Chapter 6

Interest rate models

Deterministic yield curves

- Stochastic modelling
  - Link between the different curves
  - Short-term interest rate
  - Continuous yield
  - Continuous time structure
  - Evolution of the yield curve
  - The 3 curves

- Evolution of the discrete structure
  - Link between the different curves
  - The 3 curves
  - Initial discrete structure

Cox, Ingersoll & Ross model
- Vasicek model
- Merton model
- Structure equation
- Objective, hypotheses and general scheme

Deterministic yield curves
The 3 curves

- Price, at time 0, of a zero-coupon bond paying 1 at maturity

\[
\begin{align*}
\frac{(s)^0 d}{(1 - s)^0 d} &= \frac{1 - s}{1 - s} \left( \frac{(s)^0 Y + 1}{(1 - s)^0 Y + 1} \right) \\
&= s \left( (s)^0 Y + 1 \right) \left( \frac{(1)^0 Y + 1}{(1)^0 Y} \right) \left( \frac{(1)^0 Y + 1}{(1)^0 Y} \right) = (s)^0 Y + 1
\end{align*}
\]

For example, from the other two

\[
\{ s, \ldots, 0 = \tau : (\tau)^0 Y \} \\
\{ s, \ldots, 0 = \tau : (\tau)^0 Y \} \\
\{ s, \ldots, 0 = \tau : (\tau)^0 Y \}
\]

We can express one of the different curves

\[
\begin{align*}
(s)^0 Y + 1 \cdot \cdots \cdot (1)^0 Y + 1 &= s (s)^0 Y + 1 \\
\text{ periods [0; \tau], [\tau; 2], \ldots, [s - \tau; s] for the respective short-term interest rates} \\
\text{ - The yield combines short-term interest rates} \\
\text{ - Yield : (s)^0 Y + 1 = (s)^0 Y} \\
\text{ - Yield : (s)^0 Y + 1 = (s)^0 Y} \\
\text{ - Price, at time 0, of a zero-coupon bond paying} \\
\text{ - The 3 curves}
\end{align*}
\]
Example: let us consider the yield structure:

\[(s) \cdot (1 + i) \cdot (1 + i) = \frac{(s)^2 + 1}{s}\]

\[\frac{1 - (1 - s)^2 + (s)}{1 - s(s^2 + 1)} = (s) \cdot (1 + i) \cdot (1 + i)\]

\[\frac{(s)^2 + 1}{s}\]

The yield combines short-term interest rates for the respective periods \(i; i + 1; i + 2; \ldots\) for the yield:

\[(s) \cdot (1 + i) = (s)^2\]

\[\text{Price, at time } t, \text{ of a zero-coupon bond paying}\]

<table>
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<td>4</td>
<td>7.5</td>
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<tr>
<td>5</td>
<td>7.6</td>
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The curves:

Evolution of the discrete structure
Evolution of the yield structure

\[ (s)^2 R^\vartheta = (s)^2 R + 1 \]

\[ (s)^2 R + 1 \mid n = (s)^2 R \]

so that

\[ (s)^2 R_{(1-s)}^{-\vartheta} = (s)^2 d \]

\[ \text{in the continuous time,} \]

\[ (1-s) - \left( (s)^2 R + 1 \right) = (s)^2 d \]

\[ \text{in the discrete time,} \]

\[ \text{Continuous Yield} \]

\[ s \quad i \quad 0 \]

\[ [0; \infty) \]

\[ \begin{array}{c}
\text{Continuous time structure} \\

\end{array} \]

\[ \begin{array}{c}
\text{Evolution of the yield structure} \\

\end{array} \]
Short-term interest rate
Interest rate relative to the interval 

\[ (s') + \frac{t}{s} = (t) \]

Moreover, if \((s', t') = \text{lim}_{t \to \infty} (s', t')\) is a continuous function

Rewriting the discrete formula

\[ \text{Link between the different curves} \]

\[ \text{Short-term interest rate relative to the interval } [t, t'] \]
For yield curves, $R(s)$ is a r.v. depending on time $t$.

For option models, $C_t = R(s)$ is a r.v. depending on time $t$.

Stochastic modelling?

\[
np (n) \int_s^t \frac{1 - s}{t} = (s)^{2}Y(t-s) = (s)^{2}d
\]

\[
np (n) \int_s^t \frac{1 - s}{t} = (s)^{2}Y
\]

and by taking the limit for $\Delta t \to 0$, we have:

\[
\frac{s\Delta}{1- (s)^{2}Y(t-s)} \approx \frac{s\Delta}{1 - (s)^{2}Y + \frac{1}{(s\Delta + s)^{2}Y + 1}}
\]
Objective, hypotheses and general scheme

Hypotheses

a) $e_1^n\infty;e_3^n\infty$ is a stochastic process, driven by a SDE $e_1^n\infty;e_3^n\infty = e_1^n3\infty e_3^n\infty \cdot e_1^n2 + e_1^n4\infty e_1^n4\infty$.

b) $e_1^n2\infty$ and $e_1^n4\infty$ can be considered either as stochastic processes, because they are functions of $e_1^n\infty$ or as ordinary functions of $e_1^n2, e_1^n\infty$ (in general, $e_1^n2$ will generally be fixed, the important time variable being the duration $e_1^n2 - e_1^n\infty$).

Objective

(1) For different specified SDE driving the spot rate, obtaining (deterministic) functions $(p_1^n(s))$ and $(p_2^n(s))$.

Approach

Here, we will only consider the arbitrage approach. The general scheme will be

1) Evolution of the spot rate (= state variable).

2) Portfolio of 2 bonds with different maturities with proportions such that the portfolio has no risky component. Return = risk-free rate.

3) Arbitrage free reasoning.

4) PDE (= structure PDE equation) independent of the maturity.

Getting the duration $e_1^n2 - e_1^n\infty$ involved in the structure equation.

PDE: $\frac{\partial p}{\partial t} + q \cdot \frac{\partial p}{\partial q} = q^2 \Phi \cdot \frac{\partial^2 p}{\partial q^2}$

The general scheme will be:

Hypotheses
The market price of risk is considered as a function of the two variables $(\mu_L, \mu_N)$ with
\[
\mu_N = \mu_L + \eta
\]
where
\[
\eta = \text{average instant return of the spot rate}
\]
\[
\mu_L = \text{average instant volatility of the spot rate}
\]
Applying Itô's lemma to the bond price function, we have
\[
\frac{d}{dt} q + p \left( \frac{\mu_L}{\eta} + \frac{\mu_N}{\eta} + 1 \right) = \frac{\mu_N}{\eta} \frac{\mu_L}{\eta}
\]
The return of this bond is given by
\[
\text{return of this bond} = \frac{\mu_N}{\eta} \frac{\mu_L}{\eta}
\]
where \( e^{\lambda t} \) is the average instant return of the bond.

\[
(\mu - (\tilde{\sigma}) \tilde{\sigma}) \tilde{\sigma} d = (\mu - (\tilde{\nu}) \tilde{\nu}) \tilde{\nu} d X
\]

We have, for these two equations,

\[
\lambda = \nu \quad \Leftarrow \quad 0 = \mathcal{J}
\]

By arbitrage-free principle, the return of this portfolio is equal to the risk-free rate \( \lambda \).

We chose \( \mathcal{J} \) such that this portfolio has no longer random component.

The return of this portfolio is given by

\[
\lambda = \mathcal{J} + \mathcal{J} = \mathcal{J}
\]

The value of this portfolio is

\[
(\tilde{\sigma}) \tilde{\sigma} d - (\tilde{\nu}) \tilde{\nu} d X = \lambda d
\]

The average instant volatility of the bond is

\[
\frac{d}{d} \mathcal{J} = (\tilde{\sigma} \tilde{\sigma}) d
\]

The average instant return of the bond is

\[
\frac{d}{d} \mathcal{J} = (\tilde{\nu} \tilde{\nu}) d
\]

where

\[
\lambda = \mathcal{J} + \mathcal{J} = \mathcal{J}
\]
We will have to solve this PDE for different choices

\[ I = (\mu', \gamma) \]

with the limit (terminal) condition

\[ 0 = d\mu - \frac{\partial}{\partial t} \frac{\gamma}{\xi q} + \gamma (\mu^2 \gamma + \eta) \]

and we obtain the structure equation

\[
\frac{\partial}{\partial \gamma} \frac{\gamma}{\xi q} - \frac{\partial}{\partial \mu} \frac{\gamma}{\xi q} + \gamma \mu \frac{\partial}{\partial \gamma} \frac{\gamma}{\xi q} = \gamma
\]

Structure equation

\[
\frac{(\mu')^2 \rho}{\mu - (\mu')^2 \eta} = \gamma
\]

\[
\frac{\rho}{\mu - (\xi')^2 \eta} = \gamma
\]

\[
\frac{(\gamma')^2 \rho}{\mu - (\gamma')^2 \eta} = \gamma
\]

\[
\frac{(\xi')^2 \rho}{\mu - (\xi')^2 \eta} = \gamma
\]

Eliminating \( \gamma \), we obtain

\[
\frac{(\mu')^2 \rho}{\mu - (\mu')^2 \eta} = \gamma
\]

\[
\frac{\rho}{\mu - (\xi')^2 \eta} = \gamma
\]

\[
\frac{(\gamma')^2 \rho}{\mu - (\gamma')^2 \eta} = \gamma
\]

\[
\frac{(\xi')^2 \rho}{\mu - (\xi')^2 \eta} = \gamma
\]
\[ \begin{align*}
0 &= (\mu t)^2 + \sigma^2 t \\
\mu &< 0 \quad \circ \quad \sigma \text{ is the volatility} \\
\nu &< 0 \quad \circ \quad \nu \text{ is the drift} \\
\text{where} \\
\int M_p \cdot d\nu + \sigma \cdot \nu &= M_p \\
(\nu^2) \text{ is driven by an ABM}
\end{align*} \]

**Definition**

- Merton model
- Yield curve
- Average instant return of the bond
- Consequences
- Solution of the structure equation
- Arithmetic Brownian motion
Arithmetic Brownian motion

The solution of the SDE

\[
\begin{align*}
\left( (t - s) - \frac{\varepsilon}{\varepsilon_p} \right) \cdot [\cdots] \exp = & \frac{\mu}{\nu} d \\
\left( (t - s) - \frac{\varepsilon}{\varepsilon_p} \right) \cdot [\cdots] \exp = & \frac{\varepsilon}{\nu} d \\
\left( \varepsilon (t - s) \frac{\varepsilon}{\varepsilon_p} - (t - s) \nu + \lambda \right) \cdot [\cdots] \exp = & \frac{\lambda}{\nu} d
\end{align*}
\]

so that the l.h.s. is equal to 0

\[
\begin{align*}
\frac{\varepsilon}{\varepsilon_p} (t - s) + \frac{\varepsilon}{\nu} (t - s) - \frac{\mu}{\nu} (t - s) - & \\
\frac{\lambda}{\nu} d
\end{align*}
\]

is clearly 1 is clearly satisfied and

The terminal condition

\[
\begin{align*}
\left[ \varepsilon (t - s) \frac{\varepsilon}{\varepsilon_p} + \frac{\varepsilon}{\nu} (t - s) - \mu (t - s) - & \\
\lambda d \right] \exp = & \\
(\mu, \nu)^{\frac{\lambda}{\nu}} d
\end{align*}
\]

is solution is

\[
0 = d \mu - \frac{\mu}{\nu} d \frac{\varepsilon}{\varepsilon_p} + \frac{\lambda}{\nu} \exp = \frac{\lambda}{\nu} d
\]

Structure equation : Solution of the structure equation

Conclusion : ABM is not a credible process for the behavior of the spot rate. But, historical interest...

This stochastic process has potential negative values - has a variance tending to +0 - is a straight line in mean -

So, this stochastic process

\[
\mathcal{N} \sim \mathcal{N}(\mu, \sigma^2)
\]

is given by

\[
W^0 + 0 t = \mu
\]

The solution of the SDE

Arithmetic Brownian motion
Consequences

Average instant return of the bond

\[
\omega = (\alpha', s)^{\rho} \int_{0}^{\infty} \exp(-s) \, ds
\]

\( r_0 = 10\% \) \( r_0 = 6\% \) \( r_0 = 2\% \)

Note: this instant return is constant.

Yield curve

Properties

\[
\omega(\alpha - s) \frac{g}{z^D} - (\alpha - s) \frac{z}{y} + \omega = \left( \ln(\alpha) \right)^{\frac{3-s}{t}} = (\alpha', s)^{\rho} \]

Modèle de Merton (\( \alpha = 0,07 \); \( \sigma = 0,08 \))
Vasicek model

Definition

The Vasicek model is driven by an Ornstein-Uhlenbeck process, which is a stochastic process described by the following stochastic differential equation (SDE):

\[ \frac{dp}{p} + \phi (\mu - \theta) p \, dt = \sigma \, dW \]

where:
- \( \phi \) is the force of recall,
- \( \mu \) is the average value,
- \( \sigma \) is the volatility,
- \( \theta \) is the average value of the interest rate,
- \( dW \) is a Wiener process.

The solution of the SDE is given by:

\[ p(t) = e^{-\phi \theta (t-t_0)} \left( p_0 - \frac{\sigma^2}{2 \phi^2} \right) e^{\phi \theta t} + \frac{\sigma^2}{2 \phi} \left( e^{\phi \theta (t-t_0)} - 1 \right) \]

where \( p_0 \) is the initial value of the process.

Properties

- Average instantaneous return of the bond
- Yield curve
- Average term structure
- Consequences of the structure equation
- Solution of the structure equation
- Properties

Reference


Note: The structure equation is also from this reference.
The drift coefficient is such that the trend is to "recall" to the average value when it seems to diverge:

\[ \langle \dot{x}, e_1 \rangle \geq \langle \delta, e_2 \rangle \]

This behavior is much more adapted than the ABM to model an interest rate evolution.

\[ M \cdot \delta + \theta \cdot (\nu - \theta) \eta = \nu \delta \]

The solution of the (non-stochastic) DE

\[ \nu \cdot (\nu - \theta) \eta = \nu \eta \]

is given by

\[ \nu = 0 = \eta = 0 \]

such that

\[ \nu = \eta \]

The objective is to write

\[ \nu \eta + \nu = \nu \]

Solution of the SDE
\[(1-\lambda)M - 19M \cdot 1-19\partial \sum_{u}^{1-i} \int_{\mu}^{0} \Xi_{u} = n^{\partial} \int_{1}^{0} \int \partial \]

because

\[\mathcal{N} \sim \mathcal{U} (\partial)
\]

\[n^{\partial} \int_{\mu}^{0} \int \partial = (\mathcal{U}.4n) \]

\[(18-\partial - 1)^{\partial} = (\mathcal{U}.4n) \]

Properties

\[\int_{19-\partial}^{0} \int \partial + (18-\partial - 1)^{\partial} + 18-\partial^{0} \partial = \mathcal{U} \]

is then

\[\int \partial + 18 (\mathcal{U} - \theta) \partial = 18\partial \]

The solution of the SDE

Proof

\[0 = (3\partial) \mathcal{E} \]

with

\[n^{\partial} \int \partial = (\partial \mathcal{E} - 3\partial) \]

and

\[\int \partial + 18 = \mathcal{U} \]

Write noise •

\[\int \partial = \mathcal{U} \]

\[\int \partial (\mathcal{U} - \theta) \partial = 18\partial \]

\[\int \partial (\theta + (18-\partial - 1)^{\partial} - 18-\partial^{0}\lambda) = 18\partial \partial \]

\[\int \partial (18-\partial^{0}\lambda) = 18\partial \]

\[\int \partial (18-\partial^{0}\lambda) = 18\partial \]

Proof
So, this stochastic process has the following behavioral properties:

- In mean, it is the (weighted) average of the initial value $r_0$ and the average value $\bar{r}$.
- Its variance is an increasing function of time, but is bounded: $\text{var}(r) \leq \frac{\sigma^2}{2\theta}$.
- It is not incompatible with negative values of $r$, even if the recall force of the drift term is such that this case is not frequent.

Conclusion: The Ornstein-Uhlenbeck process is a credible process for the behavior of the spot rate (maybe except for the last remark).

Solution of the structure equation:

Structure equation:

$$P_t'(s, r) = \exp\left[-\frac{\delta}{2}(s - r)\right]$$

where $k = \theta + \frac{\sigma^2}{2\delta}$ and $(s - r)(1 - e^{-\delta(s-r)})$.

The terminal condition $P_s(s, r) = 1$ is clearly satisfied and its solution is

$$P_t'(s, r) = \exp\left[-(s - r)\left(1 - e^{-\delta(s-r)}\right)\right].$$

$$P_t' + (\delta(s - r) + \lambda)P_t' = \frac{\sigma^2}{2} P_{t
n}\text{ (for } t = 0\text{)} = 0.$$
Consequences

\[ \frac{g}{\partial v} + \mu = (\mu', \alpha)^2 \eta \lim_{\tau \to \infty} (\mu - \frac{\partial}{\partial \tau}) \]

\[ \mu = (\mu', \alpha)^2 \eta \]

This instant return is an increasing but bounded function such that

\[ ((\lambda - s)\alpha - \beta) \frac{g}{\partial v} + \mu = \]

\[ (\ldots) [\ldots] \exp \left( \frac{d}{\partial v} g + \mu \right) = \]

\[ (\mu d \frac{\partial}{\partial v} - d d) \frac{d}{\partial v} = (\mu', \alpha)^2 \eta \]

Average instant return of the bond

\[ 0 = \left\{ \frac{g}{\partial v} - \mu + \theta - \frac{z \partial^2}{\partial^2} + \mu - \gamma \right\} \cdot (\ldots) [\ldots] \exp = \]

\[ \left\{ (\ldots) \left( \frac{g}{\partial v} + (\mu - \theta) - \right) \right\} \cdot [\ldots] \exp = \]

\[ \left\{ \mu - z (\ldots) \frac{z \partial^2}{\partial^2} + \mu - \left( \ldots \left( \frac{g}{\partial v} + (\mu - \theta) - \right) \right) \right\} \cdot [\ldots] \exp = \]

\[ \left\{ (\ldots) (\lambda - s) \alpha - \beta \frac{z \partial^2}{\partial^2} + (\lambda - s) \alpha - \beta (\mu - \gamma) - \gamma \right\]
\[ \frac{\gamma}{\tau} + \lambda > \rho > \frac{\gamma}{\tau} - \gamma \]

is humped if \( \rho > \frac{\gamma}{\tau} \).

is increasing if \( \rho \geq \frac{\gamma}{\tau} \).

W.r.t. \( t \), shows that the yield curve

The analysis of the variations of \( \mathcal{R}^s(t) \) shows that

\[ \lambda = (\lambda', s)^2 \]

\[ \lim_{t \to \infty} (\lambda', s) = \mathcal{R}^\infty \frac{\gamma}{\tau} \]

because

\[ 0 \sim \frac{1-s}{\tau} (1-s)^{\varphi - \vartheta -1} \]

and

\[ \sim \frac{(1-s)^g}{(1-s)^{\varphi - \vartheta -1}} \]

Properties

\[ \frac{\gamma}{\tau} + \lambda \geq \frac{\gamma}{\tau} - \gamma \]

is decreasing if \( \rho \geq \frac{\gamma}{\tau} \).
Cox-Ingersoll-Ross model

Definition

The Cox-Ingersoll-Ross model is driven by a square root process. The model is defined as:

\[ dX_t = \theta (\theta - X_t) dt + \sigma \sqrt{X_t} dW_t \]

where

- \( \lambda > 0 \) is the force of recall,
- \( \theta \) is the average value,
- \( \sigma > 0 \) is the volatility.

\[ \lambda = \lambda \left( \frac{\theta}{\theta - X_t} \right) \]

where \( \lambda > 0 \).

Consequences

- Average instant return of the bond
- Yield curve
- Term structure of interest rates


\[ 385-408 \]
The "mechanical" property of the drift term is the same as the one of Ornstein-Uhlenbeck process. Negative values of \( t \) are incompatible with the square root process: if \( t \) decreases to 0, then the SDE becomes

\[
0 = \frac{d}{dt} - \mu \frac{d}{dz} \frac{dz}{\theta} + \frac{1}{2} \sigma^2 (\mu \lambda + (\mu - \theta) \lambda) + \frac{1}{4} \sigma^4
\]

Structure equation

Solution of the structure equation

Square root process
Consequences

Average instant return of the bond

\[
\frac{e^{2n^2} - e^{3n^2}}{e^{3n^2}} = \frac{e^{2n^2} - e^{3n^2}}{e^{3n^2}}
\]

Yield curve

These curves have the same kind of properties as the Vasicek ones

\[
\begin{align*}
(\mu(s) + 3) & = \\
(\mu(s) - 3) & = (\mu s)^3
\end{align*}
\]

Yield curve

\[
\begin{align*}
((s) + 1) \cdot s = \\
(\mu \cdot s - d) \cdot \frac{d}{s} = \\
\left(\mu d + \frac{s}{d} + \mu \cdot s - \theta \cdot s + \frac{d}{s}\right) \cdot \frac{d}{s} = (\mu s)^3
\end{align*}
\]

Average instant return of the bond

Consequences