Chapter 6

Interest rate models

- Deterministic yield curves
- Objective, hypotheses and general scheme
- Structure equation
- Merton model
- Vasicek model
- Cox, Ingersoll & Ross model

Deterministic yield curves

- Initial discrete structure
  - The 3 curves
  - Link between the different curves
- Evolution of the discrete structure
  - The 3 curves
  - Evolution of the yield curve
- Continuous time structure
  - Continuous yield
  - Short-term interest rate ?
  - Link between the different curves
- Stochastic modelling ?

Initial discrete structure

The 3 curves

- Price, at time 0, of a zero-coupon bond paying 1 at maturity s (> 0) : $P_0(s)$

- Yield : $R_0(s)$

$$P_0(s) = \left(1 + R_0(s)\right)^{-s}$$

- The yield combines short-term interest rates $r(1),\ r(2), \dots,\ r(s)$ for the respective periods [0; 1], [1; 2], …, $[s-1; s]$

$$\left(1 + R_0(s)\right)^s = \left(1 + r(1)\right) \cdot \ldots \cdot \left(1 + r(s)\right)$$

Link between the different curves

We can express one of the different curves

$$\{P_0(t) : t = 0, \dots, s\}$$
$$\{R_0(t) : t = 0, \dots, s\}$$
$$\{r_0(t) : t = 0, \dots, s\}$$

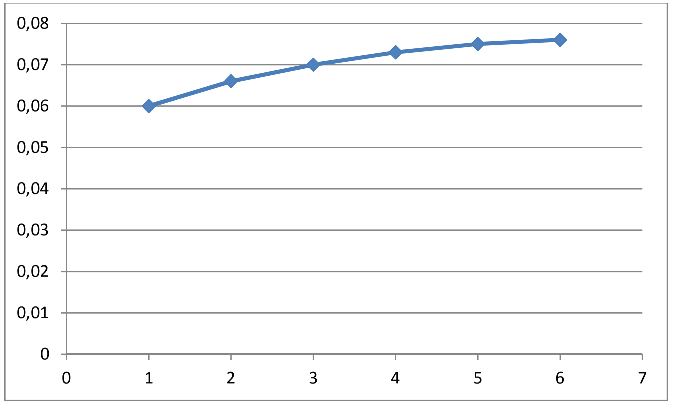
from the other two

For example,

$$1 + r(s) = \frac{\left(1+r(1)\right)\cdot\ldots\cdot\left(1+r(s-1)\right)\cdot\left(1+r(s)\right)}{\left(1+r(1)\right)\cdot\ldots\cdot\left(1+r(s-1)\right)}$$
$$= \frac{\left(1 + R_0(s)\right)^s}{\left(1 + R_0(s-1)\right)^{s-1}}$$
$$= \frac{P_0(s-1)}{P_0(s)}$$

Example : let us consider the yield structure

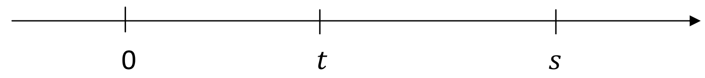
s	$R_0(s)$
1	6.0 %
2	6.6 %
3	7.0 %
4	7.3 %
5	7.5 %
6	7.6 %



Calculate the two other curves

Evolution of the discrete structure

The 3 curves



- Price, at time t, of a zero-coupon bond paying 1 at maturity s $(> t)$: $P_t(s)$

- Yield : $R_t(s)$

$$P_t(s) = \left(1 + R_t(s)\right)^{-(s-t)}$$

- The yield combines short-term interest rates $r(t+1)$, $r(t+2), \ldots$, $r(s)$ for the respective periods $[t; t+1]$, $[t+1; t+2]$, …, $[s-1; s]$

$$\left(1 + R_t(s)\right)^{s-t} = \left(1 + r(t+1)\right) \cdot \ldots \cdot \left(1 + r(s)\right)$$

so that $\quad 1 + r(s) = \dfrac{\left(1+R_t(s)\right)^{s-t}}{\left(1+R_t(s-1)\right)^{s-1-t}}$

Evolution of the yield structure

$$\begin{aligned}P_t(s) &= \left[\left(1+r(t+1)\right)\cdot\ldots\cdot\left(1+r(s)\right)\right]^{-1}\\ &= \left[\frac{\left(1+r(t)\right)\cdot\left(1+r(t+1)\right)\cdot\ldots\cdot\left(1+r(s)\right)}{1+r(t)}\right]^{-1}\\ &= \frac{\left(1+R_{t-1}(s)\right)^{-(s-t+1)}}{\left(1+R_{t-1}(t)\right)^{-1}}\\ &= \frac{P_{t-1}(s)}{P_{t-1}(t)}\\ &= \frac{P_{t-2}(s)/P_{t-2}(t-1)}{P_{t-2}(t)/P_{t-2}(t-1)}\\ &= \frac{P_{t-2}(s)}{P_{t-2}(t)}\\ &= \cdots\end{aligned}$$

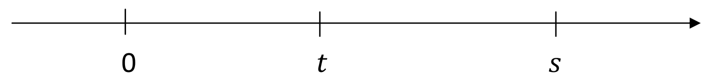
Whatever u may be $(u \leq s)$,

$$P_t(s) = \frac{P_u(s)}{P_u(t)}$$

Example : for the same data, calculate $R_2(5)$ and $P_2(5)$

Continuous time structure

Time set : $[0;\ +\infty[$



Continuous yield

- in the discrete time,

$$P_t(s) = \left(1+R_t^d(s)\right)^{-(s-t)}$$

- in the continuous time,

$$P_t(s) = e^{-(s-t)R_t(s)}$$

so that

$$\begin{cases} R_t(s) = \ln\left(1+R_t^d(s)\right) \\ 1+R_t^d(s) = e^{R_t(s)} \end{cases}$$

Short-term interest rate ?

Interest rate relative to the interval $[t_1; t_2]$:

$$r(t_1, t_2)$$

Instant-term interest rate at time t : mean of $r(t_1, t_2)$ where $t_1 = t$ and $(t_2 - t_1)$ very short

$$r(t) = \lim_{s \to t+} \frac{1}{s-t} \int_t^s r(t,u)\, du$$

Moreover, if $r(t_1, t_2)$ is a continuous function (we will suppose it here),

$$r(t) = \lim_{s \to t+} r(t; s)$$

Link between the different curves

Rewriting the “discrete” formula

$$1 + r(s) = \frac{\left(1 + R_t^d(s)\right)^{s-t}}{\left(1 + R_t^d(s-1)\right)^{s-1-t}}$$

for $s - 1 \leadsto s$ and $s \leadsto s + \Delta s$:

$$\left(1 + r(s; s+\Delta s)\right)^{\Delta s} = \frac{\left(1 + R_t^d(s+\Delta s)\right)^{s+\Delta s-t}}{\left(1 + R_t^d(s)\right)^{s-t}}$$

By Taylor expansion,

$$\left(1 + \Delta s \cdot r(s; s+\Delta s)\right) \cdot \left(1 + R_t^d(s)\right)^{s-t} \approx \left(1 + R_t^d(s+\Delta s)\right)^{s+\Delta s-t}$$

$$r(s;s+\Delta s)\cdot\left(1+R_t^d(s)\right)^{s-t} \approx \frac{\left(1+R_t^d(s+\Delta s)\right)^{s+\Delta s-t}-\left(1+R_t^d(s)\right)^{s-t}}{\Delta s}$$

and by taking the limit for $\Delta s \to 0$,

$$r(s)=\frac{\left[\left(1+R_t^d(s)\right)^{s-t}\right]_s'}{\left(1+R_t^d(s)\right)^{s-t}}$$
$$=\left[\ln\left(\left(1+R_t^d(s)\right)^{s-t}\right)\right]_s'$$
$$=\left[(s-t)\cdot\ln\left(1+R_t^d(s)\right)\right]_s'$$
$$=[(s-t)\cdot R_t(s)]_s'$$

$$\int_t^s r(u)\,du=[(u-t)\cdot R_t(u)]_{u=t}^{u=s}$$
$$=(s-t)\cdot R_t(s)$$

So,

$$R_t(s)=\frac{1}{s-t}\int_t^s r(u)\,du$$

$$P_t(s)=e^{-(s-t)R_t(s)}=e^{-\int_t^s r(u)\,du}$$

Stochastic modelling ?

- For option models, C_t = r.v. depending on time t
- For yield curves, $R_t(s)$ = r.v. depending on 2 time variables

Objective, hypotheses and general scheme

Hypotheses

a) r_t is a stochastic process, driven by a SDE

$$dr_t = a_t \cdot dt + b_t \cdot dW_t$$

b) $P_t(s)$ and $R_t(s)$ can be considered
- either as stochastic processes, because they are functions of r_t
- or as ordinary functions of (t, r)
  (s will generally be fixed, the important time variable being the duration $s - t$)

Objective

For different specified SDE driving the spot rate, obtaining (deterministic) functions ($P_t(s)$ and) $R_t(s)$

Approach

Here, we will only consider the arbitrage approach

The general scheme will be

1) Evolution of the spot rate (= state variable)
   $$dr_t = a_t \cdot dt + b_t \cdot dW_t$$
2) Portfolio of 2 bonds with different maturities with proportions such that the portfolio has no risky component
3) Arbitrage free reasoning :
   return = risk-free rate
   → the market price of risk λ_t is independent of the maturity
   → PDE (= structure PDE equation)
4) For different choices of (a_t, b_t, λ_t), solving the structure equation
   → Merton model
   → Vasicek model
   → Cox, Ingersoll & Ross model

Structure equation

- The market price of risk
- Structure equation

The market price of risk

$P_t(s, r_t)$ is considered as a function of the two variables (t, r) with

$$dr_t = a_t \cdot dt + b_t \cdot dW_t$$

where

- a_t = average instant return of the spot rate
- b_t = average instant volatility of the spot rate

Applying Itô's lemma to $P_t(s, r_t)$, we have

$$dP_t(s,r) = \left(P_t' + a_t P_r' + \frac{b_t^2}{2} P_{rr}'' \right) \cdot dt + b_t P_r' \cdot dW_t$$

The return of this bond is given by

$$\frac{dP_t(s,r)}{P_t(s,r)} = \frac{P_t' + a_t P_r' + \frac{b_t^2}{2} P_{rr}''}{P} \cdot dt + \frac{b_t P_r'}{P} \cdot dW_t = \mu_t(s,r) \cdot dt - \sigma_t(s,r) \cdot dW_t$$

where

- $\mu_t(s,r) = \frac{P_t' + a_t P_r' + \frac{b_t^2}{2} P_{rr}''}{P}$
  = average instant return of the bond
- $\sigma_t(s,r) = -\frac{b_t P_r'}{P}$
  = average instant volatility of the bond

Let us construct at time t a portfolio by

- buying X unit(s) of a bond with maturity s_1
- selling 1 unit of a bond with maturity s_2

The value of this portfolio is

$$V_t = XP_t(s_1) - P_t(s_2)$$

and

$$\begin{aligned} &dV_t \\ &= X\, dP_t(s_1) - dP_t(s_2) \\ &= X[P_t(s_1)\, \mu_t(s_1) \cdot dt - P_t(s_1)\, \sigma_t(s_1) \cdot dW_t] \\ &\quad -[P_t(s_2)\, \mu_t(s_2) \cdot dt - P_t(s_2)\, \sigma_t(s_2) \cdot dW_t] \\ &= [XP_t(s_1)\, \mu_t(s_1) - P_t(s_2)\, \mu_t(s_2)] \cdot dt \\ &\quad -[XP_t(s_1)\, \sigma_t(s_1) - P_t(s_2)\, \sigma_t(s_2)] \cdot dW_t \end{aligned}$$

The return of this portfolio is given by

$$\begin{aligned} \frac{dV_t}{V_t} &= \frac{XP_t(s_1)\, \mu_t(s_1) - P_t(s_2)\, \mu_t(s_2)}{XP_t(s_1) - P_t(s_2)}\, dt \\ &\qquad - \frac{XP_t(s_1)\, \sigma_t(s_1) - P_t(s_2)\, \sigma_t(s_2)}{XP_t(s_1) - P_t(s_2)}\, dW_t \\ &= \alpha \cdot dt + \beta \cdot dW_t \end{aligned}$$

We chose X such that this portfolio has no longer random component

By arbitrage free principle, the return of this portfolio is equal to the risk-free rate r :

$$\beta = 0 \quad \Rightarrow \quad \alpha = r$$

We have, for these two equations,

$$XP_t(s_1)\, \sigma_t(s_1) = P_t(s_2)\, \sigma_t(s_2)$$

$$XP_t(s_1)(\mu_t(s_1) - r) = P_t(s_2)(\mu_t(s_2) - r)$$

Eliminating X, we obtain

$$\frac{\mu_t(s_1) - r}{\sigma_t(s_1)} = \frac{\mu_t(s_2) - r}{\sigma_t(s_2)}$$

and so,

$$\lambda_t(s,r) = \frac{\mu_t(s,r) - r}{\sigma_t(s,r)}$$

does not depend on s

$\lambda_t(\, r) = \frac{\mu_t(s,r)-r}{\sigma_t(s,r)}$ is the market price of risk

= risk premium

= excess return w.r.t. spot rate, per unit of risk

Structure equation

$$\lambda_t = \frac{\mu_t - r}{\sigma_t}$$

$$= \frac{\dfrac{P'_t + a_t P'_r + \frac{b_t^2}{2} P''_{rr}}{P} - r}{-\dfrac{b_t P'_r}{P}}$$

$$= \frac{P'_t + a_t P'_r + \frac{b_t^2}{2} P''_{rr} - rP}{-b_t P'_r}$$

and we obtain the structure equation

$$P'_t + (a_t + \lambda_t b_t) P'_r + \frac{b_t^2}{2} P''_{rr} - rP = 0$$

with the limit (terminal) condition $P_s(s,r) = 1$

We will have to solve this PDE for different choices of (a_t, b_t, λ_t)

Merton model

- Definition
- Arithmetic Brownian motion
- Solution of the structure equation
- Consequences
  - Average instant return of the bond
  - Yield curve

Definition

- (r_t) is driven by an ABM

$$dr_t = \alpha \cdot dt + \sigma \cdot dW_t$$

where
  - α is the drift ($\alpha \in \mathbb{R}$)
  - σ is the volatility ($\sigma > 0$)

- $\lambda_t(r_t) = 0$

Arithmetic Brownian motion

The solution of the SDE

$$dr_t = \alpha \cdot dt + \sigma \cdot dW_t$$

Is given by

$$r_t = r_0 + \alpha t + \sigma W_t$$

This stochastic process is such that

a) $E(r_t) = r_0 + \alpha t$

b) $var(r_t) = \sigma^2 t$

c) $r_t \sim \mathcal{N}$

So, this stochastic process

- is a straight line in mean
- has a variance tending to $+\infty$
- has potential negative values

Conclusion : ABM is not a credible process for the behavior of the spot rate. But, historical interst …

Solution of the structure equation

Structure equation : $P'_t + \alpha P'_r + \frac{\sigma^2}{2} P''_{rr} - rP = 0$

Its solution is

$$P_t(s,r) = \exp\left[-(s-t)r - \frac{\alpha}{2}(s-t)^2 + \frac{\sigma^2}{6}(s-t)^3\right]$$

The terminal condition $P_s(s,r) = 1$ is clearly satisfied and

$$P'_t = \exp[\ldots] \cdot \left(r + \alpha(s-t) - \frac{\sigma^2}{2}(s-t)^2\right)$$
$$P'_r = \exp[\ldots] \cdot \left(-(s-t)\right)$$
$$P''_{rr} = \exp[\ldots] \cdot \left(-(s-t)\right)^2$$

so that the l.h.s. is equal to 0

Note : when $(s-t) \to +\infty$, the price also tends to infinity !

Consequences

Average instant return of the bond

$$\mu_t(s,r) = \frac{1}{P}\left(P'_t + \alpha P'_r + \frac{\sigma^2}{2}P''_{rr}\right)$$
$$= \frac{1}{P} rP$$
$$= r$$

Note : this <u>instant</u> return is constant !

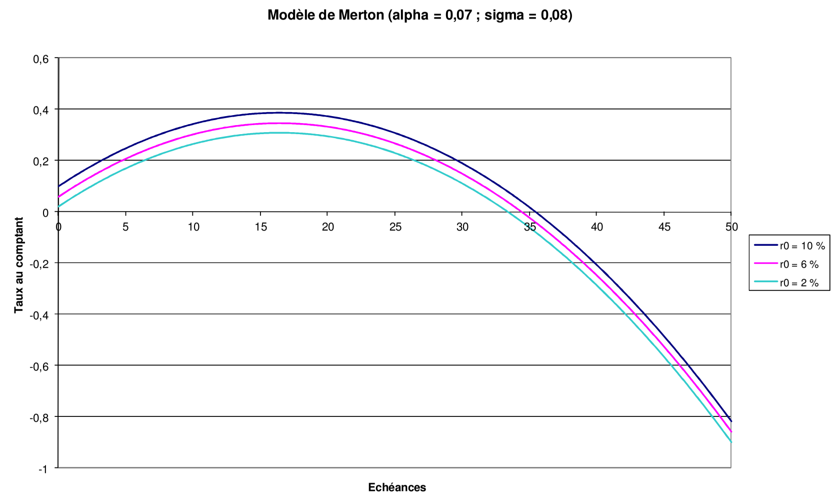
Yield curve

$$R_t(s,r) = -\frac{1}{s-t}\ln(\exp[\ldots])$$
$$= r + \frac{\alpha}{2}(s-t) - \frac{\sigma^2}{6}(s-t)^2$$

Properties

a) $R_s(s,r) = r$

b) $\lim_{(s-t)\to+\infty} R_t(s,r) = -\infty$



Vasicek model

- Definition
- Ornstein-Ühlenbeck process
  - “Mechanical” property
  - Solution of the SDE
  - Properties
- Solution of the structure equation
- Consequences
  - Average instant return of the bond
  - Yield curve

Vasicek, O. (1977) An equililbrium characterization of the term structure, *J. Financial Economics*, **5**, 177-188

Note : the structure equation is also from this reference

Definition

- (r_t) is driven by an Ornstein-Ühlenbeck process

$$dr_t = \delta(\theta - r_t) \cdot dt + \sigma \cdot dW_t$$

where $\delta, \theta, \sigma > 0$
  - δ is the force of recall
  - θ is the average value
  - σ is the volatility

- $\lambda_t(r_t)$ is a positive constant

Ornstein-Ühlenbeck process

"Mechanical" property

The drift coefficient is such that the trend is to "recall" r_t to the average value θ when it seems to diverge :

$$r_t \gg \;\Rightarrow\; \theta - r_t < 0$$
$$r_t \ll \;\Rightarrow\; \theta - r_t > 0$$

This behavior is much more adapted than the ABM to model an interest rate evolution

Solution of the SDE

$$dr_t = \delta(\theta - r_t) \cdot dt + \sigma \cdot dW_t$$

The objective is to write

$$r_t = Y_t + noise_t$$

such that

- $noise_0 = 0$ and $E(noise_t) = 0$
- Y_t is non random (and so $E(r_t) = Y_t$)

- The solution of the (non stochastic) DE

$$dY_t = \delta(\theta - Y_t) \cdot dt$$

is given by

$$Y_t = Y_0 e^{-\delta t} + \theta(1 - e^{-\delta t})$$

Proof

$$\begin{aligned} dY_t &= \left(-\delta Y_0 e^{-\delta t} + \theta\delta e^{-\delta t}\right) dt \\ &= \delta\left(-Y_0 e^{-\delta t} - \theta\left(1 - e^{-\delta t}\right) + \theta\right) dt \\ &= \delta(\theta - Y_t)\, dt \end{aligned}$$

- Write $noise_t = e^{-\delta t} Z_t$

$$r_t = Y_t + e^{-\delta t} Z_t$$

$$\begin{aligned} dZ_t &= d\left(e^{\delta t}(r_t - Y_t)\right) \\ &= \delta e^{\delta t}(r_t - Y_t)\, dt + e^{\delta t}(dr_t - dY_t) \\ &= \delta e^{\delta t}(r_t - Y_t)\, dt \\ &\quad + e^{\delta t}[\delta(\theta - r_t)dt + \sigma dW_t - \delta(\theta - Y_t)dt] \\ &= \sigma e^{\delta t}\, dW_t \end{aligned}$$

and

$$Z_t(-Z_0) = \sigma \int_0^t e^{\delta u}\, dW_u$$

with $E(Z_t) = 0$

- The solution of the SDE

$$dr_t = \delta(\theta - r_t) \cdot dt + \sigma \cdot dW_t$$

is then

$$r_t = r_0 e^{-\delta t} + \theta\left(1 - e^{-\delta t}\right) + \sigma e^{-\delta t} \int_0^t e^{\delta u}\, dW_u$$

Properties

a) $E(r_t) = r_0 e^{-\delta t} + \theta\left(1 - e^{-\delta t}\right)$

b) $var(r_t) = \sigma^2 e^{-2\delta t} \int_0^t E\left(e^{2\delta u}\right) du$
$= \frac{\sigma^2}{2\delta}\left(1 - e^{-2\delta t}\right)$

c) $r_t \sim \mathcal{N}$

because

$$\int_0^t e^{\delta u}\, dW_u = \lim_{\substack{n \to +\infty \\ \delta_n \to 0}} \sum_{i=1}^{n} e^{\delta t_{i-1}} \cdot \left(W_{t_i} - W_{t_{i-1}}\right)$$

So, this stochastic process has the following behavioral properties

- In mean, it is the (weighted) average of the initial value r_0 and the average value θ
- Its variance is an increasing function of time, but is bounded : $var(r_t) \leq \frac{\sigma^2}{2\delta}$
- It is not incompatible with negative values of r_t, even if the recall force of the drift term is such that this case is not frequent

Conclusion : the Ornstein-Ühlenbeck process is a credible process for the behavior of the spot rate (maybe except for the last remark)

Solution of the structure equation

Structure equation

$$P_t' + (\delta(\theta - r) + \lambda\sigma)P_r' + \frac{\sigma^2}{2}P_{rr}'' - rP = 0$$

Its solution is

$$P_t(s,r) = \exp\left[\begin{array}{r} -(s-t)k + \frac{k-r}{\delta}\left(1 - e^{-\delta(s-t)}\right) \\ -\frac{\sigma^2}{4\delta^3}\left(1 - e^{-\delta(s-t)}\right)^2 \end{array} \right]$$

where $k = \theta + \frac{\lambda\sigma}{\delta} - \frac{\sigma^2}{2\delta^2}$

The terminal condition $P_s(s,r) = 1$ is clearly satisfied and

$$P_t' = \exp[\ldots] \cdot \left(\begin{array}{r} k + (k-r)\left(-e^{-\delta(s-t)}\right) \\ -\frac{\sigma^2}{2\delta^2}(\ldots)\left(-e^{-\delta(s-t)}\right) \end{array} \right)$$

$$P'_r = \exp[\dots] \cdot \left(-\frac{1}{\delta}(\dots)\right)$$

$$P''_{rr} = \exp[\dots] \cdot \left(-\frac{1}{\delta}(\dots)\right)^2$$

so that the l.h.s. is equal to

$$\exp[\dots] \begin{Bmatrix} k - (k-r)e^{-\delta(s-t)} + \frac{\sigma^2}{2\delta^2} e^{-\delta(s-t)}(\dots) \\ -\left((\theta - r) + \frac{\lambda\sigma}{\delta}\right)(\dots) + \frac{\sigma^2}{2\delta^2}(\dots)^2 - r \end{Bmatrix}$$

$$= \exp[\dots] \cdot \begin{Bmatrix} (k-r)(\dots) + \frac{\sigma^2}{2\delta^2}(\dots) \\ -\left((\theta - r) + \frac{\lambda\sigma}{\delta}\right)(\dots) \end{Bmatrix}$$

$$= \exp[\dots](\dots) \cdot \left\{k - r + \frac{\sigma^2}{2\delta^2} - \theta + r - \frac{\lambda\sigma}{\delta}\right\}$$

$$= 0$$

Consequences

Average instant return of the bond

$$\begin{aligned}\mu_t(s,r) &= \frac{1}{P}\left(P'_t + \delta(\theta - r)P'_r + \frac{\sigma^2}{2}P''_{rr}\right) \\ &= \frac{1}{P}(rP - \lambda\sigma P'_r) \\ &= r + \frac{\lambda\sigma}{\delta}\frac{1}{P}\exp[\dots](\dots) \\ &= r + \frac{\lambda\sigma}{\delta}\left(1 - e^{-\delta(s-t)}\right)\end{aligned}$$

This instant return in an increasing but bounded function such that

$$\mu_s(s,r) = r$$

$$\lim_{(s-t)\to+\infty} \mu_t(s,r) = r + \frac{\lambda\sigma}{\delta}$$

Yield curve

$$R_t(s,r) = -\frac{1}{s-t}\ln(\exp[\dots])$$
$$= k - \frac{k-r}{\delta(s-t)}\left(1 - e^{-\delta(s-t)}\right)$$
$$+ \frac{\sigma^2}{4\delta^3(s-t)}\left(1 - e^{-\delta(s-t)}\right)^2$$

Properties

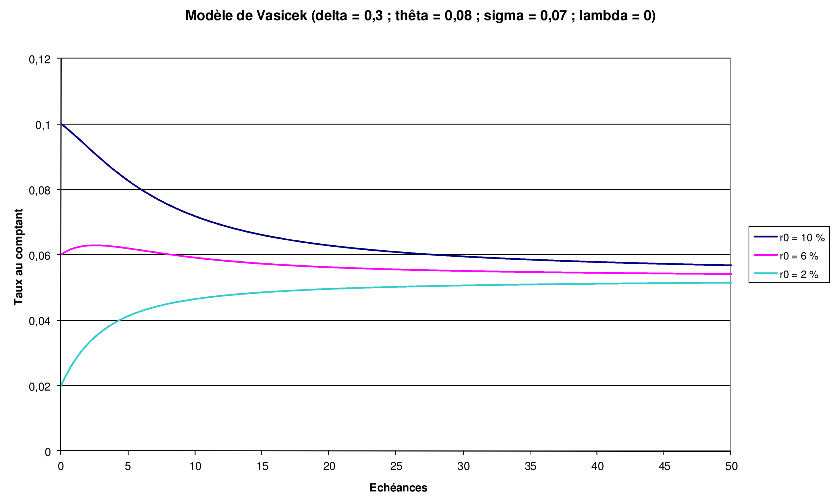
a) $R_s(s,r) = r$

because $\frac{1-e^{-\delta(s-t)}}{\delta(s-t)} \sim 1$ and $\frac{\left(1-e^{-\delta(s-t)}\right)^2}{s-t} \sim 0$

b) $\lim_{(s-t)\to+\infty} R_t(s,r) = k$

c) The analysis of the variations of $R_t(s,r)$ w.r.t. $s-t$ shows that the yield curve

- is increasing if $0 \le r \le k - \frac{\sigma^2}{4\delta^2}$
- is humped if $k - \frac{\sigma^2}{4\delta^2} < r < k + \frac{\sigma^2}{2\delta^2}$
- is decreasing if $r \ge k + \frac{\sigma^2}{2\delta^2}$



Cox-Ingersoll-Ross model

- Definition
- Square root process
- Solution of the structure equation
- Consequences
  - Average instant return of the bond
  - Yield curve

Cox, J., Ingersoll, J., Ross, S. (1985) A theory of the term structure of interest rates, *Econometrica*, **53**, 385-408

Definition

- (r_t) is driven by square root process

$$dr_t = \delta(\theta - r_t) \cdot dt + \sigma\sqrt{r_t} \cdot dW_t$$

where $\delta, \theta, \sigma > 0$

  - δ is the force of recall
  - θ is the average value
  - σ is the volatility

- $\lambda_t(r_t) = \frac{\gamma}{\sigma}\sqrt{r_t}$

where $\gamma > 0$

Square root process

- The “mechanical” property of the drift term is the same as the one of Ornstein-Ühlenbeck process

- Negative values of r_t are incompatible with the square root process : if r_t decreases to 0, then the SDE becomes

$$dr_t = \delta\theta \cdot dt$$

and r_t goes to strictly positive values with probability 1

Solution of the structure equation

Structure equation

$$P_t' + (\delta(\theta - r) + \gamma r)P_r' + \frac{\sigma^2 r}{2} P_{rr}'' - rP = 0$$

Its solution is

$$P_t(s, r) = x_t(s) \cdot e^{-y_t(s) r}$$

where

$$x_t(s) = \left(\frac{2k e^{\frac{(\delta - \gamma + k)(s-t)}{2}}}{z_t(s)} \right)^{\frac{2\delta\theta}{\sigma^2}}$$

$$y_t(s) = \frac{2\left(e^{k(s-t)} - 1\right)}{z_t(s)}$$

$$z_t(s) = (\delta - \gamma + k)\left(e^{k(s-t)} - 1\right) + 2k$$

$$k = \sqrt{(\delta - \gamma)^2 + 2\sigma^2}$$

The terminal condition $P_s(s, r) = 1$ is clearly satisfied and it is easy to verify the structure equation …

Consequences

Average instant return of the bond

$$\begin{aligned}\mu_t(s,r) &= \frac{1}{P}\left(P'_t + \delta(\theta - r)P'_r + \frac{\sigma^2 r}{2}P''_{rr}\right)\\ &= \frac{1}{P}(rP - \gamma r P'_r)\\ &= r\big(1 + \gamma y_t(s)\big)\end{aligned}$$

Yield curve

$$\begin{aligned}R_t(s,r) &= -\frac{1}{s-t}\ln\left(x_t(s)\cdot e^{-y_t(s)r}\right)\\ &= -\frac{1}{s-t}(\ln x_t(s) - y_t(s)r)\end{aligned}$$

These curves have the same kind of properties as the Vasicek ones