Part III

ACTUARIAL APPLICATIONS

7. Various topics
8. Risk process
7. Specific tools

Part III

7. Specific tools

a) (non random)

Premium

Insured

Insurance Company

Intervention

Sometimes,

- the premium is paid by a taker of insurance
- the premium is paid by a beneficiary

- the intervention is paid to a beneficiary
- the intervention is paid to an insured person

b) Randomness is present in any domain of management (e.g., in finance), and the manager try to eliminate it

Two domains where randomness is the basis of human activity:

- Gambling (consumer pays to play against the hazard)
- Insurance (consumer pays to get rid of the hazard)

(consumer pays to play against the hazard)
(consumer pays to get rid of the hazard)
d) Here: non-life insurance, even if some topics are also adapted to life insurance.

<table>
<thead>
<tr>
<th>Chapter 7</th>
<th>Specific tools</th>
<th>- Markov chains - Counting process - Random sums - Convolution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Classification life vs non-life</th>
<th>Period</th>
<th>Contract</th>
<th>Claim amount</th>
<th>Intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td>Life</td>
<td>1 year</td>
<td>Standard</td>
<td>known (in age)</td>
<td>unknown</td>
</tr>
<tr>
<td>Non-life</td>
<td>very long</td>
<td>Non-life</td>
<td>claim amount</td>
<td>intervention</td>
</tr>
</tbody>
</table>
For the m.g.f.

If $X$ and $Y$ are independent r.v.'s,

\[
\mathbb{E}(e^{\lambda T}) \cdot \mathbb{E}(e^{\lambda T}) = \mathbb{E}((\Lambda + X)e^{\lambda T})
\]

Proof

\[
\lambda^T \cdot e^{\lambda T} = (\lambda + X)^T e^{\lambda T}
\]

For the c.d.f.

Generalization

- For continuous r.v.
- For discrete r.v.
- Case of Poisson r.v.
- Case of positive integer r.v.

For the m.g.f.

Definition
For discrete r.v.

Consider two independent r.v.

Case of positive integer r.v.

For any $\gamma \in \mathbb{N}$,

$$\Pr(\xi_1 + \xi_2 = \gamma) = \Pr(\xi_1 = \gamma - 1) \cdot \Pr(\xi_2 = 1)$$

For any $\gamma \in \mathbb{N}$,

$$\Pr(\xi_1 + \xi_2 = \gamma) = \sum_{\gamma} \left(\frac{\lambda^\gamma}{\gamma!} \right) e^{-\lambda} = e^{-\lambda} \sum_{\gamma} \left(\frac{\lambda^\gamma}{\gamma!} \right)$$

Consider two independent Poisson r.v.

Case of Poisson r.v.

Consider two independent r.v.

Case of discrete r.v.
\[ \lambda p \ (\lambda) \lambda f (\lambda - 1) x f \int_1^0 = \]
\[ x p \ (x - 1) \lambda f (x) x f \int_1^0 = (1) (\lambda f \ast x f) \]

In particular, if the two r.v. are positive,'

\[ (1) (\lambda f \ast x f) = \]
\[ \lambda p \ (\lambda) \lambda f (\lambda - 1) x f \int_1^\infty = \]
\[ x p \ (x - 1) \lambda f (x) x f \int_1^\infty = (1) \lambda + x f \]

and, by differentiation w.r.t. 't,',

\[ \lambda p \ (\lambda) \lambda f \int_1^\infty (x - 1) x f \int_1^\infty = \]
\[ x p \ (x) x f \int_1^\infty (x - 1) \lambda f \int_1^\infty = (1) \lambda + x f \]

For continuous r.v.

\[ (1) (\lambda f \ast x f) = \]
\[ (\lambda) \lambda p \ (\lambda - 1) x f \int_1^\infty = \]
\[ (x) x p \ (x - 1) \lambda f \int_1^\infty = (1) \lambda + x f \]

\[ ((X - 1) \lambda f) \mathbb{E} = \]
\[ \left( (X - 1) \lambda f \right) \mathbb{E} = (1) \lambda + x f \]

so that,

\[ (X) \left( [x - 1^\lambda] \lambda f \right) \mathbb{E} = (X) [x - 1^{\lambda + x}] \mathbb{E} \]

But, by (R7),

\[ (X) [x - 1^{\lambda + x}] \mathbb{E} = \]
\[ \left( [x - 1^{\lambda + x}] \lambda f \right) \mathbb{E} = \]
\[ [\lambda \geq \lambda + x] \mathbb{E} = (1) \lambda + x f \]

For the c.d.f.
Generalization

For more than two r.v. \( X_1, X_2, \ldots, X_n \),
\[
(1) \quad (X_1 + \cdots + X_n)^k = (1)^k (X_1 + \cdots + X_n)^k \]
and, if \( X_1, X_2, \ldots, X_n \) are i.i.d., then
\[
(1)(X_1 + \cdots + X_n)^k = \quad (1)(X_1 + \cdots + X_n)^k \]

Random sums

- Variance
- Expectation
- Moments
- Moment generating function
- Cumulative distribution function
- Definition

For more than two r.v.,
A random sum is a sum of r.v. with a random number of terms. Consider a sequence of i.i.d. r.v. \( Y_1, Y_2, \ldots, Y_{N} \), and a positive integer r.v. \( N \), independent of the \( Y_i \)'s. Let \( X = \sum_{i=1}^{N} Y_i \). The random variable \( X \) is a random sum of the \( Y_i \)'s.

**Example**

- Tossing 2 coins (\( Y_i \) = number of tails)
- Throwing \( N \) dice (\( Y_i \) = sum of the points on the die)
- Tossing 2 coins (\( N \) = number of tails)

**Illustration:** Suppose we have a random situation with a finite number of possible outcomes \( \Omega = \{0, 1, \ldots, 12\} \), and the probability law of \( X \) and the \( Y_i \)'s is given by:

\[
\begin{align*}
&\Pr(X=0) = \frac{1}{12}, \quad \Pr(X=1) = \frac{1}{12}, \quad \Pr(X=2) = \frac{1}{12}, \\
&\Pr(X=3) = \frac{1}{12}, \quad \Pr(X=4) = \frac{1}{12}, \quad \Pr(X=5) = \frac{1}{12}, \\
&\Pr(X=6) = \frac{1}{12}, \quad \Pr(X=7) = \frac{1}{12}, \quad \Pr(X=8) = \frac{1}{12}, \\
&\Pr(X=9) = \frac{1}{12}, \quad \Pr(X=10) = \frac{1}{12}, \quad \Pr(X=11) = \frac{1}{12}, \quad \Pr(X=12) = \frac{1}{12}.
\end{align*}
\]

Given by the probability law of \( N \), and the number of possible outcomes \( \Omega \) is finite, we can write \( X = \sum_{i=1}^{N} Y_i \), where \( Y_i \) are independent of \( N \) and \( \{\varepsilon_{\Omega}, \varepsilon_{\Omega}, \ldots, \varepsilon_{\Omega}\} = \Omega \).

Consider a sequence of i.i.d. r.v. \( X_1, X_2, \ldots, X_n \), and \( N \) number of terms of a random sum is a sum of r.v. with a random number of terms.
Cumulative distribution function

\[ f = \sum_{i=0}^{\infty} \left( i \right)_{\nu} \frac{\lambda^i}{i!} \]

Then,

\[ \left( i \right)_{\nu} \frac{\lambda^i}{i!} L = \left( i \right)_{\nu} \frac{\lambda^i}{i!} \]

so that

\[ \left( i \right)_{\nu} \frac{\lambda^i}{i!} = \left( i \right)_{\nu} \frac{\lambda^i}{i!} \]

But, by (R7),

\[ \left( N \right)_{\left( [1^{\nu_1} + \cdots + 1^{\nu_m}] \right)} \frac{\lambda^i}{i!} = \left( N \right)_{\left( [1^{\nu_1} + \cdots + 1^{\nu_m}] \right)} \frac{\lambda^i}{i!} \]

Moment Generating Function

\[ ((i)_{\nu_1} \lambda^i)_{\nu} = \]

\[ \left( i \right)_{\nu_1} \lambda^i \]

\[ \left( i \right)_{\nu_1} \lambda^i \]

\[ \left( \left( i \right)_{\nu_1} \lambda^i \right)_{\nu} = \]

\[ \left( \left( i \right)_{\nu_1} \lambda^i \right)_{\nu} = \]

\[ \left( \left( i \right)_{\nu_1} \lambda^i \right)_{\nu} = \]

Cumulative distribution function
Moments

\[ \mathbb{E}[X] \cdot \mathbb{E}[a] + \mathbb{E}[X] \cdot \mathbb{E}[a] = \mathbb{E}[a] \cdot \mathbb{E}[X] \cdot \mathbb{E}[a] + \mathbb{E}[X] \cdot \mathbb{E}[a] = (s)\mathbb{E}[a] \]

Variance

\[ \mathbb{V}[X] \cdot \mathbb{E}[a] + \mathbb{V}[X] \cdot \mathbb{E}[a] = \mathbb{E}[a] \cdot \mathbb{V}[X] \cdot \mathbb{E}[a] + \mathbb{V}[X] \cdot \mathbb{E}[a] = (s)\mathbb{E}[a] \]

- Counting process
  - Counting process
  - Poisson Process
  - Probability distribution
  - Definition
- Time between 2 occurrences
Let us consider a counting process \( \{N_t : t \geq 0\} \) such that:

- \( N_0 = 0 \) (No occurrence at time 0)
- \( \{N_t : t \geq 0\} \) has independent and stationary increments
- No multiple occurrences

**Definition**

\[
\lim_{h \to 0} \frac{N_{t+h} - N_t}{h} = \lambda
\]

(\( \lambda \) tends to 0 more rapidly than \( h \) itself)

\[
0 = \lim_{h \to 0} \frac{N_t - N_{t-h}}{h} \iff (N)_0 = (\lambda)_f
\]

**Notation**

**Poisson Process**

**Continuous Process**
\[
\frac{iu}{u(\mathcal{Y})} = \frac{u}{u^l} \cdot \frac{i(1 - u)}{u^l} = 0 - (1)^u \mathcal{d} \gamma \vartheta
\]

\[
sp_{1-u}s \int_{1-u(\mathcal{Y})}^{0} \frac{i(1 - u)}{u^l} = sp \left( (s)^u d_{s} \gamma \vartheta \right) \int_{1}
\]

\[
\left( (1)^u d_{s} \gamma \vartheta \right) = \frac{i(1 - u)}{1-u(\mathcal{Y})} \cdot \gamma = (1)^u d \gamma + (1)^u d \cdot \gamma \vartheta
\]

\[
\frac{i(1 - u)}{1-u(\mathcal{Y})} \cdot \gamma - (1)^u d \gamma - = (1)^u d
\]

\[
\frac{\gamma}{(\gamma)^o} + (1)^{1-u} d \gamma + (1)^u d \gamma - = \frac{\gamma}{(1)^u d - (\gamma + 1)^u d}
\]

\[
(\gamma)^o + ((\gamma)^o + \eta \gamma) \cdot (1)^{1-u} d + ((\gamma)^o + \eta \gamma - 1) \cdot (1)^u d =
\]

\[
[\gamma = \mathcal{N} - \{(1)^u \mathcal{d} \gamma - \}_{\infty}^{\infty} +
\]

\[
[1 = \mathcal{N} - \{(1)^u \mathcal{d} \gamma - \}_{\infty}^{\infty} + [0 = \mathcal{N} - \{(1)^u \mathcal{d} \gamma - \}_{\infty}^{\infty} = (\gamma + 1)^u d \]
\]

\[
\mathcal{Y} - \vartheta = (1)^o d
\]

\[
\mathcal{J} \cdot \mathcal{Y} - \vartheta = (1)^o d
\]

\[
(1)^o d \gamma - = (1)^o d
\]

\[
\frac{\gamma}{(\gamma)^o} + (1)^o d \gamma - = \frac{\gamma}{(1)^o d - (\gamma + 1)^o d}
\]

\[
((\gamma)^o + \eta \gamma - 1) \cdot (1)^o d =
\]

\[
[0 = \mathcal{N} - \{(1)^u \mathcal{d} \gamma - \}_{\infty}^{\infty} \cdot (1)^o d = (\gamma + 1)^o d \]
\]

(\text{Proof by induction})

\[
\frac{iu}{u(\mathcal{Y})} \cdot \mathcal{Y} - \vartheta = [u = \mathcal{N} \mathcal{d} = (1)^u d
\]

\[
\cdots 0, 1, 2, \cdots
\]

\[
\text{Probability distribution}
\]
\[
\begin{aligned}
&\left( (\gamma)^{u(1)} \right) = (1)^{u(1) + \cdots + 1} \\cdot \ \wedge (1) \\
&\text{In particular, } \wedge \frac{\gamma - \gamma}{\gamma} = (1)^{u(1)} \\
&(\gamma > 1)
\end{aligned}
\]
A Poisson process generates an exponential renewal process. Successive durations is a renewal process.

Note: a sequence of i.i.d. r.v.'s representing

\[ \text{Exponential distribution with parameter } \lambda = 0.8, 1, 1.6 \]

\[ f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & x \leq 0 \end{cases} \]

Markov chains

- Definitions
- Properties
- Regular Markov chains
- Classification of states
- Reachability
- Homogeneous Markov chains
- Stochastic matrices

\[
\begin{align*}
9 &= \gamma \\
\lambda &= 1 \\
\frac{\gamma}{1} &= (I - L) \text{and} \\
\frac{\gamma}{1} &= (I) E
\end{align*}
\]

\[
\begin{align*}
\lambda &= 0.8, 1, 1.6 \\
\rho &= (t)^{\frac{1}{2}} f = (t)^{\rho t}
\end{align*}
\]

Then, \( T^t Y \), \( \ldots \) are i.i.d. r.v.'s.
Stochastic matrices

- **Definition**: real square matrix \( A \) such that
  \[
  \sum_{k=1}^{d} p_{ik} = 1 \quad \forall i.
  \]
  \[
  \sum_{i=1}^{d} p_{ij} \geq 0 \quad \forall j.
  \]
  \[
  \sum_{j=1}^{d} p_{ij} = 1 \quad \forall i.
  \]

- **Property**: the product of two stochastic matrices is a stochastic matrix

  Proof: every \( (PQ)_{ij} \) are positive and

- **Example**

  \[
  P = \begin{bmatrix}
  0.4 & 0.5 & 0.15 \\
  0.21 & 0.74 & 0.15 \\
  0.35 & 0.35 & 0.3
  \end{bmatrix}
  \]

  \[
  P_2 = \begin{bmatrix}
  0.45 & 0.40 & 0.15 \\
  0.16 & 0.74 & 0.14 \\
  0.2 & 0.8 & 0.15
  \end{bmatrix}
  \]

- **A stochastic matrix is regular if there exists \( k \in \mathbb{N}_0 \) such that every elements of \( p^k \) are strictly positive (and then, that is true for \( p \)).
Homogeneous Markov chains

Definitions

- Context
  - Discrete time set: \( \{0, 1, 2, \ldots\} \)
  - Finite state space: \( \mathcal{X} \)

- Markov property: \( \forall n \in \mathbb{N}, \forall i,j \in S \)

\[
[\tau = \mathbb{U}X] \mid [\tau = \mathbb{U}X]_{\tau = 0} \cup \cdots \cup [\tau = \mathbb{U}X]_{\tau = \mathbb{U}X} \mid [\tau = \mathbb{U}X]_{\tau = 0} \cup \cdots \cup [\tau = \mathbb{U}X]_{\tau = \mathbb{U}X} \]

- Interpretation: the probability of a future event, knowing the present and the past, does not depend on the past.

- Homogeneity: the Markov property is independent of \( n \).

- Homogeneous Markov chain is a stochastic process described by the previous context and satisfying the Markov property.

\( [\forall i \in \mathcal{X}] \cap \{0, 1, 2, \ldots\} = S \)

In the sequel, "MC" = homogeneous Markov chain.

At each time \( t \in \mathbb{T} \), the system is in one state.

Between two successive times, the system moves from a state to a state (and only one state).

At time \( \tau \), the system is in state \( i \): \( \{ \tau = \tau, \tau < 0 \} = S \)
Probabilities

- State probabilities:
  \( \text{Pr} = \frac{t_{i,g}^{(n)}}{g} \)

- Initial probabilities:
  \( \text{Pr} = \frac{t_{i,g}^{(0)}}{g} \)

- Transition probabilities:
  \( \text{Pr} = \frac{t_{i,g}^{(1)}}{g} \)

Example:

\[
\begin{bmatrix}
0.3 & 0.7 \\
0.8 & 0.2 \\
0.5 & 0.5
\end{bmatrix} = d
\]

Properties:

- Transition matrix is stochastic.

\[
(d \cdot (u)^i d)
\]

Evolution of the state probabilities:

\[
[\text{Pr}^{(i+1)}] = (I + u)^i [\text{Pr}^{(i)}]
\]

a) The transition matrix is stochastic.
The 2-step transition matrix \( M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \) is equal to \( M^2 = \begin{pmatrix} p_{11}^2 + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{22}^2 + p_{21}p_{12} \end{pmatrix} \). Then, it is stochastic. Then, it is stochastic.

Reachability

\( \lim_{n \to \infty} p_n = \lim_{n \to \infty} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \lim_{n \to \infty} p_n = \lim_{n \to \infty} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

Notation:

\( i \to j \) if there exists \( i \to j \) such that \( 0 < f_i d < u \) for some \( d \) and \( 0 < u < u \)

Reachability

\( (w)d \cdot (u)d = (u + w)d \)

\( (w)d \cdot (u)d \sum_{k=1}^{\infty} = (u + w)d \)

\( \sum_{k=1}^{\infty} \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \sum_{k=1}^{\infty} f_k d_{\bar{\gamma}} = \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \sum_{k=1}^{\infty} \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

The state \( i \) is reachable from \( j \) if there exists a path from \( i \) to \( j \) such that \( 0 < f_i d < u \) for some \( d \) and \( 0 < u < u \)

\( (w)d \cdot (u)d = (u + w)d \)

\( (w)d \cdot (u)d \sum_{k=1}^{\infty} = (u + w)d \)

\( \sum_{k=1}^{\infty} \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \sum_{k=1}^{\infty} \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \sum_{k=1}^{\infty} f_k d_{\bar{\gamma}} = \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \sum_{k=1}^{\infty} \begin{pmatrix} p_{11}^k & p_{12}^k \\ p_{21}^k & p_{22}^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

The finite-state transition matrix \( M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \) is equal to \( M^2 = \begin{pmatrix} p_{11}^2 + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{22}^2 + p_{21}p_{12} \end{pmatrix} \). Then, it is stochastic.
Classification of states

Definitions

Starting in state \( g_1 \), the probability that \( g_1 \) is visited at least once is denoted by \( \pi \).

Starting in state \( g_1 \), the probability that there are infinitely many occurrences of \( g_1 \) is denoted by \( \pi \).

The state \( g_1 \) is recurrent or transient according whether \( \pi \) or not, \( \pi < 1 \) would not be recurrent:

\( \pi > 1 \) if \( \pi \) would not be recurrent:

\( \pi \) is recurrent if the state \( g_1 \) is recurrent and \( \pi \) is.

Properties

\( \pi \) is recurrent if the state \( g_1 \) is recurrent and \( \pi \) is.

Definitions

Classification of states
Markov chain

- There are at least one recurrent state in a
  \[ 0 = \lim_{n \to \infty} \text{pr}(\text{state } i) \to \text{state } i \]
- \(b\) Corollaries

a recurrent state
number of times. Then, we must eventually reach
transient states is visited (with Pr 1) only a finite
So, starting in a transient state, the set of

\[ I = (I - 1)^n I \]

\(d\) Corollaries

(there are only a finite return visits to i)

\[ (I - 1)^n I = \]

(we will return to i exactly n times)

\[ I - 1 = (I - 1)^n I \] (we will never return to i)

Eventually reach a recurrent state
(c) Starting in a transient state, we must

Remain in a recurrent state
Starting in a recurrent state, we must always

\[ I = \lim_{n \to \infty} \text{pr}(\text{state } i) \to \text{state } i \]

\(d\) Corollaries

So, \(f\) is recurrent
we are certain to come back to \(f\)
we are certain to reach \(f\)
Summarizing, starting in \(f\)
A Markov chain is regular if its transition matrix is regular.

Properties

(a) In a regular Markov chain, every states are recurrent and, ∀i, j ∈ S, i → j

(b) For a regular Markov chain, we have

\[
\lim_{t \to \infty} (I - d)^t \mathbf{u} = \Pi
\]

where the rows of \(\Pi\) are identical and with strictly positive elements.

\[
\mathbf{u} = (\mathbf{w})d^\infty \mathbf{u}
\]

and

\[
\lim_{t \to \infty} d^t \mathbf{w} = \mathbf{u}
\]

is the stationary probability vector.

Having a sum equal to 1 with strictly positive elements.

\[
\mathbf{w} = \mathbf{w}d^\infty \mathbf{w}
\]

without proof.

\[
\mathbf{w} = (\mathbf{w})d^\infty \mathbf{w}
\]

\[
\lim_{t \to \infty} d^t \mathbf{w} = \mathbf{u}
\]

\[
\mathbf{w} = \mathbf{w}d^\infty \mathbf{w}
\]

\[
\lim_{t \to \infty} d^t \mathbf{w} = \mathbf{u}
\]
\[ \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.2 & 0.8 & 0.7 \\ 0.3 & 0.7 & 0.3 \\ -0.7 & -0.2 & -1.0 \\ 0.2 & 0.7 & 0.7 \\ 0.8 & 0.2 & 0.5 \\ 0.7 & 0.4 & 0.3 \\ 0.5 & 0.5 & 0.2 \\ -1.0 & 0.5 & 0.5 \\ 0.2 & 0.7 & 0.7 \\ 0.8 & 0.2 & 0.5 \\ 0.7 & 0.4 & 0.3 \\ 0.5 & 0.5 & 0.2 \end{bmatrix} \]

Moreover, the solution of

\[ \begin{bmatrix} 0 & 2.374 \\ 0.1693 & 0 \end{bmatrix} \begin{bmatrix} 0.1694 \\ 0.3376 \end{bmatrix} = 0 \]

\[ \begin{bmatrix} 0.1694 \\ 0.3376 \end{bmatrix} = \mathbf{b} \]

Example