Chapter 3

Stochastic processes

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Definitions

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Stochastic process

A stochastic process is a family of r.v., indexed by time

$$(X_t) : \ \Omega \times T \ : \ (\omega, t) \longmapsto X_t(\omega)$$

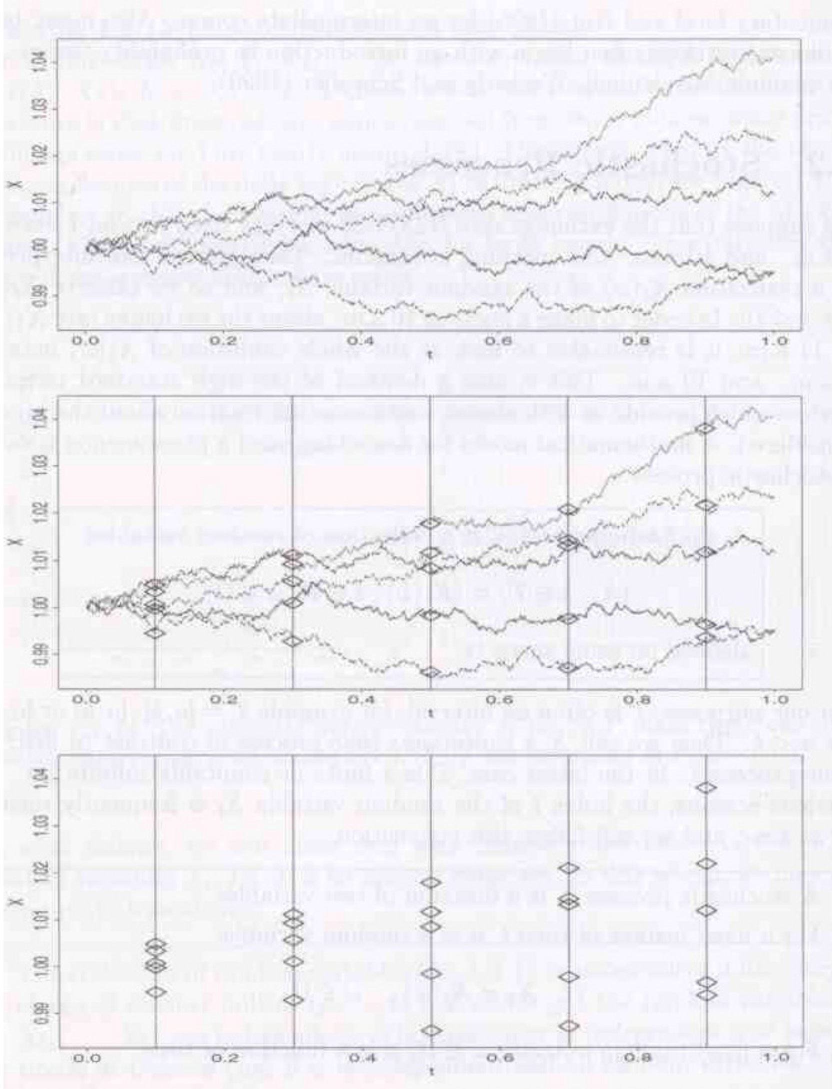
If the time set T is

- discrete, (X_t) is a discrete time stochastic process (= sequence of r.v.)
- continuous, (X_t) is a continuous time stochastic process

Let (X_t) be a stochastic process

- for any fixed $t \in T$, X_t is a r.v.
- for any fixed $\omega \in \Omega$, $X_t(\omega)$ is a path (trajectory) of the stochastic process

If $T = \mathbb{N}$ or $\mathbb{Z}$, the stochastic process (X_t) is a time series



Distribution of a stochastic process

Probability distribution

The probability distribution of a r.v. is given by $\Pr[X \in E]$ for any Borel set E

The finite-dimensional distribution of a stochastic process (X_t) is the probability distribution of

$$(X_{t_1}, \dots, X_{t_n})$$

for any $n \geq 1$, $t_1, \dots, t_n \in T$

Moments

a) Expectation : $\mu_X(t) = E(X_t) \qquad \forall t \in T$
b) Variance : $\sigma_X^2(t) = var(X_t) \qquad \forall t \in T$
c) Covariance : $c_X(s,t) = cov(X_s, X_t)$

Property : $c_X(t,t) = \sigma_X^2(t)$

Possible properties of a stochastic process

Indicate dependence structure

Strict stationarity

The stochastic process (X_t) is strictly stationary if, $\forall n \geq 1$, $t_1, \dots, t_n \in T$, h [such that $t_1 + h, \dots, t_n + h \in T$], then $(X_{t_1}, \dots, X_{t_n})$ and $(X_{t_1+h}, \dots, X_{t_n+h})$ have the same probability distribution :

$$(X_{t_1}, \dots, X_{t_n}) \triangleq (X_{t_1+h}, \dots, X_{t_n+h})$$

Property : if the stochastic process (X_t) is strictly stationary, then

$$\mu_X(t) = \mu_X(0) \qquad \forall t$$
$$\sigma_X^2(t) = \sigma_X^2(0) \qquad \forall t$$
$$c_X(s,t) = c_X(|t-s|) \qquad \forall s,t$$

Wide stationarity

(or "2nd order stationarity")

The stochastic process (X_t) is stationary in the wide sense if

$$\mu_X(t) = \mu_X(0) \qquad \forall t$$
$$c_X(s,t) = c_X(|t-s|) \qquad \forall s,t$$

Property : if the stochastic process (X_t) is stationary in the wide sense, then

$$\sigma_X^2(t) = \sigma_X^2(0) \qquad \forall t$$

Stationary increments

The stochastic process (X_t) has stationary increments (or is homogeneous) if

$$X_t - X_s \triangleq X_{t+h} - X_{s+h} \qquad \forall s,t,h$$

Independent increments

The stochastic process (X_t) has independent increments if, $\forall n \geq 1,\ t_1 < t_2 < \ldots < t_n \in T$, the r.v.

$$\left(X_{t_2} - X_{t_1}\right), \left(X_{t_3} - X_{t_2}\right), \ldots, \left(X_{t_n} - X_{t_{n-1}}\right)$$

are independent

Filtration

Definition

Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space

A filtration is an increasing family of sub-σ-fields of $\mathcal{F}$:

$$\begin{cases} \mathbf{F} = \{\mathcal{F}_t : t \in T\} \\ s < t \Longrightarrow \mathcal{F}_s \subset \mathcal{F}_t \ (\subset \mathcal{F}) \end{cases}$$

Interpretation : $\mathcal{F}_t$ is the collection of events representing the available information up to t (= the history up to t)

Note : generally, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and, in this case,

$$E(\cdot \,|\mathcal{F}_0) = E(\cdot)$$

We can now generalize the probability space when a time variable is present : $(\Omega, \mathcal{F}, \Pr, \mathbf{F})$

Filtration and stochastic process

a) Natural filtration of a stochastic process :

$$\mathcal{F}_t = \sigma(\{X_s : s \leq t\}) \qquad \forall t$$

= the history of the stochastic process

b) A stochastic process (X_t) is adapted to a filtration $\mathbf{F}$ if

$$\forall t, \quad X_t \text{ is } \mathcal{F}_t\text{-measurable}$$

c) Conditional expectation w.r.t. a filtration : if X is a r.v., let define $M_t = E(X|\mathcal{F}_t)$

Property : $\{M_t \ : \ t \in T\}$ is a stochastic process, adapted to $\mathbf{F}$

Interpretation : M_t represents the mean estimation of X, taking into account the available information up to t

Example : tossing a coin 3 times

A (symmetric) coin is tossed 3 times, independently

Let us define X_t the number of "tail(s)" up to the t-th toss $(t = 0, 1, 2, 3)$

(a) We can, for each possible outcome, associate the value of these r.v. :

Ω	TTT	TTH	THT	HTT	THH	HTH	HHT	HHH
X_0	0	0	0	0	0	0	0	0
X_1	1	1	1	0	1	0	0	0
X_2	2	2	1	1	1	1	0	0
X_3	3	2	2	2	1	1	1	0

(b) Natural filtration of the stochastic process $\{X_t : t = 0, 1, 2, 3\}$?

$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \{\emptyset, \Omega, A_T, A_H\} = \sigma(\{A_T, A_H\})$

with $A_T = \{TTT, TTH, THT, THH\}$

$A_H = \{HTT, HTH, HHT, HHH\}$

$\mathcal{F}_2 = \sigma(\{\emptyset, \Omega, A_{TT}, A_{TH}, A_{HT}, A_{HH}\})$

with $A_{TT} = \{TTT, TTH\}$

$A_{TH} = \{THT, THH\}$

$A_{HT} = \{HTT, HTH\}$

$A_{HH} = \{HHT, HHH\}$

$\mathcal{F}_3 = \mathcal{F} = \mathcal{P}(\Omega)$

(c) Conditional expectations of X_3 = number of "tail(s)" after 3 tosses ?

- w.r.t. $\mathcal{F}_0$: for any $\omega \in \Omega$,

$$\omega \longmapsto 3 \cdot \frac{1}{8} + 2 \cdot \frac{1}{8} + \cdots + 0 \cdot \frac{1}{8} = \frac{3}{2}$$

Note : $E(X_3|\mathcal{F}_0) = E(X_3)$

X_3 is independent of $\mathcal{F}_0$ (R3)

- w.r.t. $\mathcal{F}_1$

$$\omega \in A_T \longmapsto 3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 2$$
$$\omega \in A_F \longmapsto 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 1$$

- w.r.t. $\mathcal{F}_2$

$$\omega \in A_{TT} \longmapsto 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{5}{2}$$
$$\omega \in A_{TF} \longmapsto 2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{2}$$
$$\omega \in A_{FT} \longmapsto 2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{2}$$
$$\omega \in A_{FF} \longmapsto 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$$

Note 1 : $E[E(X_3|\mathcal{F}_2)|\mathcal{F}_1] = E(X_3|\mathcal{F}_1)$ (R6)

Note 2 : $E\big(E(X_3|\mathcal{F}_2)\big)$

$$= \frac{5}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{2}$$
$$= E(X_3)$$

(R2)

- w.r.t. $\mathcal{F}_3$

$$\omega = TTT \longmapsto 3$$
$$\omega = TTF \longmapsto 2$$
$$\ldots$$
$$\omega = FFF \longmapsto 0$$

Note : $E(X_3|\mathcal{F}_3) = X_3$ (R4)

Stopping time

Definition

A stopping time on $(\Omega, \mathcal{F}, \Pr, \mathbf{F})$ is a r.v. τ with values in $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$ such that

$$\forall t \in T, \qquad [\tau \leq t] \in \mathcal{F}_t$$

Interpretation : a rule which tells when to stop, based only on the knowing of the history up to the instant of stopping

Property : if τ_1 and τ_2 are two stopping times, then $\min(\tau_1, \tau_2)$ and $\max(\tau_1, \tau_2)$ are also stopping times

Example : for the short-term interest rate $r(t)$,

- the first time $r(t)$ is equal to 3 % is a stopping time
- the last time $r(t)$ is equal to 3 % is not a stopping time

A particular stopping time : the barrier

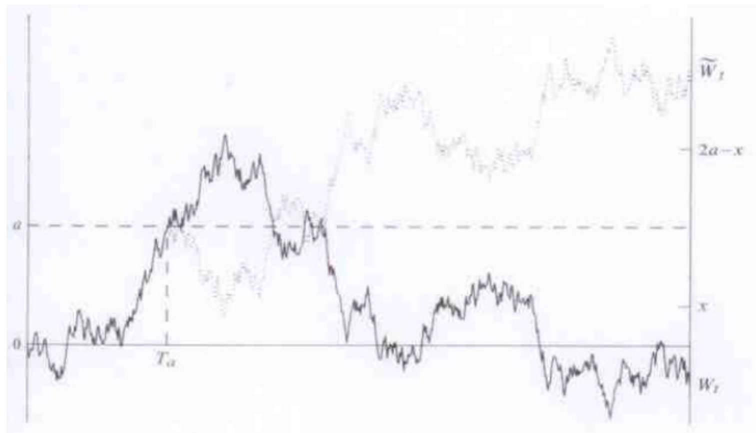
Let (X_t) be a stochastic process and $\mathbf{F}$ the natural filtration of (X_t)

$$\tau = \inf\{t \in T \ : \ X_t \geq B\}$$

is a stopping time for the barrier B

It is possible to construct the associate stochastic process (Y_t) "stopped at B" :

$$Y_t = \begin{cases} X_t & if \quad t \leq \tau \\ B & if \quad t > \tau \end{cases}$$



Example : tossing a coin 3 times

a) We stop tossing the coin when the first tail appears :

$$\tau = \min\{t : X_t = 1\}$$

Ω	TTT	TTH	THT	HTT	THH	HTH	HHT	HHH
X_0	0	0	0	0	0	0	0	0
X_1	1	1	1	0	1	0	0	0
X_2	2	2	1	1	1	1	0	0
X_3	3	2	2	2	1	1	1	0
τ	1	1	1	2	1	2	3	∞

$[\tau = 1] = \{TTT, TTH, THT, THH\} = A_T$
is $\mathcal{F}_1$-measurable

$[\tau = 2] = \{HTT, HTH\} = A_{HT}$
is $\mathcal{F}_2$-measurable

$[\tau = 3] = \{HHT\}$ is $\mathcal{F}_3$-measurable

So, τ is a stopping time

Note : $[\tau = \infty] = \{HHH\}$

b) We stop tossing the coin at the last time tail appears :

$$\tau' = \min\{t : X_t = X_3\}$$

Ω	TTT	TTH	THT	HTT	THH	HTH	HHT	HHH
X_0	0	0	0	0	0	0	0	0
X_1	1	1	1	0	1	0	0	0
X_2	2	2	1	1	1	1	0	0
X_3	3	2	2	2	1	1	1	0
τ'	3	2	3	3	1	2	3	∞

$[\tau' = 1] = \{THH\}$ is not $\mathcal{F}_1$-measurable

$[\tau' = 2] = \{TTH, HTH\}$ is not $\mathcal{F}_2$-measurable

So, τ' is not a stopping time

A discrete time stochastic process : random walk

- General case
  - Definition
  - Properties
- Special case : symetrical probabilities

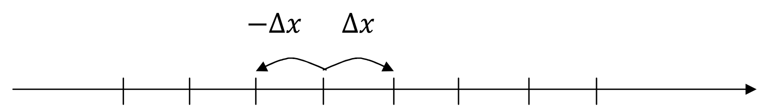
General case

Definition

Description : a particle (or an asset return, or a drunkar, …)

- starts at time $t = 0$ at the point 0
- at each step, it moves by Δx with probability p and by $-\Delta x$ with probability $q = 1 - p$
- the duration of the steps is Δt
- the different moves are independent

In this model, $T = (\Delta t) \cdot \mathbb{N} = \{0, \Delta t, 2\Delta t, \dots\}$



X_t is the position of the particle at time t

$$X_0 = 0 \qquad X_{\Delta t} \sim \begin{pmatrix} -\Delta x & \Delta x \\ q & p \end{pmatrix}$$

$$X_{2\Delta t} \sim \begin{pmatrix} -2\Delta x & 0 & 2\Delta x \\ q^2 & 2pq & p^2 \end{pmatrix} \qquad \dots$$

Properties

Let us examine the situation after n moves and denote $t = n \cdot \Delta t$

$$X_t = \sum_{k=1}^{n} Z_k \qquad Z_k \sim \begin{pmatrix} -\Delta x & \Delta x \\ q & p \end{pmatrix}$$

We have

$$E(Z_k) = \Delta x(p - q)$$

$$\begin{aligned} var(Z_k) &= (\Delta x)^2(q + p) - (\Delta x)^2(p - q)^2 \\ &= (\Delta x)^2(1 - (p - q)^2) \\ &= (\Delta x)^2 4pq \end{aligned}$$

So,

$$E(X_t) = n \cdot E(Z_k) = (p - q)\frac{\Delta x}{\Delta t} \cdot t$$

$$var(X_t) = n \cdot var(Z_k) = 4pq\frac{(\Delta x)^2}{\Delta t} \cdot t$$

Special case : symmetrical probabilities

If $p = q = \frac{1}{2}$, we have

$$E(X_t) = 0$$

$$var(X_t) = \frac{(\Delta x)^2}{\Delta t} \cdot t$$

The evolution of the particle is such that the position

- is null in mean
- with a variance proportional to time

Martingales

- Definition
- Properties
  - Expectation of a martingale
  - Interpretation as a fair game

Definition

Let us consider a probability space $(\Omega, \mathcal{F}, \Pr, \mathbf{F})$

A stochastic process (X_t) is a martingale if

- $E(|X_t|) < \infty \quad \forall t$
- (X_t) is adapted to $\mathbf{F}$
- $E(X_t|\mathcal{F}_s) = X_s \quad \forall s, t \ (s < t)$

Note 1 : A stochastic process may be a martingale w.r.t. a filtration $\mathbf{F}$, but not w.r.t. to another one $\mathbf{G}$. One use sometimes the notation $(X_t, \mathbf{F})$

Note 2 : The essential defining property $E(X_t|\mathcal{F}_s) = X_s$ means "the best prediction of (X_t) when we have information up to s $(\leq t)$ is X_s"

Note 3 : For discrete time processes, the essential defining property becomes

$$E(X_{n+1}|\mathcal{F}_n) = X_n \qquad n = 0, 1, \ldots$$

Example 1

Let $Z_1, Z_2, \dots$ be independent r.v. with null mean, then the partial sums

$$S_n = Z_1 + \cdots + Z_n \qquad n = 1, 2, \dots$$

is a martingale w.r.t. the natural filtration of (Z_n)

For the adaptation of the stochastic process,

$$\sigma(S_1, \dots, S_n) = \sigma(Z_1, \dots, Z_n) = \mathcal{F}_n$$

because they contain the same information :

$$\begin{cases} S_n = Z_1 + \cdots + Z_n \\ Z_n = S_n - S_{n-1} \end{cases}$$

Furthermore,

$$\begin{aligned} E(S_{n+1}|\mathcal{F}_n) &= E(S_n + Z_{n+1}|\mathcal{F}_n) \\ &= E(S_n|\mathcal{F}_n) + E(Z_{n+1}|\mathcal{F}_n) \\ &= S_n + E(Z_{n+1}) \\ &= S_n \end{aligned}$$

Example 2

Let Z be a r.v. such that $E(|Z|) < \infty$ and a filtration $\mathbf{F}$. Then $X_t = E(Z|\mathcal{F}_t)$ defines a martingale

For any s, t such that $s \leq t$,

$$E(X_t|\mathcal{F}_s) = E(E(Z|\mathcal{F}_t)|\mathcal{F}_s) = E(Z|\mathcal{F}_s) = X_s$$

Properties

Expectation of a martingale

The expectation function of a martingale is constant

For any s, t such that $s \leq t$,

$$\mu_X(s) = E(X_s) = E\big(E(X_t|\mathcal{F}_s)\big) = E(X_t) = \mu_X(t)$$

Interpretation as a fair game

Let (X_t) represent the winnings (or the losses) up to time t and suppose it is a martingale

If we have information up to time s $(\leq t)$, the increment of winnings $(X_t - X_s)$ during $]s;t]$ is such that

$$\begin{aligned} E(X_t - X_s|\mathcal{F}_s) &= E(X_t|\mathcal{F}_s) - E(X_s|\mathcal{F}_s) \\ &= X_s - X_s = 0 \end{aligned}$$

Gaussian process

A stochastic process (X_t) is said to be gaussian if any finite-dimensional distribution of this process is multinormal : for any $n \geq 1$, $t_1, \dots, t_n \in T$

$$\left(X_{t_1}, \dots, X_{t_n}\right)$$

is a multinormal random vector

In particular, if (X_t) is a gaussian process, then, $\forall t \in T$,

$$X_t \sim \mathcal{N}$$