Part I – PRELIMINARIES

1. General results and motivation
2. Probability theory
3. Stochastic processes

Chapter 1

General results and motivation

- Binomial model (Cox, Ross, Rubinstein)
- Fundamental theorem of risk-neutral valuation
- From discrete or deterministic to continuous and stochastic

Binomial model (Cox, Ross, Rubinstein)

- Option : definition and general properties
  - Definition
  - Value
  - General properties
- Discrete valuation model
  - Hypotheses
  - Binomial model for underlying equity
  - Binomial model for the option
  - Complete market

Option : definition and general properties

Definition

= "contingent claim"

= contract that confers to its purchaser
- the right to
- buy (call) or sell (put)
- an (underlying) asset
- at a future date (European) / up to a future date (American)
- at a price determined in advance (exercise price, strike)

by paying a certain price (premium)

Option =
- a right for the holder
- an obligation for the issuer

Here : option on an equity without dividend

The price p_t at time t of an option (C for a call, P for a put) depends on several factors :

- the price S_t of the underlying equity
- the exercise price K
- the duration $\tau = T - t$ remaining to the option maturity
- the volatility σ_R of the return of the equity
- the risk-free rate R_F

$$p_t = f(S_t, K, \tau, \sigma_R, R_F)$$

Value

Intrinsic value
= Profit obtained by the purchaser if the option is exercised at time t (without taking account of the premium)

For a call : $IV_t = \max(0, S_t - K) = (S_t - K)^+$
For a put : $IV_t = \max(0, K - S_t) = (K - S_t)^+$

Time value
= part of the price over the intrinsic value

$$TV_t = p_t - IV_t$$

General properties

a) Call-put parity relation for European option

Let us consider a portfolio at time t obtained by

	at time t	at time T
buy 1 uderlying equity	S_t	S_T
buy 1 put	P_t	P_T
sell 1 call	$-C_t$	$-C_T$
borrow an amount worth K at time T	$-K(1+R_F)^{-\tau}$	$-K$

Value at maturity : $V_T = S_T + P_T - C_T - K$

- if $S_T > K$, $V_T = S_T + 0 - (S_T - K) - K = 0$
- if $S_T \leq K$, $V_T = S_T + (K - S_T) - 0 - K = 0$

$V_T = 0$ whatever the value S_T of the equity at maturity

$\Longrightarrow$ the value V_t at time t must be equal to 0

❶

$$\boxed{P_t + S_t = C_t + (1+R_F)^{-\tau} K}$$

b) Inequalities on price of an European option

$$(S_t - (1+R_F)^{-\tau}K)^+ \leq C_t \leq S_t$$
$$((1+R_F)^{-\tau}K - S_t)^+ \leq P_t \leq (1+R_F)^{-\tau}K$$

(without proof)

c) American call = European call

If the American option is exercised at time $t(< T)$, then

$$C_t^{(a)} = S_t - K < S_t - (1+R_F)^{-\tau}K \leq C_t^{(e)}$$

Discrete valuation model

Hypotheses

- Very general hypotheses
  - Perfect market
  - Non risky asset
- Specific to valuation models
  - Arbitrage-free market

a) Perfect market

- No investor is dominant (no market makers)
- Investors are rational (prefer more to less)
- Assets infinitely divisible
- No transaction costs
- No tax
- Short sales allowed

b) Existence of a non risky asset, the same for borrowing and deposit (risk-free rate at a bank account)

c) Arbitrage-free market (= "no free lunch")

Arbitrage opportunity : asset (or portfolio) such that

- Initial (non random) value : $V_0 \leq 0$
- Final (random) value :

$$V_T(\omega) \geq 0 \quad \forall \omega$$
$$\exists \omega_0 : \ V_T(\omega_0) > 0$$

Equivalent formulation :

Arbitrage-free : if a portfolio has a final value that is non random, its return is equal to the risk-free rate

Note : already used at ❶

Justification : supply and demand law

Binomial model for underlying equity

(Random) value of the equity at time t : S_t

Evolution :

$$S_t \nearrow S_{t+1} = S_t \cdot u \qquad (\Pr = \alpha)$$

$$S_t \searrow S_{t+1} = S_t \cdot d \qquad (\Pr = 1 - \alpha)$$

α : historical probability

Possible values for (u, d) : $d < 1 < 1 + R_F < u$
(otherwise, arbitrage opportunity)

Hypothesis : α, u, d constant over time – indep. successive moves

$$S_{t+1} = S_t \cdot u \nearrow S_{t+2} = S_t \cdot u^2$$

$$S_{t+1} = S_t \cdot u \searrow S_{t+2} = S_t \cdot ud$$

$$S_{t+1} = S_t \cdot d \nearrow S_{t+2} = S_t \cdot du$$

$$S_{t+1} = S_t \cdot d \searrow S_{t+2} = S_t \cdot d^2$$

$$S_0 \to S_0 \cdot u,\ S_0 \cdot d$$

$$S_0 \cdot u \to S_0 \cdot u^2,\ S_0 \cdot ud \qquad S_0 \cdot d \to S_0 \cdot ud,\ S_0 \cdot d^2$$

$$S_0 \cdot u^3 \ \dots \qquad S_0 \cdot u^2 d \ \dots \qquad S_0 \cdot ud^2 \ \dots \qquad S_0 \cdot d^3 \ \dots$$

Generally,

$$S_T = S_0 \cdot u^N d^{T-N}$$

where N is a binomial r.v. : $N \sim \mathcal{B}(T; \alpha)$

$$\Pr[N = k] = \binom{T}{k} \alpha^k (1 - \alpha)^{T-k} \qquad (k = 0, \dots, T)$$

Note :

$$E(S_T) = \sum_{k=0}^{T} S_0 \cdot u^k d^{T-k} \binom{T}{k} \alpha^k (1-\alpha)^{T-k}$$
$$= S_0 \sum_{k=0}^{T} \binom{T}{k} (\alpha u)^k ((1-\alpha)d)^{T-k}$$
$$= S_0 \cdot (\alpha u + (1-\alpha)d)^T$$

$\leftrightarrow S_T = S_0 \cdot (1+i)^T$ for non random evolution

Binomial model for the option (CRR)

a) 1 period : $T = 1$

$$C_0 \rightarrow C_1 = C(u) = (S_0 u - K)^+ \qquad (\alpha)$$
$$C_0 \rightarrow C_1 = C(d) = (S_0 d - K)^+ \qquad (1-\alpha)$$

Let us construct a portfolio

	at time 0	at time 1
buy X equity	XS_0	?
sell 1 call	$-C_0$	?

$$V_0 = XS_0 - C_0 \rightarrow V_1 = XS_0 \cdot u - C(u)$$
$$V_0 = XS_0 - C_0 \rightarrow V_1 = XS_0 \cdot d - C(d)$$

We chose X such that V_1 is no more random :

$$V_1 = XS_0 \cdot u - C(u) = XS_0 \cdot d - C(d) \qquad (i)$$

Arbitrage-free →

$$V_1 = (XS_0 - C_0) \cdot (1 + R_F) \qquad (ii)$$

(i) : $$XS_0 = \frac{C(u)-C(d)}{u-d}$$

(ii) : $$\frac{C(u)-C(d)}{u-d} \cdot u - C(u) = \left(\frac{C(u)-C(d)}{u-d} - C_0\right) \cdot (1 + R_F)$$

$$C_0(1 + R_F) = \left(\frac{1+R_F}{u-d} - \frac{u}{u-d} + 1\right) C(u) + \left(-\frac{1+R_F}{u-d} + \frac{u}{u-d}\right) C(d)$$

$$C_0 = (1 + R_F)^{-1} \left[\frac{(1+R_F)-d}{u-d} C(u) + \frac{u-(1+R_F)}{u-d} C(d)\right]$$

$$\frac{(1+R_F)-d}{u-d} = q \qquad \frac{u-(1+R_F)}{u-d} = 1 - q$$

$$C_0 = (1 + R_F)^{-1}[q \cdot C(u) + (1 - q) \cdot C(d)]$$

C_0 = the discounted value of the expectation of C_1 with respect to q

Note 1 : this result is independent of α

Note 2 : expected value of S_1 w.r.t. q :

$$E_q(S_1) = q \cdot S_0 u + (1 - q) \cdot S_0 d = S_0 \left[\frac{(1+R_F)-d}{u-d} \cdot u + \frac{u-(1+R_F)}{u-d} \cdot d\right] = S_0(1 + R_F)$$

Evolution of the (expected value of the) risky asset under q = evolution of the non risky asset

$(q, 1 - q)$ = risk-neutral probability
= martingale probability

b) 2 periods : $T = 2$

$$C_0 \nearrow C_1 = C(u) \nearrow C_2 = C(u,u) = (S_0 u^2 - K)^+$$
$$C_1 = C(u) \searrow C_2 = C(u,d) = (S_0 ud - K)^+$$
$$C_0 \searrow C_1 = C(d) \nearrow C_2 = C(u,d) = (S_0 ud - K)^+$$
$$C_1 = C(d) \searrow C_2 = C(d,d) = (S_0 d^2 - K)^+$$

Apply twice the result for 1 period

$$\begin{cases} C(u) = (1+R_F)^{-1}[q \cdot C(u,u) + (1-q) \cdot C(u,d)] \\ C(d) = (1+R_F)^{-1}[q \cdot C(d,u) + (1-q) \cdot C(d,d)] \end{cases}$$

$$C_0 = (1+R_F)^{-1}[q \cdot C(u) + (1-q) \cdot C(d)]$$
$$= (1+R_F)^{-2} \begin{bmatrix} q^2 \cdot C(u,u) \\ +2q(1-q) \cdot C(u,d) \\ +(1-q)^2 \cdot C(d,d) \end{bmatrix}$$

C_0 = the discounted value of the expectation of C_2 with respect to the binomial law with parameters $(2, q)$

c) T periods

C_0 = the discounted value of the expectation of the value C_T at maturity T w.r.t. the binomial law $\mathcal{B}(T; q)$

$$C_0 = (1+R_F)^{-T} \sum_{j=0}^{T} \binom{T}{j} q^j (1-q)^{T-j} C(\underbrace{u, \dots, u}_{j}, \underbrace{d, \dots, d}_{T-j})$$
$$= (1+R_F)^{-T} \sum_{j=0}^{T} \binom{T}{j} q^j (1-q)^{T-j} \left(S_0 u^j d^{T-j} - K\right)^+$$

$u^j d^{T-j} = \left(\frac{u}{d}\right)^j d^T$ is increasing w.r.t. j

Let $J = \min\{j : S_0 u^j d^{T-j} - K > 0\}$

$$C_0 = (1+R_F)^{-T} \sum_{j=J}^{T} \binom{T}{j} q^j (1-q)^{T-j} \left(S_0 u^j d^{T-j} - K\right)$$

$$C_0 = S_0 \sum_{j=J}^{T} \binom{T}{j} \left(\frac{uq}{1+R_F}\right)^j \left(\frac{d(1-q)}{1+R_F}\right)^{T-j}$$
$$-(1+R_F)^{-T} K \sum_{j=J}^{T} \binom{T}{j} q^j (1-q)^{T-j}$$

$$\frac{uq}{1+R_F} + \frac{d(1-q)}{1+R_F} = \frac{u[(1+R_F)-d]+d[u-(1+R_F)]}{(1+R_F)(u-d)} = 1$$

$$\frac{uq}{1+R_F} = q' \qquad \frac{d(1-q)}{1+R_F} = 1-q'$$

$$C_0 = S_0 \cdot \Pr[B(T;q') \geq J]$$
$$-(1+R_F)^{-T} K \cdot \Pr[B(T;q) \geq J]$$

d) Estimation of the parameters J, u and d

$$J = \min\{j : S_0 u^j d^{T-j} - K > 0\}$$

$$S_0 u^j d^{T-j} - K > 0 \iff \left(\frac{u}{d}\right)^j > \frac{K}{S_0 d^T}$$
$$\iff j > \frac{\ln(K/S_0 d^T)}{\ln(u/d)}$$

S_t → $S_{t+1} = S_t \cdot u \qquad (\Pr = \alpha)$

S_t → $S_{t+1} = S_t \cdot d \qquad (\Pr = 1-\alpha)$

Return : $R = \frac{S_{t+1}-S_t}{S_t} \sim \begin{pmatrix} u-1 & d-1 \\ \alpha & 1-\alpha \end{pmatrix}$

$$E_R = \alpha(u-1) + (1-\alpha)(d-1)$$

$$\sigma_R^2 = \alpha(u-1)^2 + (1-\alpha)(d-1)^2$$
$$-\left(\alpha(u-1) + (1-\alpha)(d-1)\right)^2$$
$$= \alpha(1-\alpha)(u-1)^2 + \alpha(1-\alpha)(d-1)^2$$
$$-2\alpha(1-\alpha)(u-1)(d-1)$$
$$= \alpha(1-\alpha)(u-d)^2$$

$$\sigma_R^2 = \alpha(1-\alpha)(u-d)^2$$

With $\alpha = 1/2,\ d = 1/u$,

$$\sigma_R^2 = \frac{1}{4}\left(u - \frac{1}{u}\right)^2$$

$$u^2 - 2\sigma_R u - 1 = 0$$

$$u = \sigma_R + \sqrt{\sigma_R^2 + 1}$$

By Taylor expansion,

$$u \approx \sigma_R + \left(1 + \frac{\sigma_R^2}{2}\right) = 1 + \sigma_R + \frac{\sigma_R^2}{2} \approx e^{\sigma_R}$$

and

$$d \approx e^{-\sigma_R}$$

e) Formula for a put

Call-put parity relation :

$$\begin{aligned} P_0 &= -S_0 + C_0 + (1+R_F)^{-T}K \\ &= -S_0 + S_0 \cdot \Pr[B(T;q') \geq J] \\ &\quad -(1+R_F)^{-T}K \cdot \Pr[B(T;q) \geq J] \\ &\quad +(1+R_F)^{-T}K \end{aligned}$$

$$\begin{aligned} P_0 = -S_0 \Pr[B(T;q') < J] \\ +(1+R_F)^{-T}K \cdot \Pr[B(T;q) < J] \end{aligned}$$

f) Example

Call option (maturity : 7 months) on an equity (current value : 100 EUR) with strike price 110 EUR. Volatility : σ_R = 0.25 and risk-free rate = 4 % per year

Determine the call price and put price

Complete market

Remember ❷

	at time 0	at time 1
buy X equity	XS_0	?
sell 1 call	$-C_0$	?

We chose X such that V_1 is no more random :

$$V_1 = XS_0 \cdot u - C(u) = XS_0 \cdot d - C(d) \qquad (i)$$

Arbitrage-free →

$$V_1 = (XS_0 - C_0) \cdot (1 + R_F) \qquad (ii)$$

2 equations

- one for eliminating X
- one for calculating C_0

This portfolio
= hedging position
= replicating portfolio of the contingent claim

General formulation

> Complete market : every contingent claim can be hedged

Counter-example (non complete market) : trinomial model

$$S_0 \to S_1 = S_0 \cdot u_1 \qquad (\Pr = \alpha_1)$$
$$S_0 \to S_1 = S_0 \cdot u_2 \qquad (\Pr = \alpha_2)$$
$$S_0 \to S_1 = S_0 \cdot u_3 \qquad (\Pr = \alpha_3)$$

……….

3 equations for 2 actions (eliminating X – calculating C_0) → generally no solution

Fundamental theorem of risk-neutral valuation

Definitions

- arbitrage-free market : “no free lunch”
- complete market : “every contingent claim can be hedged”

Arbitrage-free theorem

- absence of arbitrage opportunity
- existence of risk-neutral measure

are equivalent

Completeness theorem

- complete market
- unicity of risk-neutral measure

are equivalent

Fundamental theorem of risk-neutral valuation

In a complete market without arbitrage opportunity, the price of a contingent claim is equal to the discounted value of the expectation of its final value with respect to the risk-neutral measure

From discrete or deterministic to continuous and stochastic

- For option
- Simpler : evolution of an asset value
  - From discrete time …
  - …to continuous time

For option

2 possible ways

	equity	option
discrete time	Bin	CRR
continuous time	?	B&S

	equity	option
discrete time	Bin	CRR
continuous time	?	B&S

Simpler : evolution of an asset value

	deterministic	stochastic
discrete time	$S_t = S_0 \cdot (1+i)^t$	$S_t = S_0 \cdot (1+i_1) \dots (1+i_t)$
continuous time	$S_t = S_0 \cdot e^{\delta t}$	**?**

From discrete time …

$$S_t = S_0 \cdot e^{\delta_1} \dots e^{\delta_t}$$

where $\delta_1, \dots, \delta_t$ are i.i.d. r.v. with $E(\delta_j) = \delta$

For example, $\delta_j = \delta + \sigma \cdot \varepsilon_j$ where

$$\varepsilon_j \sim \begin{pmatrix} -1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

Log-return of the asset :

$$R_t = \ln\left(\frac{S_t}{S_0}\right) = \ln\left(e^{\delta_1} \dots e^{\delta_t}\right) = \delta_1 + \dots + \delta_t$$

$$R_t = \delta \cdot t + \sigma \sum_{j=1}^{t} \varepsilon_j$$

$\delta \cdot t$: trend

σ : volatility

$\sum_{j=1}^{t} \varepsilon_j$: random noise

Let us denote $w_t = \sum_{j=1}^{t} \varepsilon_j$

$$E(w_t) = 0 \qquad var(w_t) = t$$

For the log-return,

$$R_t = \delta \cdot t + \sigma \cdot w_t$$

with

$$E(R_t) = \delta \cdot t \qquad var(R_t) = \sigma^2 \cdot t$$

… to the continuous time

How to model the random noise w_t ?

$\Longrightarrow$ Need for more advanced probability theory