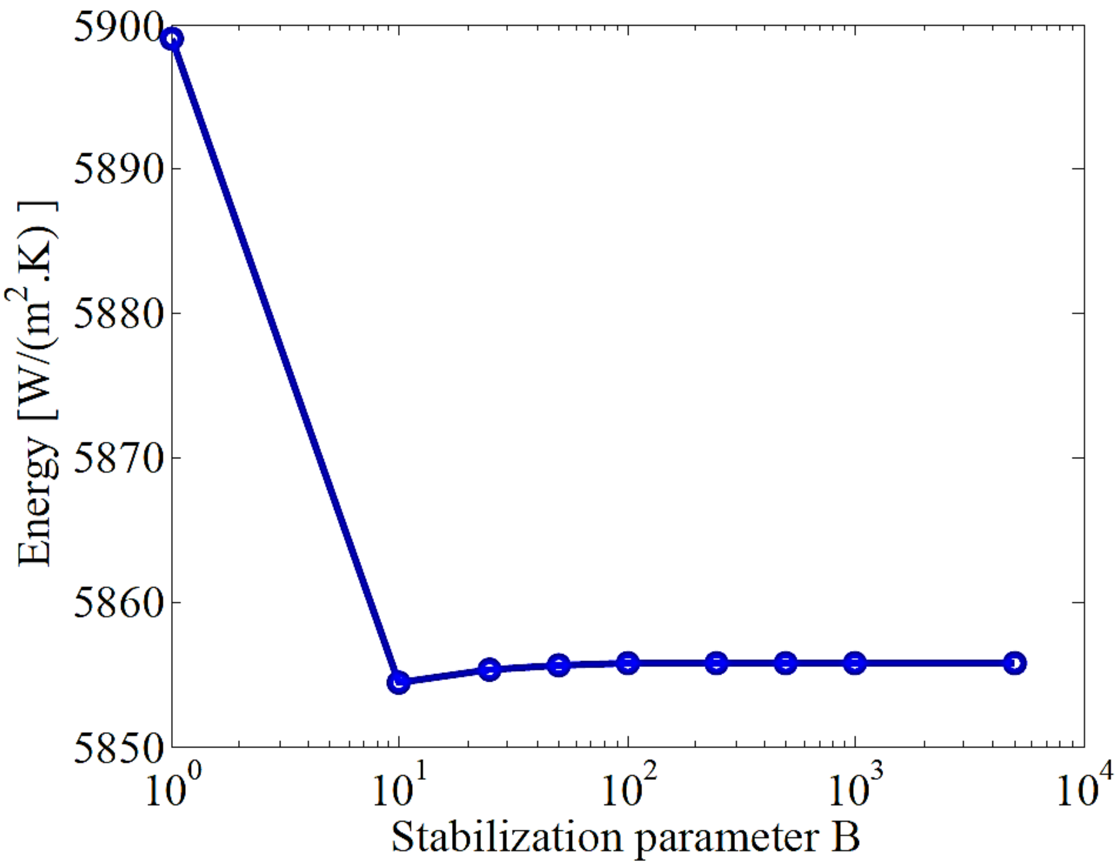




Numerical result

1-D example with two materials

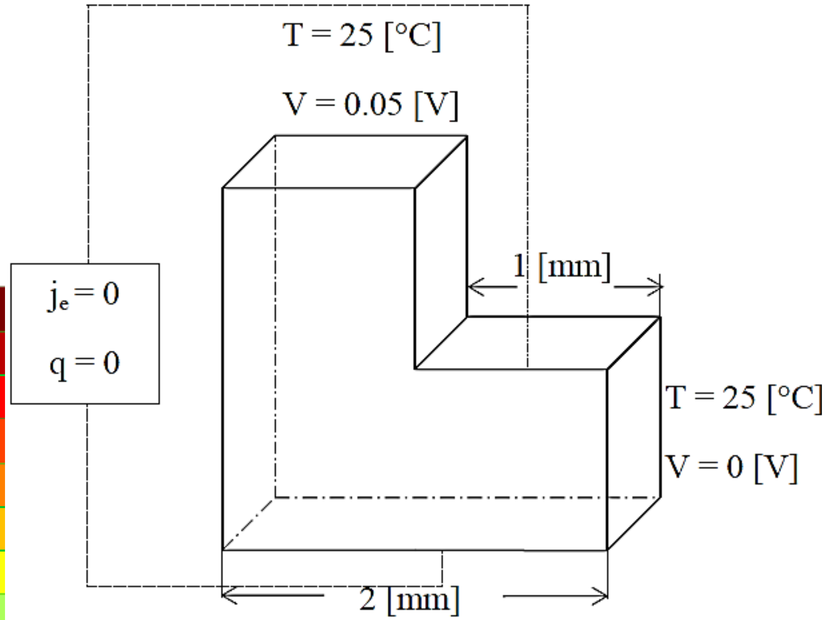
The effect of the **stabilization parameter** on the quality of the approximation

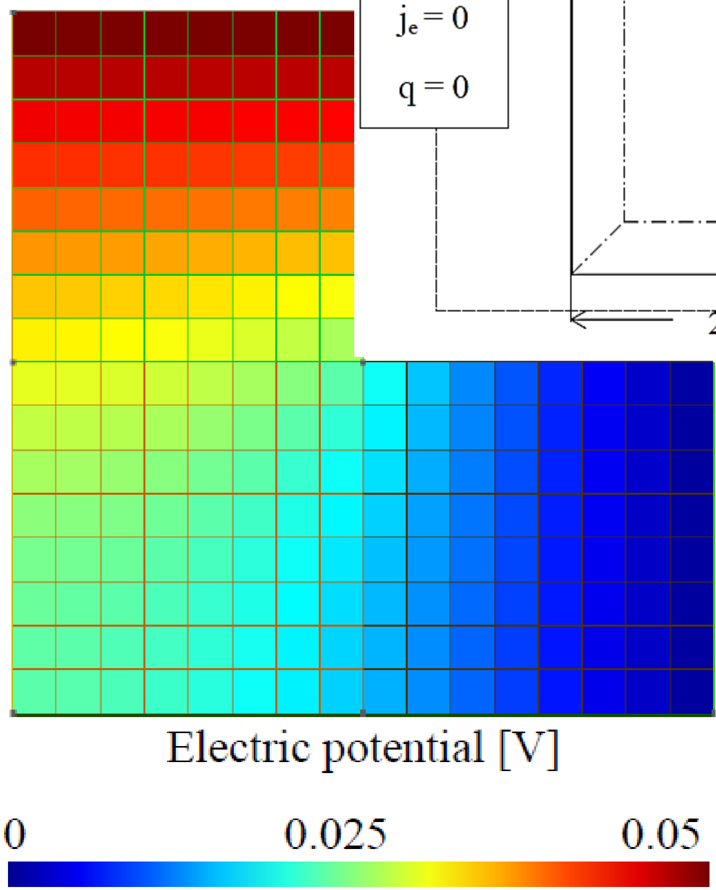


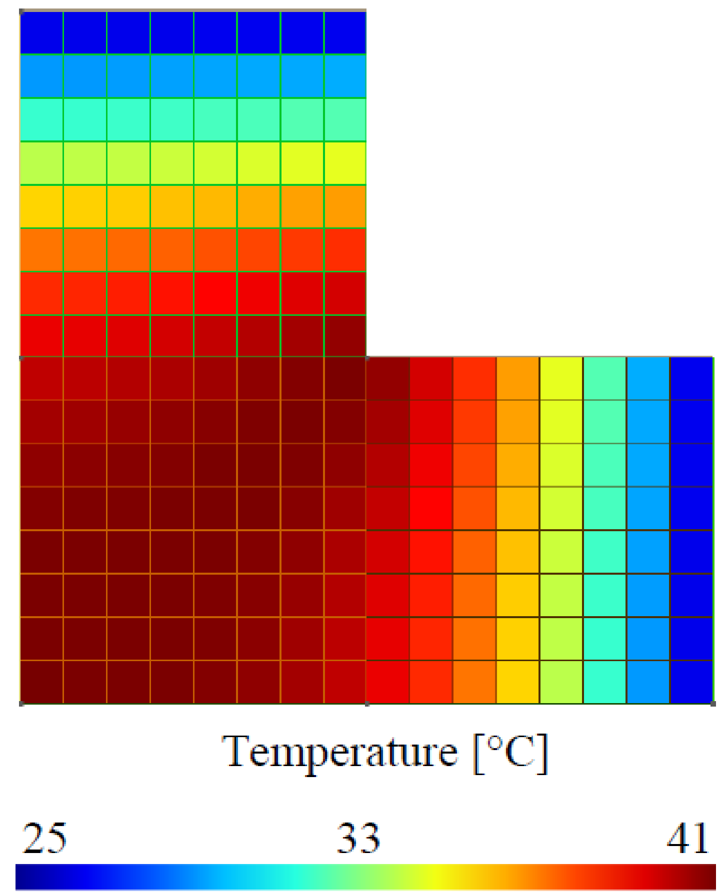
DG formulation is stable for Stabilization parameter >10

Numerical result

2-D study of convergence order





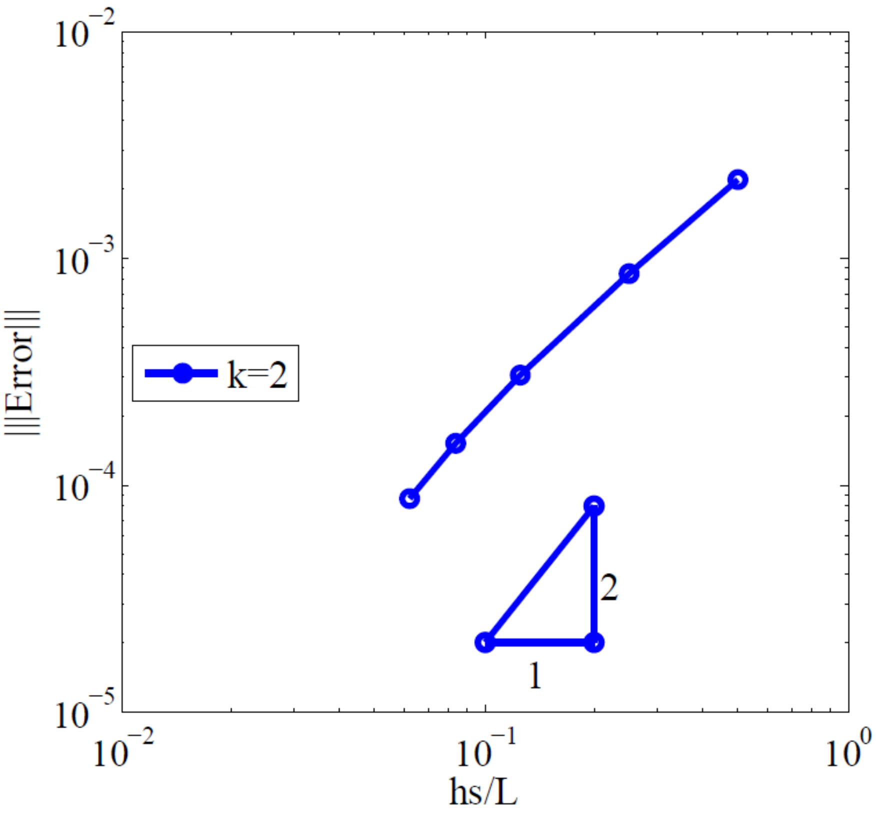


Numerical result

2-D study of convergence order

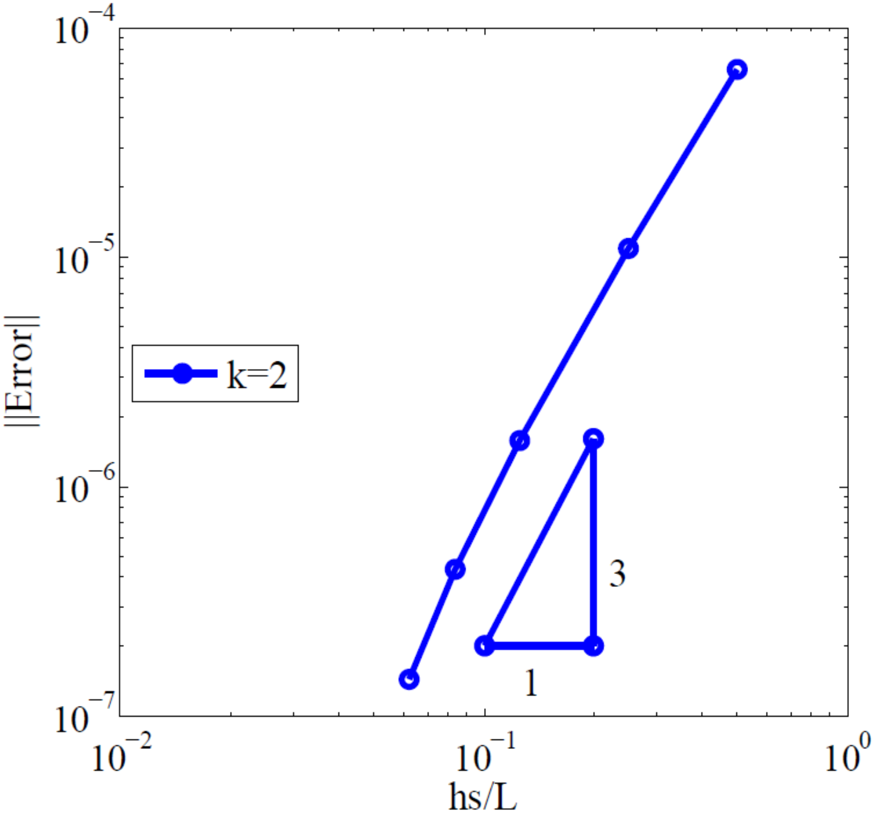
H^1-norm

Theor. converg. ord.: k



L^2-norm

Theor. converg. ord.: k+1

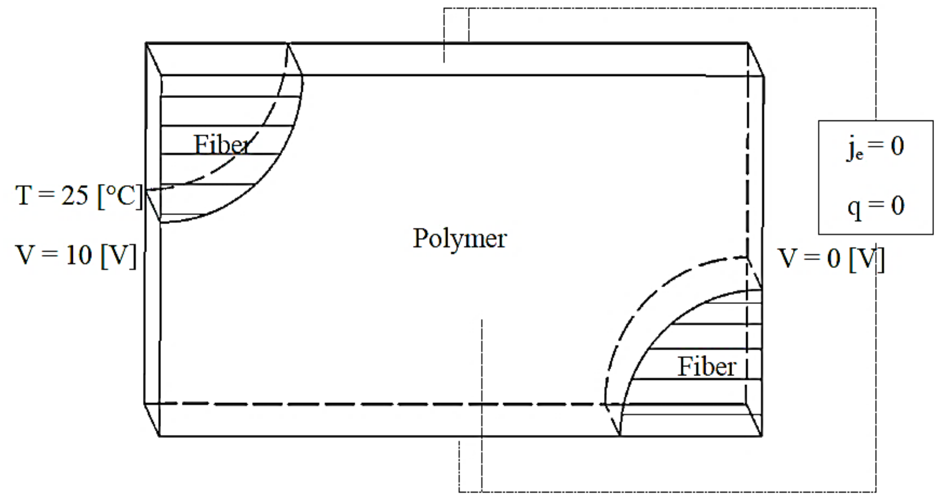


Convergence rates agree with the theoretical estimates

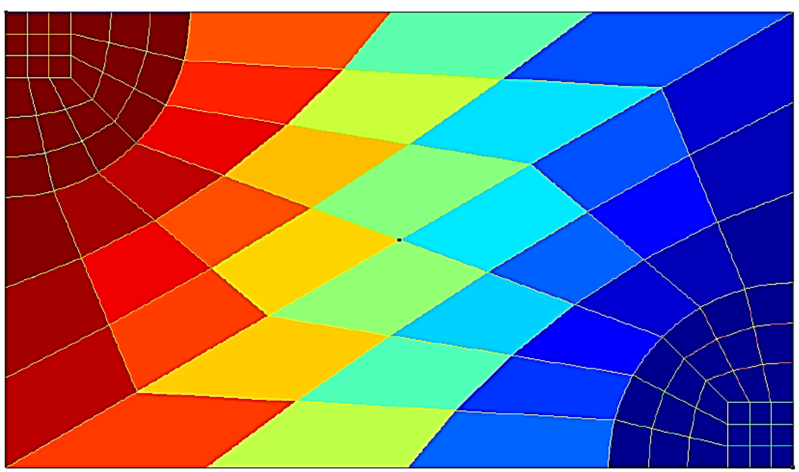
Numerical result

3-D unit cell simulation for composite material

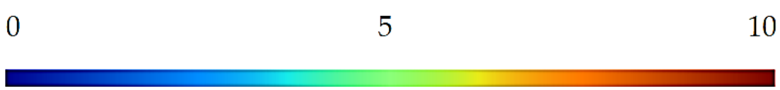
Material	**l** [S/m]	**k** [W/(K · m)]	α [V/K]
Carbon fiber	diag(100000)	diag(40)	3×10^{-6}
Polymer	diag(0.1)	diag(0.2)	3×10^{-7}

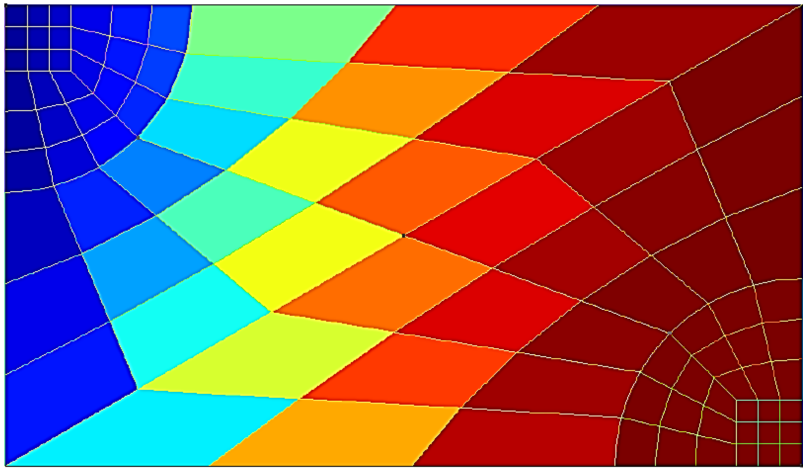


DG formulation is also applicable for irregular mesh



Electric potential [V]





Temperature [°C]

Conclusion & Perspectives

Conclusion

- A consistent and stable DG method was developed for Electro-Thermal coupled problem.
- The DG numerical properties of the Electro-Thermal coupled problem were derived:
  - Uniqueness fixed point form.
  - Optimal convergence rates in L_2, H_1-norm with respect to the mesh size.
  - Convergence rates agree with the error analysis derived in the theory.

Perspectives

- Extension to Electro-Thermo-mechanical coupled problem to recover shape memory composite material behavior.
- Take into consideration temperature dependency of the material parameters.

Thanks for your attention

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