Chapter 3

Stochastic Processes

Definitions

- A particular stopping time: the barrier
- Stopping time

- Filtration
  - Definition

- Martingales
- Walk

- A discrete time stochastic process: random

- Definitions

- Stochastic processes
A stochastic process is a family of r.v., indexed by time 

Let \( \Omega \) be a stochastic process 

For any fixed \( \omega \in \Omega \), \( X^1(\omega) \) is a path 

A stochastic process is a family of r.v., indexed by 

Time

If \( \mathcal{L} = \mathbb{N} \) or \( \mathbb{Z} \), the stochastic process \( \{X^1(t) : t \in \mathcal{L}\} \) is a sequence of r.v., 

For any fixed \( t \in \mathcal{L} \), \( X^1(t) \) is a r.v.

If the time set \( \mathcal{L} \) is

- discrete, \( \{X^1(t) : t \in \mathcal{L}\} \) is a discrete time stochastic process 
- continuous, \( \{X^1(t) : t \in \mathcal{L}\} \) is a continuous time stochastic process
Distribution of a stochastic process

The probability distribution of a r.v. is given by $\Pr(g = 0 \in \mathcal{B} = 1)$ for any Borel set $\mathcal{B}$. The finite-dimensional distribution of a stochastic process is the probability distribution of $X_{t_1}, \ldots, X_{t_n}$ for any $t_1, \ldots, t_n \in T$, where $T$ is a set of indices.

Moments

(a) Expectation: $E[X] = E[X g = 0 \in \mathcal{B} = 1]$

(b) Variance: $\text{Var}(X) = E[X^2] - E[X]^2$

(c) Covariance: $\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$

Property: If the stochastic process $X_t$ is strictly stationary, then

\[
E[X_{t_1}, \ldots, X_{t_n}] = E[X_{t_1 + u}, \ldots, X_{t_n + u}] = E[X_{t_1}, \ldots, X_{t_n}]
\]

for any $u \geq 1, t_1, \ldots, t_n \in T$. The stochastic process $X_t$ is strictly stationary if $E[X_{t_1}, \ldots, X_{t_n}]$ have the same probability distribution.

Strict stationarity

Indicate dependence structure

Possible properties of a stochastic process

Probabililty distribution of a stochastic process

\[
(i) X^g = (t, g) X^g
\]

(c) Covariance: $\text{Cov}(X, Y) = (t, g) X^g$

(b) Variance: $\text{Var}(X, g) = (t, g) X^g$

(a) Expectation: $E[X, g] = (t, g) X^g$

The stochastic process $X_t$ is strictly stationary if

such that $t_1 + u, t_2 + u, \ldots, t_n + u \in T$, $u \geq 1, t_1, \ldots, t_n \in T$. The stochastic process $X_t$ is strictly stationary if $E[X_{t_1}, \ldots, X_{t_n}]$ have the same probability distribution.

The finite-dimensional distribution of a stochastic process is given by

$\Pr[X \in \mathcal{B} \mid t_1, \ldots, t_n]$. The probability distribution of a r.v. is given by

Probabililty distribution of a stochastic process
Wide stationarity (or "2nd order stationarity")

The stochastic process \( g^{-666/g} \equiv 0 \) is stationary in the wide sense if

\[
\begin{aligned}
\forall \tau \in \mathbb{F}:
\end{aligned}
\]

Independent Increments

The stochastic process \( g^{-666/g} \equiv 0 \) has stationary increments if, \( \forall \tau \in \mathbb{F} \),

\[
\begin{aligned}
\forall \tau \in \mathbb{F}:
\end{aligned}
\]

Independent Increments

\[
\begin{aligned}
\forall \tau, \tau' \in \mathbb{F}:
\tau' + \tau - \tau' = \tau
\end{aligned}
\]

Wide stationarity

(\( 2^{nd} \) order stationarity)
Filtration

**Definition**

Let \( \Omega, \mathcal{F}, \mathbb{P} \) be a probability space. A filtration is an increasing family of sub-
fields of \( \mathcal{F} \):

\[ \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F} \]

Interpretation: \( \mathcal{F}_n \) is the collection of events representing the available information up to \( n \) (= the history up to \( n \)).

Note: generally, \( \mathcal{F}_0 = \emptyset, \Omega = \mathcal{F} \) and, in this case, \( \mathcal{F}_n = \emptyset, \Omega = \mathcal{F} \).

We can now generalize the probability space when a time variable is present:

\[ \Omega, \mathcal{F}, \mathbb{P} \]

Filtration and stochastic process

**a)** Natural filtration of a stochastic process:

\[ \mathcal{F} \]

A stochastic process is adapted to a filtration if \( X \) is a r.v., let define conditional expectation w.r.t. a filtration:

\[ \mathbb{E} (X|\mathcal{F}_t) \]

Property: \( \mathbb{E} (X|\mathcal{F}_t) \) is \( \mathcal{F}_t \)-measurable.

Interpretation: \( \mathbb{E} (X|\mathcal{F}_t) \) represents the mean estimation of \( X \), taking into account the available information up to \( t \).

**b)** Filtration of a stochastic process:

\[ \{ \mathcal{F}_t : t \geq 0 \} \]

Interpretation: \( \mathcal{F}_t \) is the collection of events representing the available information up to \( t \).

A filtration is an increasing family of sub-
fields, let \( \{ \mathcal{F}_t : t \geq 0 \} \) be a probability space.
A (symmetric) coin is tossed 3 times, independently.

Let us define for each possible outcome of a coin toss, the number of tails up to the $i$-th toss, $i = 0, 1, 2, 3$.

(a) For each possible outcome, we can associate a value of these r.v.:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$0$</th>
<th>$1$</th>
<th>$1$</th>
<th>$1$</th>
<th>$2$</th>
<th>$2$</th>
<th>$2$</th>
<th>$3$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_i$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

(b) Natural filtration of the stochastic process:

$\mathcal{F}_0 = \mathcal{F} \cap \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \mathcal{F} \cap \{\emptyset, \Omega, \{H, T\}, \{H, T, T\}\}$

$\mathcal{F}_2 = \mathcal{F} \cap \{\emptyset, \Omega, \{H, T, T\}, \{H, T, T, T\}\}$

$\mathcal{F}_3 = \mathcal{F} \cap \{\emptyset, \Omega, \{H, T, T, T\}, \{H, T, T, T, T\}\}$

(c) Let us denote $X_i$ the number of tails up to the $i$-th toss, $i = 0, 1, 2, 3$.

$X_0$ is the number of tails up to the 0th toss, which is always 0.

$X_1$ is the number of tails up to the 1st toss, which can be 0 or 1.

$X_2$ is the number of tails up to the 2nd toss, which can be 0, 1, or 2.

$X_3$ is the number of tails up to the 3rd toss, which can be 0, 1, 2, or 3.

A symmetric coin is tossed 3 times.

Example: Tossing a coin 3 times.
Conditional expectations of $X$: $E(\mathcal{F}X|\mathcal{F})$
A stopping time on $(\Omega, \mathcal{F}, P)$ is a r.v. $\tau$ with values in $\mathbb{R} \cup \{\infty\}$ such that

$$\forall \tau' \in \mathcal{F}, \{\tau' \leq \tau\} \in \mathcal{F}$$

Interpretation: a rule which tells when to stop, based only on the knowing of the history up to the instant of stopping.

Property: if $\tau_1$ and $\tau_2$ are two stopping times, then $\min(\tau_1, \tau_2)$ and $\max(\tau_1, \tau_2)$ are also stopping times.

Example: for the short-term interest rate $r(t)$,
- the first time $r(t)$ is equal to 3% is a stopping time,
- the last time $r(t)$ is equal to 3% is not a stopping time.

A particular stopping time: the barrier

Let $(\mathbb{R}, \mathcal{F})$ be a stochastic process and $\mathbb{R}$ the natural filtration of $(\mathbb{R}, \mathcal{F})$.

A stopping time on $(\mathbb{R}, \mathcal{F}, P)$ is a r.v. $\tau$ with $\tau \in \mathbb{R}^+ \cup \{\infty\}$ such that $\{\tau' \leq \tau\} \in \mathcal{F}$ for all $\tau' \in \mathbb{R}^+ \cup \{\infty\}$.

Definition
Example: tossing a coin 3 times

(a) We stop tossing the coin when the first tail appears:
\[
\{H HH\} = \{\infty = 1\} : \text{Note: } \infty \text{ is a stopping time}
\]
\[
\{HH H, HTH, HTT\} = \{2 = 1\} \text{ is } \mathcal{F}_2 - \text{measurable}
\]
\[
\mathcal{L}^X = \{HH H, HTH, HTT\} = \{2 = 1\}
\]
\[
\mathcal{L}^X = \{HH H, HTH, HTT\} = \{1 = 1\}
\]

\[
\begin{array}{cccccccc}
\infty & 3 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\
HHH & HHH & HTH & HTH & HTH & HTH & HTH & HTH \\
TTT & TTT & TTT & TTT & TTT & TTT & TTT & TTT \\
\end{array}
\]

\[
\{3X = X : \text{min} = 1\} \text{ appears: we stop tossing the coin when the last time tail appears}
\]

\[
\begin{array}{cccccccc}
\infty & 3 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\
HHH & HHH & HTH & HTH & HTH & HTH & HTH & HTH \\
TTT & TTT & TTT & TTT & TTT & TTT & TTT & TTT \\
\end{array}
\]

\[
\{\exists X = X : \text{min} = 2\} \text{ appears: we stop tossing the coin when the first tail appears}
\]

\[
\begin{array}{cccccccc}
\infty & 3 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\
HHH & HHH & HTH & HTH & HTH & HTH & HTH & HTH \\
TTT & TTT & TTT & TTT & TTT & TTT & TTT & TTT \\
\end{array}
\]

\[
\{\exists X = X : \text{min} = 3\} \text{ appears: we stop tossing a coin 3 times}
\]

\[
\begin{array}{cccccccc}
\infty & 3 & 2 & 1 & 1 & 2 & 2 & 3 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\
HHH & HHH & HTH & HTH & HTH & HTH & HTH & HTH \\
TTT & TTT & TTT & TTT & TTT & TTT & TTT & TTT \\
\end{array}
\]
A discrete time stochastic process: random walk

**General case**

**Definition**

Description: a particle (or an asset return, or a drunkard, ...)

- starts at time \( t = 0 \) at the point \( 0 \)
- at each step, it moves by \( ax \) with probability \( d \) and by \( -ax \) with probability \( 1 - d \)
- the duration of the steps is \( \Delta \)
- the different moves are independent

In this model, \( \mathcal{T} = \mathbb{N} \cdot (\forall t \in \mathbb{N}, \forall \Delta) \)

\( \cdots \{ \ldots, \forall \Delta \} \)

\( X \) is the position of the particle at time \( t \)

\[
\begin{pmatrix}
    x & \Delta \\
    x & -\Delta
\end{pmatrix}
\]

Special case: symmetrical probabilites

- o Properties
- o Definition
- General case

**General case**

**Random Walk**

A discrete time stochastic process:
Properties

Let us examine the situation after \( g_1(\cdot) \) moves and denote \( g_2 = g_1 \cdot \Delta \). Then, we have

\[
\mathcal{Z} \cdot \left( \frac{\partial}{\partial \mathcal{X}} - \frac{\partial}{\partial \mathcal{X}^-} \right) = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V} \quad \text{and} \quad \mathcal{V} = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V}.
\]

We have

\[
\mathcal{Z} \cdot \left( \frac{\partial}{\partial \mathcal{X}} - \frac{\partial}{\partial \mathcal{X}^-} \right) = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V} \quad \text{and} \quad \mathcal{V} = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V}.
\]

The evolution of the particle is such that the position is null in mean with a variance proportional to time. The special case of symmetrical probabilities is:

- If \( g_1(\cdot) = g_2 = g_1 \cdot \Delta \), we have
  \[
  \mathcal{Z} \cdot \left( \frac{\partial}{\partial \mathcal{X}} - \frac{\partial}{\partial \mathcal{X}^-} \right) = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V} \quad \text{and} \quad \mathcal{V} = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V}.
  \]

So,

\[
bd \mathcal{V} \mathcal{Z}(\mathcal{X}) =
\begin{align*}
(\mathcal{Z}(b - d) - 1) \mathcal{Z}(\mathcal{X}) = \mathcal{Z}(b - d) \mathcal{Z}(\mathcal{X}) - (d + b) \mathcal{Z}(\mathcal{X}) = (\mathcal{Z} \cdot \mathcal{X}).
\end{align*}
\]

We have

\[
(b - d) \mathcal{X} = (\mathcal{Z} \cdot \mathcal{X}) \mathcal{V}.
\]

Let us examine the situation after moves and denote \( \mathcal{V} \cdot u = (\mathcal{Z} \cdot \mathcal{X}).u \).
... \quad u = 0, \quad 1, \quad \ldots
\quad u^X = (\mathbb{E}[r^X | t+1]) \mathbb{E}^t

defining property becomes

Note 1: A stochastic process may be a martingale with respect to a filtration \( \mathbb{F} \), but not with respect to another one.

Let us consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

\begin{align*}
\text{Expectation of a martingale} & \\
\text{Interpretation as a fair game} & \\
\text{Properties of a martingale} & \\
\end{align*}

Note 2: The essential defining property means "the best prediction of \( s^X \) when we have information up to \( s \) is \( \mathbb{E}[\mathbb{E}[r^X | \mathcal{F}_s] | \mathcal{F}_s] \)."

Note 3: For discrete time processes, the essential defining property becomes

\begin{align*}
( t > s ) \quad A, \delta \quad s^X = (\mathbb{E}[r^X | \mathcal{F}_s]) \mathbb{E}^t
& \text{ is adapted to } (\mathbb{F}) \\
& \text{ if } \mathbb{E}[r^X | \mathcal{F}_s] \text{ is a martingale.}
\end{align*}
Example 1  Let \( X \sim \mathcal{N}(0, \sigma^2) \) be independent r.v. with null mean, then the partial sums
\[
S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mathcal{N}(0, \sigma^2)
\]
are martingales w.r.t. the natural filtration of \( \mathcal{F}_n \).

For the adaptation of the stochastic process,
\[
\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i = \sum_{i=1}^{n} \mathcal{N}(0, \sigma^2)
\]

Furthermore,
\[
\tilde{S}_n - S_n = \mathcal{N}(0, \sigma^2)
\]

Example 2  Let \( X \sim \mathcal{N}(0, \sigma^2) \) be a r.v. such that
\[
\tilde{X}_i = \sum_{j=1}^{i} \mathcal{N}(0, \sigma^2)
\]
and a filtration \( \mathcal{F}_n \). Then \( X \) is a martingale w.r.t. the natural filtration of \( \mathcal{F}_n \).

For the adaptation of the stochastic process,
\[
\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i = \sum_{i=1}^{n} \mathcal{N}(0, \sigma^2)
\]

Let \( Z \) be a r.v. with null mean, then the partial sums
\[
\sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} \mathcal{N}(0, \sigma^2)
\]

Example 1
Properties

**Expectation of a martingale**

The expectation function of a martingale is constant. For any $s \leq t$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, the increment of winnings $\{X_t\}_{t \geq 0}$ during $[s, t]$ is

The expectation up to time $s$ is a martingale to time $t$ and suppose it is a martingale.

Interpretation as a fair game

Let $(\{X_t\}_{t \geq 0})$ represent the winnings (or the losses) up to time $t$. Then, the expectation function of a martingale is

**Gaussian process**

A stochastic process $\{X_t\}_{t \geq 0}$ is said to be Gaussian if any finite-dimensional distribution of this process is multinormal. In particular, if $(\{X_t\}_{t \geq 0})$ is a Gaussian process, then,

$$N \sim \mathcal{N}$$

For any $s, t$ such that $s \leq t$,

$$E \{X_s | \mathcal{F}_t\} = E \{X_s\} = (s \mathcal{F})$$

The expectation function of a martingale is

**Properties**