Chapter 8

Risk process

Chapter 8

- Comparison of the two models
- Collective model
- Individual model

Generally, the risk process is the stochastic process (S(t)) representing the cumulative claim amounts. For the company’s portfolio of contracts, the risk process S(t) is the stochastic process representing the cumulative claim amounts for the company, of a portfolio of contracts, for the company, of a portfolio of contracts up to time t. In this chapter, we will only need the r.v. S(t).
Definition and hypotheses

We consider here the portfolio as the sum of independent risks (= contracts) $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$

with the hypothesis of independence of

$$\Lambda_1 I + \cdots + \Lambda_n I = S$$

Then,

$$[I = I] \text{ claim amount conditionally to } \Lambda$$

$$\left( \begin{array}{cc} b & d \\ 1 & 0 \end{array} \right) \sim I$$

$$P_{[I \text{ claim}]} = \frac{d - 1}{d} = \frac{b}{d}$$

and we define for each risk (the $j$-th), the claim amount is a r.v. $\Lambda_j$

and independent risks (= contracts) $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$

We consider here the portfolio as the sum of $n$ independent risks (i.e., $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$)

- Example
- Particular case: degenerated claim amounts
  - Moments
  - Moment-generating function
  - Probability distribution of the risk process
- Definition and hypotheses
Example: if \( q \) is the probability that no claim occur, then

\[
(\lambda_f) \int_{\lambda_f} b + \int_{\lambda_f} d = (I = f) \lambda \cdot ((I = f) I \geq \lambda f) + (0 = f) \lambda \cdot ((0 = f) I \geq \lambda f) + (I \geq \lambda f) \lambda = (I) \lambda \lambda_f
\]

for \( t \geq 0 \),

\[
(\lambda) \left( \lambda \lambda_f \lambda_f \cdots \lambda_f \lambda \right) = (\lambda) \lambda_f
\]

Cumulative distribution function

Probability distribution of the risk process

Example of shape of the c.d.f. of the company

\( W \) is the maximum intervention of the case that no claim occur, the height of the jump at 0 is the probability

Where

\( \lambda \) is the c.d.f. of the company
\begin{align*}
\{(\lambda)_{\mathcal{D}}\mathcal{E}f^{\prime}d + (\lambda)_{\mathcal{W}}a\} \sum_{u}^{1} f = (S)_{\mathcal{W}}a
\end{align*}

\begin{align*}
\{(\lambda)_{\mathcal{D}}\mathcal{E}f^{\prime}d + (\lambda)_{\mathcal{W}}a\} \sum_{u}^{1} = (S)_{\mathcal{W}}a
\end{align*}

\begin{align*}
\left( (\lambda)^{\prime}_{\mathcal{W}}m^{\prime} + f^{d} \right) \sum_{u}^{1} = (S)_{\mathcal{W}}m
\end{align*}

\begin{align*}
\left( (\lambda)^{\prime}_{\mathcal{W}}m^{\prime} + f^{d} \right) \sum_{u}^{1} = (S)_{\mathcal{W}}m
\end{align*}

But, for an $\omega$ such that

\begin{align*}
\left( (\lambda)^{\prime}_{\mathcal{W}}m^{\prime} + f^{d} \right) \sum_{u}^{1} = (S)_{\mathcal{W}}m
\end{align*}

Moments

Moments Generating Function
Particular case: degenerated claim amounts

If \( g_1 \geq 18 \) and \( g_3 \geq 37 \),

\[ \expit^{-1}(g_1 + g_3) \geq 37 \]

for any possible \( g_1 \leq 6 \),

- c.d.f.: \( \expit^{-1}(g_1 + g_3) \leq 37 \)
- m.g.f.: \( \expit^{-1}(g_1 + g_3) = F(g_1 g_3) \)
- moments

\[ \expit^{-1}(g_1 + g_3) \]

Note: even in this particular case, convolutions are not easy to calculate. If \( g_1 \geq 2 \),

\[ (g_1 g_3 + g_1 d + g_3 d + d) \sim S \]

are not easy to calculate. If \( g_1 = u \) with \( g_1 > 1 \),

\[ \expit^{-1}(g_1 + g_3) \]

for any possible \( g_1 \leq 6 \),

- c.d.f.: \( \expit^{-1}(g_1 + g_3) \leq 37 \)
- For any \( g_1 \leq \frac{395}{114475} \)

Example

A portfolio of 14 risks, with degenerated claim amounts, sorted in 3 categories (5 different values)

<table>
<thead>
<tr>
<th>Risk amount</th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>95</td>
<td>15</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>60</td>
<td>95</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>50</td>
<td>95</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

Values of \( S \) : \( \{0, 100, 200, \ldots, 4000\} \)

It is not easy to obtain c.d.f. or m.g.f. (possible values of \( g_1 \leq 15 \))

Note: even in this particular case, convolutions are not easy to calculate.
Collective model

Definition and hypotheses

The claims are no more generated individually: they are generated by the portfolio during the time interval. Occurrences of claim is a counting process $N$. 

$N$ is independent of the $X_i$'s, $X_1, X_2, \ldots$ are i.i.d. r.v., where

$$N X + \cdots + X_1 = S$$

Then $S$ is the number of claims during 1 year, $X_i$ is the claim amount for the claim occurring at time $t_i$.

Then thanks to Panjer's recursion formula in the original counting process, the probability distribution of the risk process is

\[ 
\begin{array}{cccccc}
\text{time} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{occurrences of claim} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

"Collective model"
Such a process is a compound counting process\( (X) \in \mathbb{E} \cdot (N) \in \mathbb{N} + (X) \in \mathbb{N} \cdot (N) \in \mathbb{E} = (S) \in \mathbb{N} \).

\( (X) \in \mathbb{E} = (X) \in \mathbb{E} \cdot (N) \in \mathbb{E} = (S) \in \mathbb{E} \).

### Moments

\[
(1-(t(X)\in \mathbb{N}y)^d = (1-(t(X)\in \mathbb{N}y)^d = (1)^{\mathbb{N}y} \quad \text{Moment Generating Function}
\]

\[
\frac{\chi}{\kappa y} \cdot (1)^{\chi} \sum_{j=0}^{\infty} \frac{\chi}{j} = (1)^{\mathbb{N}y} \quad \text{Cumulative distribution function}
\]

### Probability distribution of the risk process

\[
\text{Cumulative distribution function of the risk process}
\]

The risk process is then a compound Poisson process with parameter \( \mathbb{N}y \).

### Additional hypothesis: the competing process is a Poisson process with parameter \( \mathbb{N}y \).

Various conditions can affect these factors in different ways:

- Daytime headlights affects more than \( N \).
- Seat belt affects more than \( N \).
- Different ways:
  - \( X \): severity of the claim
  - \( N \): claim frequency

So, in practice:

- Study the two components separately.
- Put them together in the model.
Panjer's recursion formula

The probability distribution of \( g_1 \) is again not easy to calculate, because of convolutions. Solution for the case where the possible values of \( g_1 \) are positive integers: Panjer's formula

\[
\Pr \left[ g_1 = 0 \right] = \exp \left\{ - \sum_{i=1}^{g_1} (1 - \Pr \left[ g_i = 0 \right]) \right\}
\]

and, for \( s \in \mathbb{N}_0 \),

\[
\Pr \left[ g_1 = s \right] = \left( \sum_{i=1}^{g_1} \Pr \left[ g_i = s \right] \right) \Pr \left[ g_1 = 0 \right]
\]

Proof for \( g_1 = 0 \)

\[
\Pr \left[ g_1 = 0 \right] = \exp \left\{ - \sum_{i=1}^{g_1} (1 - \Pr \left[ g_i = 0 \right]) \right\}
\]

The \( X_i \)'s are positive integers: Panjer's formula

Solution for the case where the possible values of \( g_1 \) are again not easy to calculate, because of convolutions.

Panjer's recursion formula
Proof for \( g_{1871} \geq 1 \)

\[
\Pr \left[ X \right] = \sum_{s=1}^{\infty} \frac{1}{s} \sum_{\gamma=1}^{\infty} \Pr \left[ s = \gamma X + \cdots + \gamma X \right] \cdot f \left( x \right) \cdot \frac{f}{s} \left( x \right)
\]

Then,

\[
\Pr \left[ s = \gamma X + \cdots + \gamma X \right] \cdot \Pr \left[ x \right] = \sum_{s=1}^{\infty} \frac{1}{s} \sum_{\gamma=1}^{\infty} \Pr \left[ s = \gamma X + \cdots + \gamma X \right] \cdot \Pr \left[ x \right]
\]

But for \( i \in \{1, \ldots, n\} \),

\[
\Pr \left[ s = \gamma X + \cdots + \gamma X \right] \cdot \Pr \left[ x \right] = \sum_{s=1}^{\infty} \frac{1}{s} \sum_{\gamma=1}^{\infty} \Pr \left[ s = \gamma X + \cdots + \gamma X \right] \cdot \Pr \left[ x \right]
\]
Other probability distributions?
\[
[f - s = S]p \cdot [f = X]p \cdot \left( \frac{s}{\gamma q} + p \right)^{f - \gamma q} = \left[ s = S \right]p
\]

and, for \( s \in \mathbb{N}_0 \),

\[
0 < [0 = X]p \int_0 ! \left( ([0 = X]p|\lambda) \lambda^N \right) \lambda = [0 = S]p
\]

\[
[1 - \gamma = N]p \cdot \left( \frac{\gamma}{q} + p \right) = [\gamma = N]p
\]

In fact, Panjer's formula may be generalized for any distribution such that

\[
[1 - \gamma = N]p \gamma = \lambda \gamma^{\gamma - \gamma} = [\gamma = N]p
\]

The Poisson distribution is such that

Thanks to Panjer's formula

Note 1:

\[
(N)\mu = (V)\mu < (N)\mu
\]

\[
(V)\var + (V)\mu = ((V|N)\var) \mu + (V|N)\var \mu = (N)\var
\]

\[
(V)\mu = ((V|N)\mu) \mu = (N)\mu
\]

Moments

\[
(T - 1)\mu^m =
\]

\[
((1 - 1)\mu) \mu =
\]

\[
((V|N)\mu) \mu =
\]

\[
(N)\mu^m = (1)^m
\]

Moment Generating Function

\[
\text{if } \gamma \sim \text{exponential, then } V \sim N
\]

\[
\text{if } \gamma \sim \text{gamma, then } V \sim N
\]

\[
\text{if } \gamma \sim \text{negative binomial, then } V \equiv V
\]

\[
\text{if } \gamma \text{ degenerated then } V
\]
Comparison of the two models

Example

- Variance
- Expected values
- Principle

Quality of modeling

Comparison of the two probability laws

Principle
- Expected values
- Variances

Example

$(d':u) \sim \mathcal{N}$

and $u \in \mathbb{N}_0$.

If $d > 0$ then

\[
\frac{d-\lambda}{(1+\mu)d} = q \quad \text{and} \quad \frac{d-\lambda}{d-\lambda} = a
\]

with

\[
(\lambda - \mu)(d - \lambda) = q \quad \text{and} \quad d - \lambda = a
\]

If $a < 0$, then

$\gamma = q \quad \text{and} \quad 0 = a$.
Quality of modeling

Individual model is concerned by situations where only 0 or 1 claim occur during the time interval $o$

- No problem for life insurance
- For non-life insurance, the model is wrong, except if the claim probability is such that $\Pr \geq 2/xni \geq 9$ claims $xni \geq 9$ for one risk is negligible

Collective model is adaptable to much more situations

- The collective model is easier to handle, thanks to Panjer's formula
- Collective model is adaptable to much more situations

We identify the two first moments of the weighted average of the $X_s$ for the collective model.

We identify the two first moments of

$$E = \left( \frac{\sum_{i=1}^{1} x_i}{\sum_{i=1}^{n} x_i} \right)$$

Comparison of the two probability laws

Principle

- For non-life insurance, the model is
  - No problem
- For life insurance, the model is
  - Invalid

where only 0 or 1 claim occur during the time interval

- Individual model is concerned by situations

Individual model
The collective model is more "careful" because of its greater dispersion.

\[
\{(\hat{\theta})_{z} \in \mathbb{E} f + (\hat{\lambda})_{wa} \} f b \sum_{u} <
\]

\[
\{(\hat{\phi})_{z} \in \mathbb{E} f + (\hat{\lambda})_{wa} \} f b \sum_{u} =
\]

\[
(\hat{\phi})_{z} \in \mathbb{E} f b \sum_{u} =
\]

\[
\left(\int_{u} f b \sum_{u} \frac{1}{b} \mathbb{E} \right) \mathbb{E} =
\]

\[
(\sum_{u} X) \mathbb{E} = (\sum_{u} y) \mathbb{E}
\]

Expected Values

Variances
Example

With the same data (portfolio with 4 risks)

\[
\Pr[X = 200] = \frac{1}{3} \cdot 0.05 + 0.15
\]

\[\gamma = \frac{1}{3} \approx 0.33\]

By Panjer’s formula,

\[
\begin{array}{cccc|cc|c|c|c}
\hline
\text{N° cat. cl} & \text{amount} & p_0 & q_0 & p_1 & q_1 & p_2 & q_2 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0.10 & 0.90 & 0.30 & 0.70 & 0.20 & 0.80 \\
3 & 2 & 0.15 & 0.85 & 0.45 & 0.55 & 0.30 & 0.70 \\
4 & 3 & 0.20 & 0.80 & 0.50 & 0.50 & 0.40 & 0.60 \\
5 & 4 & 0.25 & 0.75 & 0.55 & 0.45 & 0.50 & 0.50 \\
6 & 5 & 0.30 & 0.70 & 0.60 & 0.40 & 0.60 & 0.40 \\
7 & 6 & 0.35 & 0.65 & 0.65 & 0.35 & 0.70 & 0.30 \\
8 & 7 & 0.40 & 0.60 & 0.70 & 0.30 & 0.80 & 0.20 \\
9 & 8 & 0.45 & 0.55 & 0.75 & 0.25 & 0.90 & 0.10 \\
10 & 9 & 0.50 & 0.50 & 0.80 & 0.20 & 1.00 & 0.00 \\
11 & 10 & 0.55 & 0.45 & 0.85 & 0.15 & 1.00 & 0.00 \\
12 & 11 & 0.60 & 0.40 & 0.90 & 0.10 & 1.00 & 0.00 \\
13 & 12 & 0.65 & 0.35 & 0.95 & 0.05 & 1.00 & 0.00 \\
14 & 13 & 0.70 & 0.30 & 1.00 & 0.00 & 1.00 & 0.00 \\
\hline
\end{array}
\]
Distribution du coût cumulé des sinistres