Part II
FINANCIAL APPLICATIONS

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Chapter 4
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Definition

Let us consider a (discrete time) symmetrical random walk \((X_t)\)

\[
X_t = \sum_{k=1}^{n} Z_k \quad Z_k \sim \begin{pmatrix} -\Delta x & \Delta x \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\]

with
- \(t = n \cdot \Delta t\)
- independent moves
- \(X_0 = 0\)

We know that

\[
E(X_t) = 0
\]

\[
\text{var}(X_t) = \frac{(\Delta x)^2}{\Delta t} \cdot t
\]
We want to define
- a continuous time stochastic process
- with positive constant instantaneous variance
  o if $\text{var}(X_t) \to \infty$, too “explosive”: the fluctuations will grow to infinity
  o if $\text{var}(X_t) \to 0$, no more random

So, we have to
- let $\Delta t$ tend to 0
- in such a manner that $\frac{(\Delta x)^2}{\Delta t} \cdot t \to C \cdot t$

We can choose $C = 1$: if we want another constant $\sigma$, we will consider $(\sigma X_t)$

Thanks to the CLT, we have

$$X_t = \sum_{k=1}^{n} Z_k \to \mathcal{N}(0; t)$$

Furthermore, a random walk has independent and stationary increments ...

**Definition**

A continuous time stochastic process $(W_t)$ is a standard brownian motion (SBM) if
- $W_0 = 0$
- $(W_t)$ has independent increments
- $(W_t)$ has stationary increments
- $W_t \sim \mathcal{N}(0; t)$

The notation “$W$” is for Wiener

Strictly speaking, a Wiener process on a probability space $(\Omega, \mathcal{F}, \text{Pr}, \mathcal{F})$ is a SBM adapted to the filtration $\mathcal{F}$
Properties

Elementary properties

a) A SBM is a Gaussian process

b) If \( s < t \), \( (W_t - W_s) \triangleq W_{t-s} \sim \mathcal{N}(0; t - s) \)

c) We have \( E(W_t) = 0 \), \( \text{var}(W_t) = t \) and
\[
\text{cov}(W_s, W_t) = \min(s, t)
\]

Proof: if \( s < t \),
\[
\text{cov}(W_s, W_t) = \text{cov}(W_s, W_t - W_s + W_s)
= \text{cov}(W_s, W_t - W_s) + \text{cov}(W_s, W_s)
= 0 + s
\]

Quadratic variation of a SBM

Let us consider a partition \( \mathcal{P}_n \) of the time interval \([0; t] \) (\( 0 = t_0 < t_1 < \cdots < t_n = t \)) such that
\[
\delta_n = \max\{t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1}\}
\]
tends to 0 when \( n \to \infty \)

We define the quadratic variation of the SBM \( W_t \),
associated with the partition \( \mathcal{P}_n \), by
\[
Q_n(t) = \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2
\]
Property: when \( n \to \infty \), we have \( Q_n(t) \xrightarrow{q.m.} t \)

Lemma: if \( X \sim \mathcal{N}(0; \sigma^2) \), then \( \text{var}(X^2) = 2\sigma^4 \)

Since \( \mu_4 = 3\sigma^4 \), we have

\[ \text{var}(X^2) = E(X^4) - E^2(X^2) = 3\sigma^4 - (\sigma^2)^2 \]

Proof

\[ E(Q_n(t)) = \sum_{i=1}^{n} E \left( (W_{t_i} - W_{t_{i-1}})^2 \right) = \sum_{i=1}^{n} (t_i - t_{i-1}) = t \]

\[ \text{var}(Q_n(t)) = \sum_{i=1}^{n} \text{var} \left( (W_{t_i} - W_{t_{i-1}})^2 \right) \\
= 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \leq 2 \delta_n \sum_{i=1}^{n} (t_i - t_{i-1}) \\
= 2t\delta_n \to 0 \]

so that \( E((Q_n(t) - t)^2) \to 0 \)

**Regularity properties**

a) The paths of a SBM are continuous

We have to prove that \( \lim_{\Delta t \to 0} W_{t+\Delta t} = W_t \)

We give a proof for limit in probability. Let us choose an arbitrary \( \varepsilon > 0 \). We will prove that

\[ \lim_{\Delta t \to 0} \Pr[|W_{t+\Delta t} - W_t| > \varepsilon] = 0 \]

Since (Chebyshev’s inequality)

\[ \Pr[|W_{t+\Delta t} - W_t - 0| > \varepsilon \sqrt{\Delta t}] \leq \frac{1}{\varepsilon^2} \]

we have

\[ \Pr[|W_{t+\Delta t} - W_t| > \varepsilon] \leq \frac{\Delta t}{\varepsilon^2} \to 0 \]
b) The paths of a SBM are nowhere derivable

\[ W_{t+\Delta t} - W_t \sim \mathcal{N}(0; \Delta t) \triangleq \sqrt{\Delta t} \cdot Z \]

with \( Z \sim \mathcal{N}(0; 1) \)

\[ \frac{W_{t+\Delta t} - W_t}{\Delta t} \triangleq \frac{Z}{\sqrt{\Delta t}} \]

that tends to \( \pm \infty \), depending on the sign of \( Z \)

Interpretation of this property: a SBM is unpredictable over short time intervals

c) A SBM has unbounded variations. More precisely (with the same notations as for quadratic variation),

\[ V_t = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}| = +\infty \quad a.s. \]

If \( V_t \) were finite (= \( C \), say), then, for any partition \( \mathcal{P}_n \),

\[ Q_n(t) = \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 \]

\[ \leq \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}| \cdot \max_{j=1,\ldots,n} |W_{t_j} - W_{t_{j-1}}| \]

\[ \leq C \cdot \max_{j=1,\ldots,n} |W_{t_j} - W_{t_{j-1}}| \]

and the 2\textsuperscript{nd} factor tends to \( 0 \) by continuity of the paths of the SBM. This is incompatible with the property of quadratic variation: \( Q_n(t) \to t \)
d) Self-similarity of a SBM

(= scaling effect = “fractals” property)

By definition, a stochastic process is $H$-self-similar if, for any $n \geq 1$, $t_1, \ldots, t_n \in T$ and $\lambda > 0$,

$$
(X_{\lambda t_1}, \ldots, X_{\lambda t_n}) \sim (\lambda^H X_{t_1}, \ldots, \lambda^H X_{t_n})
$$

$H$ is the Hurst index of the stochastic process

Property: a SBM is $\frac{1}{2}$-self-similar:

$$
(W_{\lambda t_1}, \ldots, W_{\lambda t_n}) \sim (\sqrt{\lambda} W_{t_1}, \ldots, \sqrt{\lambda} W_{t_1})
$$

Proof (for $n = 1$):

$$
W_{\lambda t} \sim \mathcal{N}(0; \lambda t) \equiv \sqrt{\lambda} \cdot \mathcal{N}(0; t) \sim \sqrt{\lambda} \cdot W_t
$$

Interpretation: the pattern of any path of a SBM has a similar shape, independently of the length of the time interval

Simulation of a SBM

It is easy to obtain pseudo-random values for the law of $X$ from $\mathcal{U}(0; 1)$ pseudo-random values:

$$
F_X^{-1}(U) \sim X
$$

$$
\Pr[F_X^{-1}(U) \leq t] = \Pr[U \leq F_X(t)] = F_X(t)
$$
For simulating a path of a SBM, we discretize the time variable: let the time interval $[0; t]$ be partitioned in $n$ sub-intervals of length $\Delta t$:

$$t = n \cdot \Delta t$$

We know that

$$W_{\Delta t}, (W_{2\Delta t} - W_{\Delta t}), ..., (W_{n\Delta t} - W_{(n-1)\Delta t})$$

are i.i.d. r.v. $\sim \mathcal{N}(0; \Delta t)$

Algorithm:

- Generate $n$ pseudo-random values $u_1, ..., u_n$
  values of a $\mathcal{U}(0; 1)$ r.v.
- Take the reciprocal of these values to obtain pseudo-random normal values
  $$W_{j\Delta t} - W_{(j-1)\Delta t} = F_{\mathcal{N}}^{-1}(u_j ; 0, \Delta t)$$
- Cumulate these values
  $$W_{k\Delta t} = \sum_{j=1}^{k} (W_{j\Delta t} - W_{(j-1)\Delta t})$$
- Using continuity of the path, connect the points by line segments
**Associated BM**

**Arithmetic BM**

An ABM with drift $\alpha (\in \mathbb{R})$ and volatility $\sigma (> 0)$, associated to the SBM $(W_t)$, is a stochastic process $(X_t)$ defined by

$$X_t = \alpha t + \sigma W_t$$

Properties
- An ABM is a Gaussian process
- Moments :
  $$\mu_X(t) = \alpha t$$
  $$\sigma_X^2(t) = \sigma^2 t$$
  $$c_X(s, t) = \sigma^2 \min(s, t)$$

This process can be generalized for beginning at a value $x_0$ instead of 0 :

$$X_t = x_0 + \alpha t + \sigma W_t$$

**Brownian bridge**

A Brownian bridge over the time interval $[0; 1]$, associated to the SBM $(W_t)$, is a stochastic process $(X_t)$ defined by

$$X_t = W_t - tW_1$$

Properties
- A Brownian bridge is a Gaussian process
- $X_0 = X_1 = 0$
- Moments :
  $$\mu_X(t) = 0$$
  $$\sigma_X^2(t) = t(1 - t)$$
  $$c_X(s, t) = \min(s, t) - st$$

For the covariance function,

$$c_X(s, t) = \text{cov}(W_s - sW_1, W_t - tW_1)$$
$$= \min(s, t) - s \min(1, t)$$
$$= -t \min(s, 1) + st \min(1, 1)$$
$$= \min(s, t) - st$$
Brownian motion and martingales

Let us consider a probability space \((\Omega, \mathcal{F}, \Pr, \mathbf{F})\)
where \(\mathbf{F}\) is the natural filtration of a SBM \((W_t)\)

(In this section, we will suppose \(0 \leq s < t\))

Examples of martingales

a) \((W_t)\) is a martingale

\[
E(W_t | \mathcal{F}_s) = E(W_t - W_s + W_s | \mathcal{F}_s)
\]
\[
= E(W_t - W_s | \mathcal{F}_s) + E(W_s | \mathcal{F}_s)
\]
\[
= E(W_t - W_s) + E(W_s | \mathcal{F}_s)
\]
\[
= 0 + W_s
\]

b) \((W_t^2 - t)\) is a martingale

\[
E(W_t^2 - t | \mathcal{F}_s) = E(W_t^2 - W_s^2 + W_s^2 | \mathcal{F}_s) - t
\]
\[
= E(W_t^2 - W_s^2 | \mathcal{F}_s) + E(W_s^2 | \mathcal{F}_s) - t
\]
\[
= E(W_t^2 - W_s^2 | \mathcal{F}_s) + W_s^2 - t
\]

But, \(W_t^2 - W_s^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s)\)

so that

\[
E(W_t^2 - W_s^2 | \mathcal{F}_s)
\]
\[
= E((W_t - W_s)^2 | \mathcal{F}_s) + 2E(W_s(W_t - W_s) | \mathcal{F}_s)
\]
\[
= (t - s) + 2W_s E(W_t - W_s | \mathcal{F}_s)
\]
\[
= (t - s) + 2W_s E(W_s - W_s)
\]
\[
= t - s
\]

and we have

\[
E(W_t^2 - t | \mathcal{F}_s) = (t - s) + W_s^2 - t
\]
\[
= W_s^2 - s
\]
c) Counter-example: \((W_t^3)\) is not a martingale

We know that

\[
E((W_t - W_s)^3 | \mathcal{F}_s) = E((W_t - W_s)^3) = 0
\]

\[
0 = E(W_t^3 - 3W_t^2W_s + 3W_tW_s^2 - W_s^3 | \mathcal{F}_s)
= E(W_t^3 | \mathcal{F}_s) - 3W_sE(W_t^2 | \mathcal{F}_s)
+ 3W_s^2E(W_t | \mathcal{F}_s) - W_s^3
= E(W_t^3 | \mathcal{F}_s) - 3W_sE(W_t^2 - t) + t | \mathcal{F}_s)
+ 3W_s^2W_s - W_s^3
= E(W_t^3 | \mathcal{F}_s) - 3W_s[(W_s^2 - s) + t] + 2W_s^3
= E(W_t^3 | \mathcal{F}_s) - W_s^3 + 3W_s(s - t)
\]

so that

\[
E(W_t^3 | \mathcal{F}_s) = W_s^3 - 3W_s(s - t) \neq W_s^3
\]

Reciprocal (without proof)

If a stochastic process \((X_t)\) is such that \((X_t)\) and \((X_t^2 - t)\) are martingales, then \((X_t)\) is a SBM

**Exponential Brownian motion**

An EBM is a stochastic process \((X_t)\) defined by

\[
X_t = e^{\sigma W_t - \frac{\sigma^2 t}{2}}
\]

with \(\sigma > 0\)

Property: an EBM is a martingale

\[
E(e^{\sigma W_t} | \mathcal{F}_s) = E(e^{\sigma(W_t - W_s)} \cdot e^{\sigma W_s} | \mathcal{F}_s)
= e^{\sigma W_s} \cdot E(e^{\sigma(W_t - W_s)} | \mathcal{F}_s)
= e^{\sigma W_s} \cdot \frac{e^{\sigma^2(t-s)}}{2}
\]

so that

\[
E(X_t | \mathcal{F}_s) = E\left(e^{\sigma W_t - \frac{\sigma^2 t}{2}} | \mathcal{F}_s\right)
= e^{\sigma W_s} \cdot e^{-\frac{\sigma^2(s-t)}{2}}
= e^{\sigma W_s} \cdot e^{-\frac{\sigma^2 s}{2}}
= X_s
\]
Particular case: if $s = 0$, 

$$E\left(e^{\sigma W_t - \frac{\sigma^2 t}{2}}\right) = X_0 = 1$$

**Using BM as a “noise”**

Objective: express a stochastic process $(X_t)$ as the “superposition” of
- a deterministic function $f_t$
- a non predictable “noise” (= martingale)

We can use

a) a SBM as an additive random noise:

$$X_t = f_t + \sigma W_t$$

b) An EBM as a multiplicative random noise:

$$X_t = f_t \cdot e^{\sigma W_t - \frac{\sigma^2 t}{2}}$$

In both case, $E(X_t) = f_t$

**Hitting time for a SBM**

**Definition and property**

For any fixed $a > 0$, we define the hitting time $T_a$ as the first time the SBM $W_t$ hits the value $a$:

$$\min\{t \in T : W_t = a\}$$

(and $+\infty$ if $W_t \neq a \ \forall t \in T$)

Property: the hitting time is a stopping time
Reflection principle

By symmetry, knowing that $T_a \leq t$, the events $[W_t > a]$ and $[W_t < a]$ have the same probability:

$$\Pr([W_t > a] | [T_a \leq t]) = \Pr([W_t < a] | [T_a \leq t])$$

$$= \frac{1}{2}$$

Distribution of hitting time and maximum

- By total probabilities formula,

$$\Pr[W_t > a] = \Pr([W_t > a] | [T_a \leq t]) \cdot \Pr[T_a \leq t]$$

$$+ \Pr([W_t > a] | [T_a > t]) \cdot \Pr[T_a > t]$$

$$= \frac{1}{2} \Pr[T_a \leq t]$$

So,

$$F_{T_a}(t) = \Pr[T_a \leq t]$$

$$= 2 \Pr[W_t > a]$$

$$= 2 \left( 1 - \Phi \left( \frac{a}{\sqrt{t}} \right) \right)$$

$$= 2 \Phi \left( -\frac{a}{\sqrt{t}} \right)$$

- If we define $M_t = \max\{W_s : 0 \leq s \leq t\}$,

$$\Pr[M_t \geq a] = \Pr[T_a \leq t] = 2 \Phi \left( -\frac{a}{\sqrt{t}} \right)$$
Stochastic integral

- Definition
  o Motivation
  o Classical Riemann integral
  o Stieltjes-Riemann integral
  o Generalization?
  o Choice of a definition
  o Definition

- Properties
  o Conditions of existence
  o Properties

Motivation

- The definition of the integral of a function
  \( f(x) \) is concerned with small variations of the variable \( x \)

- The definition of the differential of a function
  \( df(x) = f'(x) \cdot dx \) is also concerned with small variations of the variable \( x \)

Here, we will look at the time variations “through a SBM”, which has
- unbounded variations
- non differentiable paths

The convergence being no more defined in the classical way, we have to give new definitions
The Riemann integral is defined by

$$\int_a^b f(u) \, du = \lim_{n \to +\infty} \sum_{i=1}^n f(u_i) \cdot \Delta_i$$

It can be prove that if \( f \) is sufficiently “regular” (continuous by parts e.g.), this integral
- exists
- is independent of \( \mathcal{P}_n \)
- is independent of the choice of \( u_i \) in \( ]t_{i-1}; t_i[ \)
**Stieltjes-Riemann integral**

This is the same notion as ordinary Riemann integral, but the measure along horizontal axis is no more the length of segments, but the length through another function $g$

\[
\int_{a}^{b} f(u) \, dg(u) = \lim_{n \to +\infty} \sum_{i=1}^{n} f(u_i) \cdot (g(t_i) - g(t_{i-1}))
\]

This integral has the same properties as the ordinary Riemann integral (with, furthermore, regularity conditions for $g$)

**Example**

\[
\int_{-\infty}^{+\infty} u \, dF_X(u) = \lim_{n \to +\infty} \delta_{n \to 0} \sum_{i=1}^{n} u_i \cdot \Pr[t_{i-1} < X \leq t_i] = E(X)
\]

Note: from now on, the interval of integration becomes $[0; T]$ instead of $[a; b]$
**Generalization ?**

Let \((X_t)\) be a stochastic process and \((W_t)\) a SBM. How can we define \(\int_0^T X_u \, dW_u\) ?

**Problems**

a) Convergence “point by point” is the convergence a.s. (incompatible with the unbounded variation of the SBM)

\[\Rightarrow \text{ Solution: give a definition with another convergence mode (q.m.)}\]

b) The definition is no more independent of the choice of \(u_i\) in \([t_{i-1}, t_i]\)

\[\Rightarrow \text{ Solution: make a choice for } u_i\]

Let us examine the particular case of

\[\int_0^T W_u \, dW_u = \lim_{n \to +\infty} \sum_{i=1}^n W_{u_i} \cdot (W_{t_i} - W_{t_{i-1}})\]

We will need the following lemma

\[
\begin{align*}
a(b - a) &= \frac{1}{2} [(b^2 - a^2) - (b - a)^2] \\
b(b - a) &= \frac{1}{2} [(b^2 - a^2) + (b - a)^2]
\end{align*}
\]

- First choice: \(u_i = t_{i-1}\)

\[\int_0^T W_u \, dW_u = \lim_{n \to +\infty} \sum_{i=1}^n W_{t_i} \cdot (W_{t_i} - W_{t_{i-1}})\]

\[= \lim_{\delta_n \to 0} \frac{1}{n} \sum_{i=1}^n \left\{ (W_{t_i}^2 - W_{t_{i-1}}^2) - (W_{t_i} - W_{t_{i-1}})^2 \right\}\]

\[= \frac{1}{2} \lim_{\delta_n \to 0} \left( W_{T_n}^2 - Q_n(T) \right)\]

\[= \frac{1}{2} (W_T^2 - T)\]

(this last convergence is in q.m.)
Second choice : $u_i = t_i$

$$\int_0^T W_u \, dW_u = \lim_{n \to +\infty} \sum_{i=1}^n W_{t_i} \cdot (W_{t_i} - W_{t_{i-1}})$$

$$= \frac{1}{2} \lim_{n \to +\infty} \sum_{i=1}^n \left\{ \left(W_{t_i}^2 - W_{t_{i-1}}^2\right) + \left(W_{t_i} - W_{t_{i-1}}\right)^2 \right\}$$

$$= \frac{1}{2} \lim_{n \to +\infty} \left(W_T + Q_n(T)\right)$$

$$= \frac{1}{2} (W_T^2 + T)$$

Third choice : $u_i = \frac{t_{i-1} + t_i}{2}$

It can be shown that

$$\int_0^T W_u \, dW_u = \frac{1}{2} W_T^2$$

Choice of a definition

- Stratonovich integral give the same result as in the deterministic case : if $f(0) = 0$, by integrating by parts,

  $$\int_0^T f(u) \, df(u) = \frac{1}{2} f^2(T)$$

- Itô integral has two interesting properties
  a) Non-anticipativity : for the $i$-th interval $]t_{i-1}; t_i[\}$, the integrand $X_t$ is known at time $t_{i-1}$
  b) We know that the stochastic process $(W_t^2 - t)$ is a martingale ; so is the Itô integral

  $\Rightarrow$ Itô integral is chosen for applications in finance

Note
- First choice : Itô integral
- Third choice : Stratonovich integral
Definition

Let \((X_t)\) be a stochastic process adapted to the natural filtration of the SBM \((W_t)\). We define

\[
I_T = \int_0^T X_u \, dW_u = \lim_{\delta_n \to 0} \frac{1}{\delta_n} \sum_{i=1}^{\delta_n^{-1}} X_{t_{i-1}} \cdot (W_{t_i} - W_{t_{i-1}})
\]

where

\[
I_T^{(n)} = \sum_{i=1}^{n} X_{t_{i-1}} \cdot (W_{t_i} - W_{t_{i-1}})
\]

More precisely, it can be prove that there exists a r.v. \(I_T\) such that

\[
\lim_{n \to +\infty} \lim_{\delta_n \to 0} \mathbb{E} \left[ \left( I_T^{(n)} - I_T \right)^2 \right] = 0
\]

so that \(I_T^{(n)}\) converges in q.m. to \(I_T\).

Note: the hypothesis implies that \(X_{t_{i-1}}\) is independent of \((W_{t_i} - W_{t_{i-1}})\)

Properties

Condition of existence

If \((X_t)\) is a stochastic process adapted to the natural filtration of the SBM \((W_t)\), then

\[
\int_0^T X_u \, dW_u
\]

exists if

- paths of \((X_t)\) are continuous
- \(\mathbb{E} \left( \int_0^T X_u \, du \right)\) is finite
Properties

a) \( \int_0^T (\lambda_1 X_u^{(1)} + \lambda_2 X_u^{(2)}) \, dW_u \)
\[= \lambda_1 \int_0^T X_u^{(1)} \, dW_u + \lambda_2 \int_0^T X_u^{(2)} \, dW_u \]

b) \( E \left( \int_0^T X_u \, dW_u \right) = 0 \)

Proof:
\[E \left( X_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \right) = E(X_{t_{i-1}}) \cdot E(W_{t_i} - W_{t_{i-1}}) = 0 \]

c) \( \text{var} \left( \int_0^T X_u \, dW_u \right) = \int_0^T E(X_u^2) \, du \)

Proof:
\[\text{var} \left( \int_0^T X_u \, dW_u \right) = E \left[ \left( \int_0^T X_u \, dW_u \right)^2 \right] \]

\[= \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{E \left( X_{t_{i-1}}^2 \left( W_{t_i} - W_{t_{i-1}} \right)^2 \right)}{\delta_n} \]
\[+ 2 \lim_{n \to +\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} E \left( X_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right) \cdot X_{t_{j-1}} \left( W_{t_j} - W_{t_{j-1}} \right) \right) \]

But
\[E \left( X_{t_{i-1}}^2 \left( W_{t_i} - W_{t_{i-1}} \right)^2 \right) = E(X_{t_{i-1}}^2) \cdot E \left( \left( W_{t_i} - W_{t_{i-1}} \right)^2 \right) = E(X_{t_{i-1}}^2) \cdot (t_i - t_{i-1}) \]
and the first term is equal to \( \int_0^T E(X_u^2) \, du \)

Furthermore, for \( i < j, \)
\[E \left( X_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right) \cdot X_{t_{j-1}} \left( W_{t_j} - W_{t_{j-1}} \right) \right) = E \left( X_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right) X_{t_{j-1}} \left( W_{t_j} - W_{t_{j-1}} \right) \right) \]
\[= E \left( X_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right) X_{t_{j-1}} \right) \cdot E \left( W_{t_j} - W_{t_{j-1}} \right) \]
\[= 0 \]
d) The stochastic process \((I_t)\) for \(t \in [0; T]\) is a martingale.

For \(s < t\),

\[
E(I_t | F_s) = \lim_{n \to +\infty} \sum_{i=1}^{n} E(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | F_s)
\]

- If \(s, t \in ]t_{k-1}; t_k]\)

\[
E(I_t | F_s) = I_s + E \left( X_{t_{k-1}}(W_t - W_s) | F_s \right)
\]

\[
= I_s + X_{t_{k-1}} \cdot E(W_t - W_s | F_s)
\]

\[
= I_s + X_{t_{k-1}} \cdot E(W_t - W_s)
\]

\[
= I_s
\]

- If \(s \in ]t_{j-1}; t_j]\) and \(t \in ]t_{k-1}; t_k]\) with \(j < k\)

\[
E(I_t | F_S) = I_s + E \left( X_{t_{j-1}}(W_t - W_s) | F_s \right)
\]

\[
+ \sum_{i=j+1}^{k-1} E \left( X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | F_s \right)
\]

\[
+ E \left( X_{t_{k-1}}(W_t - W_{t_{k-1}}) | F_s \right)
\]

\[
= I_s + (a) + (b) + (c)
\]

\[
(a) = X_{t_{j-1}} \cdot E(W_{t_j} - W_s | F_s)
\]

\[
= X_{t_{j-1}} \cdot E(W_{t_j} - W_s)
\]

\[
= 0
\]

\[
(b) : E \left( X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) | F_s \right)
\]

\[
= E \left( X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \right)
\]

\[
= E \left( X_{t_{i-1}} \right) \cdot E(W_{t_i} - W_{t_{i-1}})
\]

\[
= 0
\]

\[
(c) = 0 : \text{ same reasoning as } (b)
\]

e) The stochastic process \((I_t)\) has continuous paths (without proof)
**Stochastic differential**

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| \[ dX(t) = f(t) \cdot dt \]
| \[ \Leftrightarrow X(t) = X(0) + \int_0^t f(u)du \] |

**Properties**

- Formal multiplication rules
- Properties

**Examples**

- Simple examples
- Arithmetic Brownian motion
- Geometric Brownian motion

**Use of the stochastic differential**

- Evolution of financial variables
- Classical stochastic differentials in finance

**In the stochastic case**

If the stochastic processes \( (a_t) \) and \( (b_t) \) are integrables and adapted to the natural filtration of the SBM \( (W_t) \), we define

\[ dX_t = a_t \cdot dt + b_t \cdot dW_t \]

by

\[ X_t = X_0 + \int_0^t a_u \, du + \int_0^t b_u \, dW_u \]
**Properties**

**Formal multiplication rules**

We will neglect terms smaller than $dt = o(dt)$

- $(dt)^2 \approx 0$

- $dt \times dW_t \approx 0$

$$E(dt \cdot dW_t) = dt \cdot E(dW_t) = 0$$
$$\text{var}(dt \cdot dW_t) = (dt)^2 \cdot \text{var}(dW_t) = (dt)^3$$

- $(dW_t)^2 \approx dt$

$$E((dW_t)^2) = \text{var}(dW_t) = dt$$
$$\text{var}((dW_t)^2) = 2(\text{var}(dW_t))^2 = 2(dt)^2$$

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**Properties**

a) Linearity: if $(X_t^{(1)})$ and $(X_t^{(2)})$ are defined w.r.t. the same SBM $(W_t)$,

$$d \left( \lambda_1 X_t^{(1)} + \lambda_2 X_t^{(2)} \right) = \lambda_1 dX_t^{(1)} + \lambda_2 dX_t^{(2)}$$

b) Product: if

$$dX_t^{(k)} = a_t^{(k)} \cdot dt + b_t^{(k)} \cdot dW_t \quad (k = 1, 2)$$

then

$$d \left( X_t^{(1)} X_t^{(2)} \right) = X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + b_t^{(1)} b_t^{(2)} dt$$
Proof

Taylor formula for \( n \) variables \( x = (x_1, \ldots, x_n) \)

\[
df(x) \approx \sum_{i=1}^{n} f'_{x_i} \, dx_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f''_{x_i x_j} \, dx_i \, dx_j
\]

applied to \( f(x_1, x_2) = x_1 x_2 \) give

\[
d \left( X_t^{(1)} X_t^{(2)} \right) = X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + \frac{1}{2} \cdot 2 \left( dX_t^{(1)} \cdot dX_t^{(2)} \right)
\]

and

\[
dX_t^{(1)} \cdot dX_t^{(2)} = \left( a_t^{(1)} \, dt + b_t^{(1)} \, dW_t \right) \left( a_t^{(2)} \, dt + b_t^{(2)} \, dW_t \right) = b_t^{(1)} b_t^{(2)} (dW_t)^2
\]

\[ c) \text{ Compound function (= Itô’s lemma)} \]

If \( dX_t = a_t \cdot dt + b_t \cdot dW_t \) and if \( f(t, x) \) is a deterministic function, derivable (one time w.r.t. \( t \) and twice w.r.t. \( x \)), then

\[
df(t, X_t) = \left( f'_t(t, X_t) + a_t f'_x(t, X_t) + \frac{b_t^2}{2} f''_{xx}(t, X_t) \right) \cdot dt + b_t f'_x(t, X_t) \cdot dW_t
\]

Proof : by Taylor,

\[
df(t, X_t) = f'_t \, dt + f'_x \, dX_t + \frac{1}{2} \left[ f''_{tt} \, (dt)^2 + 2 f''_{tx} \, (dt)(dX_t) + f''_{xx} \, (dX_t)^2 \right] = f'_t \, dt + f'_x \, dX_t + \frac{1}{2} f''_{xx} \, (dX_t)^2
\]

and

\[
(dX_t)^2 = (a_t \cdot dt + b_t \cdot dW_t)^2 = b_t^2 \, dt
\]
Examples

Simple examples

a) $f(t, x) = g(t)x$ and $X_t = W_t$

$$d(g(t)W_t) = g'(t)W_t \, dt + g(t) \, dW_t$$

$$\int_0^T d(g(t)W_t) = g(T)W_T = \int_0^T g'(t)W_t \, dt + \int_0^T g(t) \, dW_t$$

$$\int_0^T g(t) \, dW_t = g(T)W_T - \int_0^T g'(t)W_t \, dt$$

(= integration by parts)

b) $f(t, x) = x^2$ and $X_t = W_t$

$$d(W_t^2) = \frac{1}{2} 2 \, dt + 2W_t \, dW_t$$

$$\int_0^T d(W_t^2) = W_T^2 = \int_0^T dt + 2 \int_0^T W_t \, dW_t$$

$$\int_0^T W_t \, dW_t = \frac{1}{2}(W_T^2 - T)$$

c) $f(t, x) = e^x$ and $dX_t = a_t \, dt + b_t \, dW_t$

$$d(e^{X_t}) = (a_t e^{X_t} + \frac{b_t^2}{2} e^{X_t}) \, dt + b_t e^{X_t} \, dW_t$$

$$= e^{X_t} \left[ (a_t + \frac{b_t^2}{2}) \, dt + b_t \, dW_t \right]$$

$$= e^{X_t} \left[ dX_t + \frac{b_t^2}{2} \, dt \right]$$
Arithmetic Brownian motion

Definition: $X_t = X_0 + \alpha t + \sigma W_t$

$$dX_t = \alpha \, dt + \sigma \, dW_t$$

Geometric Brownian motion

Definition: $S_t = S_0 \, e^{\mu t + \sigma W_t}$

$$f(t,x) = S_0 \, e^{\mu t + \sigma x} \text{ and } X_t = W_t$$

$$dS_t = left(\mu S_t + \frac{\sigma^2}{2} S_t right) dt + \sigma S_t \, dW_t$$

$$= \delta S_t \, dt + \sigma S_t \, dW_t$$

with $\delta = \mu + \frac{\sigma^2}{2}$

So, the GBM can be written

$$S_t = S_0 \, e^{\left(\delta - \frac{\sigma^2}{2}\right) t + \sigma W_t}$$

Moments: $e^{\left(\delta - \frac{\sigma^2}{2}\right) t + \sigma W_t}$ being a log-normal r.v.,

$$E(S_t) = S_0 \, e^{\left(\delta - \frac{\sigma^2}{2}\right) t + \frac{\sigma^2 t}{2}} = S_0 \, e^{\delta t}$$

$$\text{var}(S_t) = S_0^2 \, e^{2\left(\delta - \frac{\sigma^2}{2}\right) t + \sigma^2 t} \left(e^{\sigma^2 t} - 1\right)$$

$$= S_0^2 \, e^{2\delta t} \left(e^{\sigma^2 t} - 1\right)$$
**Use of the stochastic differential**

**Evolution of a financial variable**

\[ dX_t = a_t \, dt + b_t \, dW_t \]

is an equation that describe the evolution of a financial variable

- For an equity, we have solved the equation:
  GBM
- For an option, we will solve it
- For a yield curve, the evolution of a state variable \( r_t \) will be describe by a stochastic differential and we will deduce \( R_t(s) \)

However, we will not study the techniques for solving a general SDE

**Classical stochastic differentials in finance**

For an Itô stochastic differential, the stochastic processes \( (a_t) \) and \( (b_t) \) are deterministic functions of \( t \) and \( X_t \)

Here, these functions do not depend explicitly on the time variable \( t \)

\[ a_t = a(X_t) \quad b_t = b(X_t) \]

- Arithmetic Brownian motion
  \[ dX_t = \alpha \, dt + \sigma \, dW_t \]
- Geometric Brownian motion
  \[ dX_t = \delta X_t \, dt + \sigma X_t \, dW_t \]
- Ornstein-Uhlenbeck process
  \[ dX_t = \delta (\theta - X_t) \, dt + \sigma \sqrt{X_t} \, dW_t \]
- Square-root process
  \[ dX_t = \delta (\theta - X_t) \, dt + \sigma \sqrt{X_t} \, dW_t \]
Change of probability measure

- Radon-Nikodym theorem
  - Discrete case
  - General case
- Girsanov theorem
  - Girsanov theorem
  - Generalization

Radon-Nikodym theorem

Discrete case

Let \( \Omega = \{\omega_1, \omega_2, ..., \omega_n, ... \} \) be the set of possible outcomes in a random situation with probability measure \( \Pr \) :

\[
\Pr(\{\omega_i\}) = p_i \quad (\sum p_i = 1)
\]

Let \( Q \) be another probability measure for this random situation :

\[
Q(\{\omega_i\}) = q_i \quad (\sum q_i = 1)
\]

The r.v. \( L \) is defined by

\[
L(\omega_i) = \frac{q_i}{p_i}
\]
This r.v. has the following properties
- $L$ positive
- $E_p(L) = \sum_{i} \frac{q_i}{p_i} p_i = 1$
- For any r.v. $X$,

\[ E_q(X) = \sum X(\omega_i)q_i = \sum X(\omega_i) \frac{q_i}{p_i} p_i = E_p(L \cdot X) \]

and, in the particular case where $X = 1_A$,

\[ Q(A) = E_p(L \cdot 1_A) \]

**General case**

Let $Pr$ and $Q$ be two probability measures on $(\Omega, \mathcal{F})$

We say that $Q$ is absolutely continuous w.r.t. $Pr$ $(Q \ll Pr)$ if

\[ \forall A \in \mathcal{F}, \quad Q(A) = 0 \quad \Rightarrow \quad Pr(A) = 0 \]

If $Q \ll Pr$ and $Pr \ll Q$, the two measures are said equivalent
Radon-Nikodym theorem

\( Q \) is absolutely continuous w.r.t. \( \Pr \) if and only if there exist a positive r.v. \( L \) such that

\[
\forall A \in \mathcal{F}, \quad Q(A) = \int_A L(\omega) \, d\Pr(\omega)
\]

or, equivalently,

\[
Q(A) = E_Q(1_A) = E_{\Pr}(L \cdot 1_A)
\]

\( L \) is named Radon-Nikodym derivative and one writes

\[
L = \frac{dQ}{d\Pr}
\]

Property: by putting \( A = \Omega \), we have

\[
1 = Q(\Omega) = \int_{\Omega} L(\omega) \, d\Pr(\omega) = E_{\Pr}(L)
\]

---

Girsanov theorem

The definition of a SBM depends heavily on the probability measure: independent and stationary increments, normal distribution, ...

Let us consider a SBM \((W_t)\) on \((\Omega, \mathcal{F}, \Pr)\) for the time interval \([0; T]\).

The stochastic process \((\widetilde{W}_t)\), defined by \(\widetilde{W}_t = W_t + qt\), is an ABM, but no more a SBM:

\[
E(\widetilde{W}_t) = qt \neq 0
\]

The EBM \(L_t = e^{-qw_t - \frac{q^2t}{2}}\) is a positive stochastic process, martingale, with \(E_{\Pr}(L_t) = 1\). We will use it as a Radon-Nikodym derivative
Girsanov theorem

- The function

\[ Q(A) = \int_A L_T(\omega) \, d\Pr(\omega) \quad (A \in \mathcal{F}) \]

is a probability measure

- The \( Q \) measure is equivalent to the \( \Pr \) measure

- Under \( Q \), \((\tilde{W}_t)\) is a SBM, adapted to the natural filtration of \((W_t)\)

The \( Q \) measure is the equivalent martingale measure

---

**Generalization**

Let \((W_t)\) be a SBM on \((\Omega, \mathcal{F}, \Pr)\) for the time interval \([0; T]\) and \((\tilde{W}_t)\) the associated ABM with drift \(\mu\) and volatility \(\sigma\):

\[ \tilde{W}_t = \mu t + \sigma W_t \]

Then, \((\tilde{W}_t)\) is an ABM with drift \(\nu\) and volatility \(\sigma\) under the probability measure

\[ Q(A) = \int_A L_T(\omega) \, d\Pr(\omega) \quad (A \in \mathcal{F}) \]

where

\[ L_t = e^{\frac{\nu-\mu}{\sigma^2} \tilde{W}_t - \frac{(\nu^2-\mu^2)}{2\sigma^2} t} \]