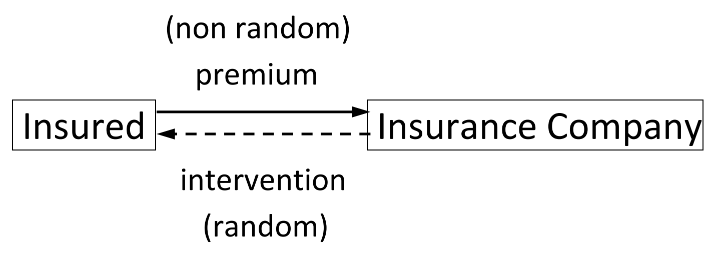
Part III
ACTUARIAL APPLICATIONS

a)



Sometimes,

- the premium is paid by a taker of insurance (≠ insured person)
- the intervention is paid to a beneficiary (≠ insured person)

b) Randomness is present in any domain of management (e.g. in finance), and the manager try to eliminate it

Two domains where randomness is the basis of human activity

- gambling
  (consumer pays to play against the hazard)
- insurance
  (consumer pays to get rid off the hazard)

c) Classification life vs non-life

	Non-life	Life
Period	1 year	(very) long
Contract	standard	specific (function of age)
Intervention	unknown (claim amount)	generally known (in the contract)
Nb of claims	any positive integer	0 or 1

d) Here : non-life insurance, even if some topics are also adapted to life insurance

Chapter 7

Specific tools

- Convolution
- Random sums
- Counting process
- Markov chains

Convolution

- Definition
- For the m.g.f.
- For discrete r.v.
    - Case of positive integer r.v.
    - Case of Poisson r.v.
- For the c.d.f.
- For continuous r.v.
- Generalization

Definition

The problem is to determine the probability distribution of the sum of independent r.v.

For the m.g.f.

If X and Y are independent r.v.,

$$m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$$

Proof

$$\begin{aligned} E\left(e^{t(X+Y)}\right) &= E(e^{tX} \cdot e^{tY}) \\ &= E(e^{tX}) \cdot E(e^{tY}) \end{aligned}$$

For discrete r.v.

Case of positive integer r.v.

Consider two independent r.v.

$$X \sim \begin{pmatrix} 0 & 1 & \dots & n & \dots \\ p_0 & p_1 & \dots & p_n & \dots \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & \dots & n & \dots \\ q_0 & q_1 & \dots & q_n & \dots \end{pmatrix}$$

For any $k \in \mathbb{N}$,

$$\begin{aligned} \Pr[X + Y = k] &= \sum_{j=0}^{k} \Pr([X = j] \cap [Y = k - j]) \\ &= \sum_{j=0}^{k} \Pr[X = j] \cdot \Pr[Y = k - j] \\ &= \sum_{j=0}^{k} p_j \, q_{k-j} \end{aligned}$$

Case of Poisson r.v.

Consider two independent Poisson r.v.

$$X_i \sim \mathcal{P}(\lambda_i) \qquad i = 1, 2$$

For any $k \in \mathbb{N}$,

$$\begin{aligned} \Pr[X_1 + X_2 = k] &= \sum_{j=0}^{k} e^{-\lambda_1} \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \frac{\lambda_2^{k-j}}{(k-j)!} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!} \end{aligned}$$

So,

$$X_1 + X_2 \sim \mathcal{P}(\lambda_1 + \lambda_2)$$

For the c.d.f.

$$\begin{aligned} F_{X+Y}(t) &= \Pr[X+Y \le t] \\ &= E\left(\mathbf{1}_{[X+Y\le t]}\right) \\ &= E\left(E\left(\mathbf{1}_{[X+Y\le t]}|X\right)\right) \end{aligned}$$

But, by (R7),

$$E\left(\mathbf{1}_{[X+Y\le t]}|X\right) = E\left(E_Y\left(\mathbf{1}_{[Y\le t-X]}\right)|X\right)$$

so that,

$$\begin{aligned} F_{X+Y}(t) &= E\left(E_Y\left(\mathbf{1}_{[Y\le t-X]}\right)\right) \\ &= E\left(F_Y(t-X)\right) \end{aligned}$$

$$\begin{aligned} F_{X+Y}(t) &= \int_{-\infty}^{+\infty} F_Y(t-x)\, dF_X(x) \\ &= \int_{-\infty}^{+\infty} F_X(t-y)\, dF_Y(y) \\ &= (F_X * F_Y)(t) \end{aligned}$$

For continuous r.v.

$$\begin{aligned} F_{X+Y}(t) &= \int_{-\infty}^{+\infty} F_Y(t-x)\, f_X(x)\, dx \\ &= \int_{-\infty}^{+\infty} dx\, f_X(x) \int_{-\infty}^{t-x} f_Y(y)\, dy \end{aligned}$$

and, by derivation w.r.t. t,

$$\begin{aligned} f_{X+Y}(t) &= \int_{-\infty}^{+\infty} f_X(x) f_Y(t-x)\, dx \\ &= \int_{-\infty}^{+\infty} f_X(t-y) f_Y(y)\, dy \\ &= (f_X * f_Y)(t) \end{aligned}$$

In particular, if the two r.v. are positive,

$$\begin{aligned} (f_X * f_Y)(t) &= \int_0^t f_X(x) f_Y(t-x)\, dx \\ &= \int_0^t f_X(t-y) f_Y(y)\, dy \end{aligned}$$

Generalization

For more than two r.v.,

$$\begin{aligned} &F_{X+Y+Z}(t) \\ &\quad = F_{(X+Y)+Z}(t) \\ &\quad = F_{X+(Y+Z)}(t) \\ &\quad = (F_X * F_Y * F_Z)(t) \end{aligned}$$

and, if $X_1, X_2, \dots, X_n$ are i.i.d., then

$$F_{X_1+\cdots+X_n}(t) = \left(F_{X_1} * \dots * F_{X_n}\right)(t) = F_X^{*n}(t)$$

Random sums

- Definition
- Cumulative distribution function
- Moment generating function
- Moments
  - Expectation
  - Variance

Definition

A random sum is a sum of r.v. with a random number of terms

Consider a sequence of i.i.d. r.v. $X_1, \ldots, X_n, \ldots$ and a positive integer r.v. N, independent of the X_k's

$$S = \sum_{k=1}^{N} X_k$$

Illustration : if the random situation has a finite number of possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and if the probability law of N and the X_k's is given by

	N	X_1	X_2	X_3	X_4	…
ω_1	1	x_{11}	x_{21}	x_{31}	x_{41}	…
ω_2	2	x_{12}	x_{22}	x_{32}	x_{42}	…
ω_3	3	x_{13}	x_{23}	x_{33}	x_{43}	…

$$S(\omega_1) = x_{11}$$
$$S(\omega_2) = x_{12} + x_{22}$$
$$S(\omega_3) = x_{13} + x_{23} + x_{33}$$

Example

a) Tossing 2 coins (N = number of tails)

b) Throwing N dice

→ S = sum of the points of the dice

$$N \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad X_k \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$S[\Omega] = \{0, 1, 2, \ldots, 12\}$$

$$\Pr[S = 0] = \Pr[N = 0] = \frac{1}{4}$$
$$\Pr[S = 1] = \Pr([N = 1] \cap [X_1 = 1]) = \frac{1}{2} \cdot \frac{1}{6}$$
$$\Pr[S = 2] = \Pr([N = 1] \cap [X_1 = 2]) + \Pr([N = 2] \cap [X_1 = 1] \cap [X_2 = 1])$$
$$= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{6}$$

$$S \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \frac{36}{144} & \frac{12}{144} & \frac{13}{144} & \frac{14}{144} & \frac{15}{144} & \frac{16}{144} & \frac{17}{144} & \frac{6}{144} & \frac{5}{144} & \frac{4}{144} & \frac{3}{144} & \frac{2}{144} & \frac{1}{144} \end{pmatrix}$$

$$E(S) = \frac{7}{2} \qquad var(S) = \frac{217}{24}$$

Cumulative distribution function

$$\begin{aligned} F_S(t) &= \Pr[X_1 + \cdots + X_N \le t] \\ &= E\left(\mathbf{1}_{[X_1+\cdots+X_N \le t]}\right) \\ &= E\left(E\left(\mathbf{1}_{[X_1+\cdots+X_N \le t]} \middle| N\right)\right) \end{aligned}$$

But, by (R7),

$$\begin{aligned} E\left(\mathbf{1}_{[X_1+\cdots+X_N \le t]} \middle| N\right) &= E\left(E_X\left(\mathbf{1}_{[X_1+\cdots+X_N \le t]}\right) \middle| N\right) \\ &= F_X^{*N}(t) \end{aligned}$$

so that

$$F_S(t) = E\left(F_X^{*N}(t)\right)$$

Then,

$$F_S(t) = \sum_{k=0}^{\infty} F_X^{*k}(t) \cdot \Pr[N = k]$$

Moment generating function

$$\begin{aligned} m_S(t) &= E\left(e^{t(X_1+\cdots+X_N)}\right) \\ &= E\left(E\left(e^{t(X_1+\cdots+X_N)} \middle| N\right)\right) \\ &= E\left(m_{X_1+\cdots+X_N}(t)\right) \\ &= E\left(m_X^N(t)\right) \\ &= E\left(e^{N \cdot \ln m_X(t)}\right) \\ &= m_N(\ln m_X(t)) \end{aligned}$$

Moments

Expectation

$$\begin{aligned} E(S) &= E\big(E(X_1 + \cdots + X_N | N)\big) \\ &= E\big(N \cdot E(X)\big) \\ &= E(N) \cdot E(X) \end{aligned}$$

Variance

$$\begin{aligned} var(S) &= E\big(var(S|N)\big) + var\big(E(S|N)\big) \\ &= E\big(N \cdot var(X)\big) + var\big(N \cdot E(X)\big) \\ &= E(N) \cdot var(X) + var(N) \cdot E^2(X) \end{aligned}$$

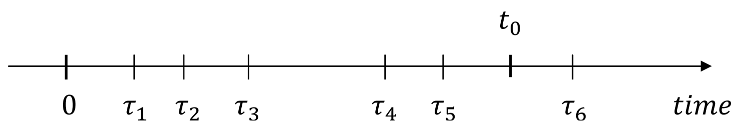
Counting process

- Counting process
- Poisson Process
  - Definition
  - Probability distribution
  - Time between 2 occurrences

Counting process

= "pure jump process"

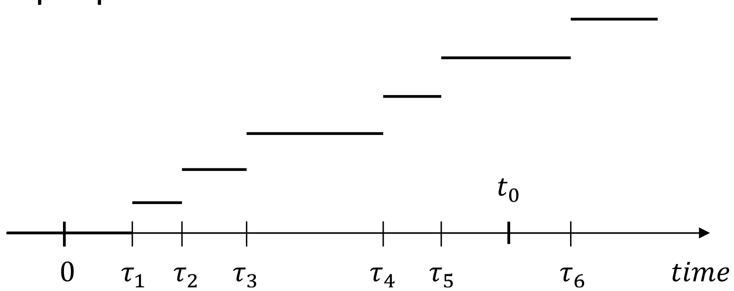
Let us assume that "events" occur during time (accident claims for an insurance portfolio e.g.)



= representation of one outcome (ω)
τ_k = epoch of the k-th event

Let N_t denote the number of occurred events up to time t (in the example, $N_{t_0} = 5$)

Sample path :



Poisson process

Notation

$$f(h) = o(h) \quad \Leftrightarrow \quad \lim_{h\to 0} \frac{f(h)}{h} = 0$$

($f(h)$ tends to 0 more rapidly than h itself)

Definition

Let us consider a counting process $\{N_t : t \geq 0\}$ such that

- $N_0 = 0$
- (N_t) has independent and stationary increments
- No multiple occurrences

$$\Pr[N_{t+h} - N_t \geq 2] = o(h)$$

- Occurring with rate λ (> 0)

$$\Pr[N_{t+h} - N_t = 1] = \lambda h + o(h)$$

Consequence :

$$\Pr[N_{t+h} - N_t = 0] = 1 - \lambda h + o(h)$$

Probability distribution

For $n = 0, 1, 2, \ldots$

$$p_n(t) = \Pr[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

Proof (by induction)

- $$\begin{aligned} p_0(t+h) &= p_0(t) \cdot \Pr[N_{t+h} - N_t = 0] \\ &= p_0(t) \cdot (1 - \lambda h + o(h)) \end{aligned}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0(t) + \frac{o(h)}{h}$$

$$p_0'(t) = -\lambda p_0(t)$$

$$p_0(t) = e^{-\lambda t} \cdot C$$

$$p_0(t) = e^{-\lambda t}$$

- $$\begin{aligned} p_n(t+h) &= p_n(t) \cdot \Pr[N_{t+h} - N_t = 0] \\ &\quad + p_{n-1}(t) \cdot \Pr[N_{t+h} - N_t = 1] \\ &\quad + \sum_{k=2}^{\infty} p_{n-k}(t) \cdot \Pr[N_{t+h} - N_t = k] \\ &= p_n(t) \cdot \big(1 - \lambda h + o(h)\big) \\ &\quad + p_{n-1}(t) \cdot \big(\lambda h + o(h)\big) + o(h) \end{aligned}$$

$$\frac{p_n(t+h) - p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(h)}{h}$$

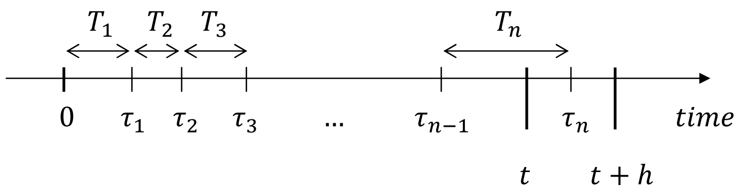
$$p_n'(t) = -\lambda p_n(t) + \lambda \cdot e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$\begin{aligned} e^{\lambda t} \cdot \big(p_n'(t) + \lambda p_n(t)\big) &= \lambda \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \\ &= \left(e^{\lambda t} p_n(t)\right)' \end{aligned}$$

$$\int_0^t \left(e^{\lambda s} p_n(s)\right)' ds = \frac{\lambda^n}{(n-1)!}\int_0^t s^{n-1} ds$$

$$e^{\lambda t} p_n(t) - 0 = \frac{\lambda^n}{(n-1)!} \cdot \frac{t^n}{n} = \frac{(\lambda t)^n}{n!}$$

Time between 2 occurrences



$$\begin{aligned}\Pr[t < \tau_n \le t+h] &= F_{\tau_n}(t+h) - F_{\tau_n}(t)\\ &= p_{n-1}(t)\cdot \Pr[N_{t+h} - N_t \ge 1]\\ &= p_{n-1}(t)\cdot\big(\lambda h + o(h)\big)\\ &= p_{n-1}(t)\cdot \lambda h + o(h)\end{aligned}$$

$$\frac{F_{\tau_n}(t+h) - F_{\tau_n}(t)}{h} = \lambda p_{n-1}(t) + \frac{o(h)}{h}$$

$$f_{\tau_n}(t) = \lambda p_{n-1}(t) = \lambda^n e^{-\lambda t}\frac{t^{n-1}}{(n-1)!}$$

$$\begin{aligned}m_{\tau_n}(t) &= \int_0^{+\infty} e^{tx} f_{\tau_n}(x)\,dx\\ &= \lambda^n \int_0^{+\infty} e^{(t-\lambda)x}\frac{x^{n-1}}{(n-1)!}\,dx\\ &= \lambda^n \cdot I_{n-1}\end{aligned}$$

$(t < \lambda)$

$$\begin{aligned}I_{n-1} &= \int_0^{+\infty}\left(\frac{e^{(t-\lambda)x}}{t-\lambda}\right)'\frac{x^{n-1}}{(n-1)!}\,dx\\ &= \left[\frac{e^{(t-\lambda)x}}{t-\lambda}\cdot\frac{x^{n-1}}{(n-1)!}\right]_0^{+\infty} - \frac{1}{t-\lambda}\int_0^{+\infty} e^{(t-\lambda)x}\frac{x^{n-2}}{(n-2)!}\,dx\\ &= \frac{1}{\lambda-t}\cdot I_{n-2}\\ &= \frac{1}{(\lambda-t)^2}\cdot I_{n-3}\\ &= \cdots\\ &= \frac{1}{(\lambda-t)^{n-1}}\cdot I_0\\ &= \frac{1}{(\lambda-t)^{n-1}}\int_0^{+\infty} e^{(t-\lambda)x}\,dx\\ &= \frac{1}{(\lambda-t)^n}\end{aligned}$$

So,

$$m_{\tau_n}(t) = m_{T_1+\cdots+T_n}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$$

In particular, $m_{T_1}(t) = \frac{\lambda}{\lambda-t}$ and

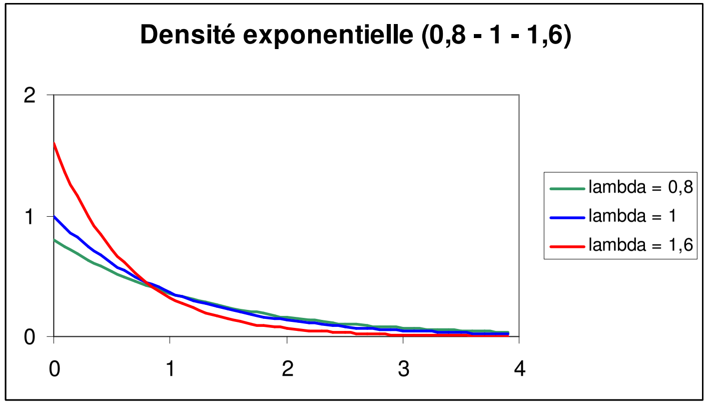
$$m_{T_1+\cdots+T_n}(t) = \left(m_{T_1}(t)\right)^n$$

Then, $T_1, T_2, \ldots$ are i.i.d. r.v. with

$$f_T(t) = f_{\tau_1}(t) = \lambda e^{-\lambda t} \qquad (t > 0)$$

= exponential distribution with parameter λ

$$E(T) = \frac{1}{\lambda} \qquad var(T) = \frac{1}{\lambda^2} \qquad \gamma_1 = 2 \qquad \gamma_2 = 6$$



Note : a sequence of i.i.d. r.v. representing successive durations is a renewal process

A Poisson process generate an exponential renewal point process, and reciprocally

Markov chains

- Stochastic matrices
- Homogeneous Markov chains
  - Definitions
  - Properties
  - Reachability
- Classification of states
  - Definitions
  - Properties
- Regular Markov chains
  - Definition
  - Properties

Stochastic matrices

- Definition : real square matrix P such that

$$\begin{cases} p_{ij} \geq 0 & \forall i,j \\ \sum_j p_{ij} = 1 & \forall i \end{cases}$$

- Property : the product of two stochastic matrices is a stochastic matrix

Proof : every $(PQ)_{ij}$ are positive and

$$\begin{aligned} \sum_j (PQ)_{ij} &= \sum_j \sum_k p_{ik} q_{kj} \\ &= \sum_k p_{ik} \sum_j q_{kj} \\ &= \sum_k p_{ik} \\ &= 1 \end{aligned}$$

- A stochastic matrix is *regular* if there exists $k \in \mathbb{N}_0$ such that every elements of P^k are strictly positive (and then, that is true for $P^l \quad \forall l \geq k$).

- Example

$$P = \begin{bmatrix} . & 0,5 & 0,5 \\ 0,2 & 0,8 & . \\ 0,7 & . & 0,3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0,45 & 0,40 & 0,15 \\ 0,16 & 0,74 & 0,10 \\ 0,21 & 0,35 & 0,44 \end{bmatrix}$$

P (and P^2) is stochastic
P is regular

Homogeneous Markov chains

Definitions

- Context
  - Discrete time set : $T = \{0, 1, 2, \dots\}$

$$(X_t) : \quad X_0, X_1, \dots, X_n, \dots$$

  - Finite state space :

$$S = \{i, j, \dots\} = \bigcup_{t=0}^{\infty} X_t[\Omega]$$

  - At each time $t \in T$, the "system" is in one (and only one) state
  - Between two successive times, the "system" moves from a state to a state

« At time n, the system is in state i » : $[X_n = i]$

- Markov property : $\forall n \in \mathbb{N}, \ \forall i, j, k, l \in S$

$$\Pr([X_{n+1} = j] | [X_n = i] \cap [X_{n-1} = k] \cap \dots \cap [X_0 = l]) = \Pr([X_{n+1} = j] | [X_n = i])$$

Interpretation : the probability of a future event, knowing the present and the past, does not depend on the past → modelling a dependence phenomenon with a first order memory

- Homogeneity : the Markov property is independent of n.

- Definition : an homogeneous Markov chain is a stochastic process described by the previous « context » and
  - satisfying the Markov property ;
  - homogeneous

In the sequel, « MC » = homogeneous Markov chain

- Probabilities
  - state probabilities : $p_i(n) = \Pr[X_n = i]$
  - initial probabilities : $p_i(0) = \Pr[X_0 = i]$
  - transition probabilities :

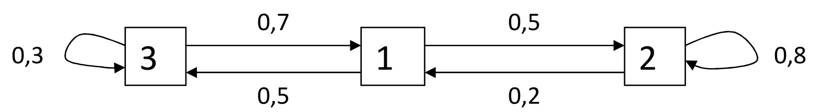
$$p_{ij} = \Pr([X_{n+1} = j] | [X_n = i])$$

➔ vector $p(n)$

➔ matrix P

- Example

$$P = \begin{bmatrix} . & 0{,}5 & 0{,}5 \\ 0{,}2 & 0{,}8 & . \\ 0{,}7 & . & 0{,}3 \end{bmatrix}$$



Properties

a) The transition matrix is stochastic

$$\sum_j p_{ij} = \sum_j \Pr([X_{n+1} = j] | [X_n = i])$$
$$= \Pr([X_{n+1} \in S] | [X_n = i])$$
$$= 1$$

b) Evolution of the state probabilities :

$$p^t(n+1) = p^t(n) \cdot P$$

$$(p^t(n) \cdot P)_j$$
$$= \sum_i p_i(n) p_{ij}$$
$$= \sum_i \Pr[X_n = i] \cdot \Pr([X_{n+1} = j] | [X_n = i])$$
$$= \Pr[X_{n+1} = j]$$

c) The m-step transition matrix $P^{(m)}$

$$p_{ij}^{(m)} = \Pr([X_{n+m} = j] \mid [X_n = i])$$

Is equal to P^m. Then, it is stochastic.

$$\begin{aligned}
p_{ij}^{(2)} &= \Pr([X_{n+2} = j] \mid [X_n = i]) \\
&= \sum_k \Pr([X_{n+2} = j] \cap [X_{n+1} = k] \mid [X_n = i]) \\
&= \sum_k \frac{\Pr([X_{n+2} = j] \cap [X_{n+1} = k] \cap [X_n = i])}{\Pr[X_n = i]} \\
&\qquad \times \frac{\Pr([X_{n+1} = k] \cap [X_n = i])}{\Pr([X_{n+1} = k] \cap [X_n = i])} \\
&= \sum_k \Pr([X_{n+1} = k] \mid [X_n = i]) \\
&\qquad \cdot \Pr([X_{n+2} = j] \mid [X_{n+1} = k] \cap [X_n = i]) \\
&= \sum_k p_{ik} p_{kj} \\
&= (P^2)_{ij}
\end{aligned}$$

d) We have

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$$

$$p^t(m+n) = p^t(n) \cdot P^{(m)}$$

Reachability

The state j is *reachable* from i if there exists $n > 0$ such that $p_{ij}^{(n)} > 0$

Notation : $i \to j$

Classification of states

Definitions

Starting in state i, the probability that i is « visited » at least once is denoted by f_i

Starting in state i, the probability that there are infinitely many occurrences of i is denoted by $f_i^{(\infty)}$

The state i is *recurrent* or *transient* according wether $f_i = 1$ or $f_i < 1$

Note : the probability that, starting in state i, the event A occurs is denoted by

$$\Pr(A|[X_0 = i]) = p_i(A)$$

Properties

a) If the state i is recurrent and $i \to j$, then j is recurrent

• $p_j(\text{we will eventually reach } i) = 1$ for iff not, i would not be recurrent : $f_i < 1$

In particular, $j \to i$

• We define

$$\alpha = p_i(\text{we will return to } i \text{ without ever hitting } j)$$

$\alpha < 1$ because $j \to i$

$$p_i(\text{we will make } n \text{ return visits to } i \text{ without ever hitting } j) = \alpha^n$$

$$p_i(\text{we will eventually reach } j) = \lim_{n\to\infty}(1 - \alpha^n) = 0$$

- Summarizing, starting in j,
  - we are certain to reach i
  - we are certain to come back to j

so, j is recurrent

b) Corollaries,

- i is recurrent iff $f_i^{(\infty)} = 1$
- Starting in a recurrent state, we must always remain in recurrent states

c) Starting in a transient state i, we must eventually reach a recurrent state

$$p_i(\text{we will never return to } i) = 1 - f_i$$

$$p_i(\text{we will return to } i \text{ exactly } n \text{ times}) = f_i^n(1 - f_i)$$

$$p_i(\text{there are only a finite return visits to } i) = \sum_{k=0}^{\infty} f_i^k(1 - f_i) = 1$$

So, starting in a transient state, the set of transient states is visited (with Pr 1) only a finite number of times. Then, we must eventually reach a recurrent state

d) Corollaries

- i is transient iff $f_i^{(\infty)} = 0$
- There are at least one recurrent state in a Markov chain

Regular Markov chains

Definition

A Markov chain is *regular* if its transition matrix is regular

Properties

a) In a regular Markov chain, every states are recurrent and, $\forall i, j \in S,\ i \to j$

b) For a regular Markov chain, we have

- $\lim_{m\to\infty} P^{(m)} = \Pi$
  where the rows π^t of Π are identical and with strictly positive elements

- π is the stationary probability vector : independently of $p(0)$,
  $$\lim_{n\to\infty} p^{(n)} = \pi$$

- π is the unique solution of the system $\pi^t(P - I) = 0$ with strictly positive elements having a sum equal to 1

(without proof)

- Example

$$P = \begin{bmatrix} . & 0,5 & 0,5 \\ 0,2 & 0,8 & . \\ 0,7 & . & 0,3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0,45 & 0,40 & 0,15 \\ 0,16 & 0,74 & 0,10 \\ 0,21 & 0,35 & 0,44 \end{bmatrix}$$

...

$$P^{16} = \begin{bmatrix} 0,2373 & 0,5931 & 0,1696 \\ 0,2372 & 0,5934 & 0,1694 \\ 0,2374 & 0,5929 & 0,1697 \end{bmatrix}$$

Moreover, the solution of

$$[\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} -1,0 & 0,5 & 0,5 \\ 0,2 & -0,2 & . \\ 0,7 & . & -0,7 \end{bmatrix} = [0 \quad 0 \quad 0]$$

with $\pi_1 + \pi_2 + \pi_3 = 1$ is

$$\begin{cases} \pi_1 = \dfrac{14}{59} = 0,237288 \\ \pi_2 = \dfrac{35}{59} = 0,593220 \\ \pi_3 = \dfrac{10}{59} = 0,169492 \end{cases}$$