Iconic virtues of diagrams

Peirce on ampliative reasoning

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The present paper is part of a larger research on the role of diagrams in mathematical "ampliative" reasoning. Leaning on Charles Sanders Peirce's work we will here focus on the semiotic aspects of the question and refer to other papers for more details on the historical and epistemological issues.

1. Historical issues[1]

1.1. Kant's doctrine of pure intuition and schematism

In his *Critique of pure reason*, Immanuel Kant (1781) notoriously claimed that mathematical knowledge is both synthetic and *a priori*. Unlike statements such as "Every triangle has three angles" or "7 is the successor of 6", which merely rest on the analysis of the definition of the concepts involved, statements such as "No side of a triangle can be longer than the sum of the other two" or "7+5=12" provide some information which could not be found by conceptual analysis only[2]; they are "synthetic" statements, i.e. they attribute to their subject a property which was not already involved in its representation. Now, according to Kant, this requires that the new property be given through perception or another kind of "intuition". In the case of mathematical statements, which appear to be necessarily rather than inductively true, such an intuition *"must be encountered in us* a priori*, i.e. prior to all perception of an object, thus it must be pure, not empirical intuition"*[3].

Geometrical knowledge, for instance, is grounded on some *"formal intuition"*[4], namely the representation of the form of space, which is provided through sense intuition[5], yet not in the *material content* of *some* singular experience but in the very *form* of *any* experience[6]. Geometrical knowledge, Kant says, gets its necessity and *a prioricity* from the fact that it rests on the formal conditions of any experience of spatial objects:

> If there did not lie in you a faculty for intuiting *a priori*; if this subjective condition regarding form were not at the same time the universal *a priori* condition under which alone the object of this (outer) intuition is itself possible; if the object (the triangle) were something in itself without relation to your subject: then how could you say that what necessarily lies in your subjective conditions for constructing a triangle must necessarily pertain to the triangle in itself ?[7]

This notion of "construction" gives an insight into what pure intuition is. When I claim to state some property of triangles, I can lean on the image of some triangle, either by drawing one or by imagining one. Yet in both cases what matters is not the singular figure which is drawn or imagined but the general notion – the concept – according to which this triangle is drawn or imagined[8]. The notion of "construction" makes the link between singularity of the intuition and universality of the conception. Even though geometry is a conceptual discipline, construction is what provides to its abstract concepts the intuitive meaning they require in order to ground synthetic *a priori* statements[9]

[1] For more details and references on this historical aspects, see Leclercq (2016).
[2] *CPR*, A25/B39
[3] *CPR*, B41
[4] *CPR*, B161, B457
[5] *CPR*, A25/B39-41, A47-48/B64-65
[6] *CPR*, A27
[7] *CPR*, A48/B65
[8] *CPR*, A714/B742
[9] *CPR*, A240/B299

and therefore to allow for some increase of knowledge[10]. Reversely, geometry gains universal value from the fact that what can be seen is constrained by *universal* "conditions of construction"[11].

The same is true of arithmetic[12]. A schema is *"a general procedure of the imagination for providing a concept with its image"*[13]. Numbers do not reduce to sets of dots, but drawing dots in a row is a general procedure for providing an image to any natural number[14]. Being *"the representation that summarizes the successive addition of one (homogeneous) unit to another"*[15], number is the pure schema of the category of magnitude. And, according to Kant, the whole set of fundamental laws of natural numbers' arithmetic is involved in this "general procedure": if to a row of seven dots I add another five dots I will get a row of twelve dots. Of course, if it was limited to such intuitive images of magnitudes, arithmetic could not progress very far. But number, as a schema, gathers together all conceivable images within a single general procedure that ensures that such a construction can be made even when sense intuition is exceeded[16].

1.2. Logicist objections to Kant's doctrine

Thanks to the way schemas relate pure concepts to the conditions of their construction, i.e. to the (general) conditions of their representation in sense intuition, arithmetical and geometrical propositions are, for Kant, both synthetic and necessary.

Bernard Bolzano notoriously raised strong objections against Kant's views on mathematics. In an appendix to his *Contributions to a better grounded presentation of mathematics*, Bolzano tackles Kant's doctrine of "construction of concepts through intuitions" as well as his "dubious" theory of pure intuition. According to Bolzano (1810, § 9), intuition on its own cannot ground the universal and necessary propositions of arithmetic and geometry, since intuition is by nature singular and contingent. Surely intuitive representation can be useful to support mathematical thought on a *psychological* level, but it cannot ground it on a *logical* level.

For Bolzano (1810, § 11), mathematical knowledge involves generality and even necessity, which can only rest on concepts rather than on singular representations such as intuitions (1810, § 7). Such a view leads Bolzano's own work, as when he looks for purely analytical proofs to theorems that had traditionally be justified by so-called geometrical "evidences" (Bolzano 1817). Against Kant's thesis that mathematical reasoning could be grounded on pure intuitions, Bolzano insists on the purely deductive nature of pure conceptual sciences such as mathematics. Despite what Kant believed, "7+2=9" can be deduced from the mere analysis of the meanings of "7", "2" and "9" and from the associative property of addition (Bolzano 1810, § 8). And, as will soon be shown by non-Euclidian geometry, the geometrical statement that "The sum of the angles of a triangle equals 180°" depends on the definition of such notions as "triangle" and "plane".

The requirement for deductive rigor lies at the centre of Bolzano's whole theory of science. Science requires the ability to exhibit which are the basic truths of a theory and which are their consequences as well as to show precisely how the second ones can be proved on the ground of the first ones. Logic must reveal the architecture of a theory, just as in Greek axiomatics (Bolzano 1837, § 394). And this is why, for Bolzano, it is a task for logic to ground mathematical reasoning. Mathematical truths should not so much be constructed as be deduced from the basic propositions of the theory.

Now, it is well known that such a stand would later be taken and defended by Gottlob Frege (for arithmetic), Bertrand Russell (for the whole of mathematics) and Rudolf Carnap (for the whole rational frame of science). Using the tools of formal logic, logicists would show in detail how the

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[10] *CPR*, A718/B746
[11] *CPR*, A716/B744
[12] *CPR*, B15
[13] *CPR*, A140/B179
[14] *CPR*, A140-142/B179-181
[15] *CPR*, A142/B182
[16] *CPR*, A140/B179

basic notions of mathematics could be defined in purely logical terms; how, being rephrased this way, the basic truths of mathematics could be seen as mere logical theorems; and how all mathematical reasonings could be rephrased in such a way that only logical rules of inference are used. From Frege to Carnap, an explicit goal of the logicist program consists in showing that Kant was wrong in believing that mathematics is grounded on pure intuition and that its propositions are synthetic *a priori* rather than analytic[17].

1.3. Peirce's rejoinder

It is however noteworthy that, being a contemporary of Frege, Charles Sanders Peirce, who also took part to the development of formal logic and studied at length its relations to algebra, did not defend logicism at all but rather took a stand to defend Kant's views. As his published and unpublished work shows, the early Peirce was a great admirer of Kant. And most of what Peirce has said about the semiotics of mathematical proofs was in step with Kant's claim that, even though they are singular, sense intuitions, including images, can have a general meaning and provide new knowledge of a general value. Surely Peirce was well aware of the kind of objections that could be raised from a Bolzanian point of view. He discusses for instance Francis Abbott's claim that singular sense intuitions, whether real or imaginary, cannot carry nor provide general knowledge[18]. But he maintains the possibility of some formal intuition, i.e. some sense intuition where only form matters notwithstanding its singular sense content.

According to Peirce, only "ampliative" reasoning – a notion which aims at both embedding and overcoming Kant's synthetic/analytic distinction – deserves interest[19]. How reasoning can lead to gain of knowledge is strongly linked to the way contents of thought are expressed. This is why, for Peirce, logic must be some part of "Semeiotics"[20], which consists in the investigation of the way signs can refer ("stecheotic" or "speculative grammar"), tell the truth ("critical logic" or "critic") and be efficient in both these tasks ("rhetorics" or "methodeutic").

All knowledge, Peirce says, is knowledge of relations, which is the reason why logic of relatives is required for the analysis of most complex reasonings. Semiotics reveals that some increase of knowledge can rest on the fact that signs somehow exhibit (structural) relations between contents of thought in such a way that manipulation of the signs leads to the discovery of new relations in the structure. And this is especially the case in the recombination of algebraic signs (which Leibniz studied) as well as in the construction of geometric figures (which Kant studied).

Algebraic formulae are icons[21] which exhibit the formal relations between the data of a problem (or sometimes between them and some unknown). By recombining the signs according to fixed rules, algebraic reasoning helps to exhibit other formal relations that where not obvious at first sight. Similarly, construction of geometric figures helps to exhibit some formal relations between spatial elements. But what is of course important is that, as Kant rightly pointed, such a construction allows to make general suppositions out of individual cases[22]. On this point, Kant's idea was that "pure intuition" and "schemas" have some general value because they do not only express singular experiences but the very conditions of any experience. Here lies the important significance of the notions of "any", "whatever" or "in general" in mathematics.

Formal relations have to be *seen* as much as to be *conceived*. In this prospect, pure intuition is abstractive observation, i.e. observation of "forms" in the double meaning of perceptual *Gestalten* and intellectual *Ideas*[23]; *Ideas* actually are perceived through *Gestalten*. According to Peirce, this is the

[17] See for instance Frege (1884, §§ 3, 17, 87-91), Russell (1903, § 4) (1959, chap. VII), Carnap R. (1928, § 106).
[18] Peirce C.S., *Writings of Charles S. Peirce, A Chronological Edition*, Peirce Edition Project (eds.), Indiana University Press, Bloomington and Indianapolis, 1982- , vol. 1, pp. 156-157 (hereafter W1/156-157), W1/242-243.
[19] W3/244
[20] Peirce C.S., *Collected Papers of Charles Sanders Peirce*, vols. 1-6, Charles Hartshorne and Paul Weiss (eds.), 1931-1935, vols. 7-8, Arthur W. Burks (ed.), 1958, Cambridge, Harvard University Press, vol. 1, § 444 (hereafter CP1/444).
[21] CP2/279.
[22] W2/389
[23] CP2/227, CP5/161-162

case when Euclid formulates his theorems in abstract terms but proves them by constructing figures[24]. But this is also the case when Euler draws diagrams to exhibit some logical rules of inference[25]. And again algebra also has its own schemas[26]. Inferences then rely on the construction, manipulation and observation of diagrams (Kant's schemes) according to some conditions, prescriptions or rules, which provide them with a general value[27]. Allowed by eliminated premises or by general rules, inferences are mostly symbolic substitutions and these are mainly a question of form, which can be *seen* on graphs or formula. Inferences and their leading principles are meant to be *exhibited*[28]. Algebraic transformation rules are icons, and so are schematic presentations of syllogisms.

Now, what matters most for Peirce is that such symbolic constructions and manipulations help to explore concepts by somehow going "outside of them". And this of course contrasts with a purely internal analysis of these concepts. Mathematics is both deductive and inventive (as opposed to trivial, tautological or "identical") because it is actively "constructive"[29]. Diagrams, Peirce says, "evolve what was involved"[30]; they extract contents that were already contained and yet could not be seen at first sight. Because he had no idea of the logic of relatives, Kant was not aware of the distinction between two kinds of deductions (corollarial and theorematic) and he stuck to the analytic *versus* synthetic distinction without seing that some deductions are ampliative. Unlike "corollarial" deductions, "theorematic" deductions[31] require some "theoretical step", i.e. the introduction of new ideas – for instance in some lemma, which is "a demonstrable proposition about something outside the subject of inquiry"[32] – and the investigation of their consequences. This step aside (and beyond) what is already known characterizes the mathematician's daring.

According to Kant, reasoning evolves the meaning involved in the concepts of the premises. For Peirce, the meaning of a concept is the set of its consequences; deductions therefore reveal the meaning of the concepts of the premises. That diagrams, or other tools for ampliative reasoning, "evolve what was involved" should thus be understood as claiming that they show that the meaning of the concepts was richer than intended at first thought. It can now be understood why figuring concepts into intuition can do better than mere internal analysis of their definition: by allowing some manipulations and exo-constructions, it helps to exhibit features which were there but could not be seen; added lines exhibit wholes to which the previous lines belonged[33].

2. Semiotic issues

According to Peirce, the best quality of a logical notation is to favour such ampliative reasoning. The wish for a "calculus ratiocinatior" is not sufficient. Indeed, reasoning is not just computing, and Peirce tackles the dullness of reasoning "machines"[34]. Ratiocination involves two tasks that are not mechanical: colligation of relevant premises[35] and (inventive) construction. With its semiotic, logic has to help providing useful notations or diagrams for this inventive construction[36]. Now, according to Peirce, unlike Euler's circles and Hamilton's algebra[37], Boole's algebra is ampliative (even though it's not perfect)[38].

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[24] CP2/55. See also CP6/568.
[25] CP4/362sq.
[26] CP4/246, See also CP4/368
[27] W6/37, W6/257-259 , W8/24, CP1/54, 238-240, CP2/601, 778
[28] W4/250-251, CP3/324sq.
[29] W5/164 , CP3/363, 556
[30] CP4/86
[31] CP4/23
[32] CP7/204
[33] CP2/175-179
[34] W6/69-70, W8/201-202, CP3/641, CP4/141, 611
[35] CP2/442, CP4/45
[36] W5/331
[37] W1/224-225.
[38] W1/225

Let's now stress three semiotic features of diagrams which enable them to support ampliative reasoning.

2.1. Diagrams are purely formal

A first one is that they are purely formal. As is well known, Peirce distinguishes three kinds of signs : indices or marks (which refer by ostension of, or causal relation to, their referent), symbols or tokens (which are conventionally linked to general notions) and icons or analogues (which exhibit forms)[39]. In order to say something (of a general nature) about some (singular) things, propositions both require predicates and proper names, i.e. descriptive and demonstrative signs[40]. But, Peirce says, the syntactic relation between these two is an icon :

> Symbols and Indices together are generally not enough. The arrangement of the words in the sentence, for instance, must serve as *Icons*, in order that the sentence may be understood. The chief need for the Icons is in order to show the Forms of the synthesis of the elements of thought.[41]

We could perhaps understand this by comparison to Frege and Russell's views. Propositions, they say, are made of concepts (that are of a general and functional nature) and objects (that are of an individual and substantial nature). But what makes a proposition out of them is the syntactic "saturation" of functions by their arguments; in this lies the general form of the proposition :

Fa	(a is Fool)
Hab	(a Hates b)
Gabc	(a Gives b to c)

Notwithstanding the (attributive or relational) concepts that are here expressed by *F*, *G*, *H*, *L* and notwithstanding the objects that are referred to by *a*, *b*, *c*, these formal expressions exhibit the logical structure of some propositions in which they can appear. And such an exhibition is the iconic role of the formal expression. The same is of course true of more complex propositions such as $Fa\wedge\neg Fb$, $\exists x(Fx)$, $\forall x(Fx\supset Lx)$, … The logical structure of thoughts, Wittgenstein will later say, cannot be *said* (*gesagt*, *vertreten*), i.e. described in the conceptual content of propositions, but only *shown* (*gezeigt*) by the syntactic form of propositions.

That syntax – of ordinary language, of algebra, of graphs – is iconic is an important claim by Peirce[42]; it reveals that mathematical forms are mainly depicted by the syntax of mathematical expressions[43], be they algebraic formula or geometrical figures. Algebraic expressions[44], Peirce says, are icons exhibiting a "rheme", namely a general form involving functions[45] with variables (x,y), i.e. places for indices, with "x" and "y" meaning "any"[46]. Similarly, diagrams involve indices (individuals, universes of discourse, …) as well as conventional signs (e.g. Venn adds umbrage on Euler's diagrams)[47], but their syntax is iconic [48]; diagrams represent general forms, i.e. complex relations between terms[49] that can be considered notwithstanding their singular reference and thus taken as "abstractions"[50].

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[39] Peirce changed his mind on these topics. See his comments in CP1/564, CP2/340. On this see also Morris (1938, p. 102).
[40] W4/402-403, W5/111-112, 164, CP2/312, 318, 321, 337, 369, 438, CP4/56-58, 544, CP6/350, CP8/41
[41] CP4/544
[42] CP4/544
[43] CP5/550, CP6/360
[44] CP2/279, 305
[45] CP2/279, 305, 439 CP3/398, 420, 433, 445, 459-460, 630, CP4/3, 391
[46] CP4/354
[47] W5/163-164, CP3/362-363, 419, CP4/274, 418, 531
[48] CP2/305
[49] W8/64, CP2/227, 281-282, CP4/433
[50] CP3/509, CP4/234-235, 372. See also Chauviré (2008, p. 142).

2.2. Diagrams are two-dimensional

Insofar as they exhibit forms and relations by their own syntactic structure, diagrams can support mathematical conception and reasoning. Now, this first feature of diagrams is what gives importance to a second of their feature, which has been much discussed in contemporary semiotics: unlike most linguistic notations, diagrams are two-dimensional and therefore able to exhibit more complex relations than linear structures can. The complexity of mathematical problems requires them to be exhibited in complex diagrams, which show relations in different dimensions[51]. Geometrical figures, Peirce says, help us to see relations which algebra did not make apparent[52]. But algebraic languages themselves are not purely linear; not only do they express some structures by the use of brackets, but they also gain a genuine two-dimensional nature by the use of fractions, exponents, and so on :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Peirce himself shows how the notation for logic of relatives can be simplified by "spreading the formulae over two dimensions"[53].

That two-dimensionality is the main advantage of diagrams is however doubtful. The complexity of mathematical problems often involves multiple kinds of relations between its objects so that two-dimensionality is not sufficient to exhibit all of them on the same diagram. In this respect, i.e. as regards visualisation of n-dimension problems, two-dimensionality is almost as limited as one-dimensionality is[54]. By using specific symbolic notations, algebraic formulas can exhibit n-dimensional structures[55] even better than graphs[56].

2.3. Diagrams pertain to imagination

A third feature of diagrams, which is also linked to their iconicity, seems to be more significant. Unlike indices, which refer to real objects (to which they are causally related), icons pertain to imagination. Symbols, Peirce says, are characterized by rules of interpretation and therefore have a general meaning[57]; they denote by the way of connoting, which means that their extension is fixed by their comprehension. In this regard, indices and icons could be considered as "degenerate signs"[58]. An index denotes without connoting, i.e. its reference to an object is direct and does not go through a general meaning or comprehension. On the contrary, an icon connotes without denoting[59], i.e. it is informational without necessarily being referential. Unlike a photograph, which is both iconic and indexical, a painting is not necessarily causally linked to any referent; it exhibits forms but not necessarily the forms of real objects. And this is why icons do not provide knowledge on singular objects but help to explore possibilities without any interest in their reality[60] :

> A proposition is not a statement of perfectly pure mathematics until it is devoid of all definite meaning, and comes to this — that a property of a certain icon is pointed out and is declared to belong to anything like it, of which instances are given. (…) The pure mathematician deals exclusively with hypotheses. Whether or not there is any corresponding real thing, he does not care.

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[51] CP3/406

[52] CP4/137, CP6/175

[53] W4/394,CP4/430. The multiple dimension of relations is considered by Peirce as the "valence" of relatives and it should be presented like valence in chemistry (W4/391-392,CP1/292,CP3/469, CP5/469). Russell also pleaded for some spatial representation of relations (on this see Lemon & Pratt (1997).

[54] There are, for example, limits of expressivity of Euler and Venn's diagrams (Mancosu 2005, p. 25). Lemon and Pratt (1997) show how the representation abilities of a language can be measured by complexity theory.

[55] W5/110

[56] CP3/618

[57] CP2/292-293, 297-298, 301

[58] CP2/92, CP3/361-362

[59] W1/272

[60] CP1/184, CP2/65, CP3/428, 558-559, CP4/232-233, 238, 431, 447, CP8/110. For Peirce, there is experimentation in mathematics just like in natural science, even though the reality of what is observed does not matter in mathematics (Chauviré 2008, pp. 35-36, 49, 110, 152, 204).

> His hypotheses are creatures of his own imagination; but he discovers in them relations which surprise him sometimes.[61]

Making hypotheses and investigating the consequences they *would* have is the main task of mathematics; and here we do not talk only about logical possibilities (which are ruled by consistency and analyticity) but also about some "substantive"[62] possibilities such as the ones of space (which, as phenomenologists will say, are ruled by some "material" *a priori*). In his obsession for triads[63], Peirce even claims that deduction rests on symbols, induction rests on indices and reasoning by hypothesis rests on icons[64].

In the imaginary exploration of possibilities which icons permit some new knowledge can be gained[65]. Since they include indices, propositions ("dicisigns") convey information on real objects. Because they do not refer, icons do not *convey* such information[66], but paradoxically they help to *derive* new information on what could and could not be, and therefore also on some necessary properties of real situations. Soldiers who explore possibilities by fictitiously moving battalions and battery on military maps[67] surely gain knowledge on the actual situation and on some of the actual relations that were not easy to see at first sight. Similarly, a chess player who would be allowed to try some moves on the chessboard and to investigate their consequences would gain knowledge on the present state of the game[68]. And this is why a step aside (and beyond) what is already known can lead to new knowledge: imaginary geometrical constructions, introduction of imaginary numbers in the algebraic calculus, …

Another way of stating this lies in the Peircean notion of "superfluous comprehension". Extension and comprehension are normally linked by a principle of inverse proportionality : "tiger" has a richer comprehension but a smaller extension than "mammal", which has itself a richer comprehension but a smaller extension than "animal"; and similarly for "square", "rectangle" and "polygon". Denotative or extensive propositions investigate reference: by learning (through perception) that this public park is a square, I gain more knowledge on it than when I learn that it is a rectangle or that it is a polygon. Connotative or analytic propositions investigate comprehension : that something which is a square is also rectangular can be learned by mere conceptual analysis, so that I do not really gain knowledge of that thing by learning that it is a *rectangular* square (as I would by learning that it is a *green* square). Now, informative or synthetic intensive propositions seem to investigate both reference and comprehension: by learning (through mathematical construction) that diagonals of a square bisect each other, I gain knowledge both on the denotation and on the conotation of "square".

Since it does not "add" anything to the comprehension of "square", conceptual analysis does not narrow its extension – there are no less rectangular squares than there are squares –, reason why learning that this park lies in the extension of "rectangular square" was no increase of knowledge with regard to learning that it lies in the extension of "square". The information that its diagonals bisect each other, however, was not included in the definition of a square; it was not a constitutive part of its comprehension. There is thus a gain in knowledge of this comprehension when I learn about this property of squares that could not be seen in their definition (but could only be exhibited by some construction which would somehow go "outside" the definition). What is important here is that comprehension is enriched by some investigation led by imagination on actual and possible squares,

[61] CP5/567
[62] CP3/527, CP4/67
[63] On Peirce's triad-mania, see W1/46-49, 525, W5/294-309, W6/166sq., CP1/23, 211sq., 300sq., 354sq., 417sq., 568sq., CP2/234sq. According to Pierce, this obsession comes from Kant (W5/242, W6/182) and Hegel (W6/179, CP1/42, 491, 524, 544, CP2/87, 386, CP5/436)
[64] W1/281-282, 485. That this pairing of the two triads does not work is shown by the fact that, later on, Peirce will name reasoning by hypothesis "abduction" and include in it lots of reasonings whose link with icons is not clear at all.
[65] Jaakko Hintikka (1973) rigorously defines information as exclusion of possibilities (p. 153) and he distinguishes depth information from surface information (pp. 224sq.), which grows when analytical consequences are drawn (pp. 186-187).
[66] CP2/309, 314
[67] CP4/530
[68] CP3/516

i.e. on its possible extension. And here the principle of inverse proportionality of comprehension and extension is violated: extension is not narrowed – there are no less squares whose diagonals bisect each other than there are squares – even though comprehension is enriched. It is as if I had learned something by mere induction on real objects – that all ravens are black, so that there are no less black ravens than there are ravens – but, since I have not taken into consideration a set of *given* squares but investigated through imagination all *possible* squares, i.e. the possibilities and impossibilities which are linked to the construction of a square[69], I'm no more talking about the *actual* but about the *possible* extension – all squares *must* have bisecting diagonals, while ravens are black but *could* be brown –, reason why the comprehension is also enriched.

Surely this new information or "superfluous comprehension"[70] was already involved in the initial comprehension of "square", and is derived from it, but it was not a constitutive part of it as was the property of rectangularity. It had to be "discovered" via some imaginary constructions that, unlike conceptual analysis, would somehow go "outside" the definition. Diagrams play that role; they "evolve what was involved"[71].

3. Epistemological issues[72]

3.1. Practical usefulness and theoretical dispensability of diagrams

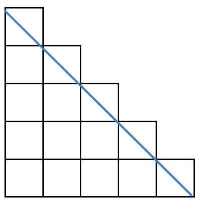
The claim that, despite their practical usefulness, diagrams are theoretically dispensable has been much debated in contemporary Philosophy of Mathematics. All will agree on the fact that diagrams have cognitive[73] virtues and therefore heuristic[74], time-saving (Tappenden 2005, pp. 181-182), pedagogical[75] or rhetorical (Edeline 2011) values. But the question remains whether or not these values are essential to the proof and whether or not diagrams could or even should *de jure* be replaced by analytical proofs.

In the line of Bolzano, many 19th century mathematicians will claim that visualisation of geometrical figures is not essential to mathematical nor even to geometrical proofs[76]. This claim, however, has been questioned in recent epistemology.

[69] A good example of how possibilities and necessities can appear through sense experience via imaginative exploration is provided by the visual proof of the following theorem (taken from Brown J.R., « Naturalism, Pictures and Platonic Intuitions », art. cit., pp. 64-66) :

Theorem : $1+2+3+\ldots+n = n^2/2 + n/2$

Proof :



Besides what is explicitly seen, imagination helps to check what could be seen, and therefore, by distinguishing accidental and essential properties of what is seen, helps to highlight necessity (Chauviré 2008, p. 67).

[70] W1/187, 275-277, 286-287, 459-467, W2/7-10, 59, 70sq., 328, W3/88-89, CP1/559, CP2/364, 418, 432. In a pragmatist view, information is the set of propositions which can be inferred from a notion (CP2/364)

[71] W1/277-278

[72] For more details and references on this epistmological aspects, see Leclercq (forthcoming).

[73] At the Ecole Normale Supérieure de Paris, Giuseppe Longo, Jean Petitot and Bernard Teissier worked on the cognitive foundations of geometry in a research project entitled "Géométrie et cognition"; they studied the natural processes of stimuli segmentation, comparisons, constitution of invariants and forms (*Gestalten*) which organize visual perception and ground geometrical thought.

[74] See what David Hilbert (1909) says of the Minkowskian art of discovery.

[75] As Hermann Weyl (1913) says, we want to understand the "idea of the proof". On the thesis, that "knowability requires comprehensibility", see also Krabbe (2008) or Coleman (2009).

[76] See Mancosu (2005, pp. 14-15). In a brilliant paper Ivahn Smadja (2012) however underlines some tension between these statements and Hilbert's considerations (in the very same talk) on the way abstract reasoning requires some visual support in signs and diagrams.

3.2. The dichotomy betwwen logic of discovery and logic of justification questioned

A first aspect of the claim that diagrams can always (*de jure*) be replaced by analytical proofs lies in the thesis, which Yehuda Rav (1999, p. 35) (2007, p. 302) names “Hilbert’s thesis”, that all mathematical proofs can be wholly formalized and reduced to calculi. According to Jody Azzouni (2004), diagrams or informal proofs only indicate the formal proof which constitutes the genuine proof.

Mathematical proofs however involve some semantic dimension which is not easily reducible to pure syntactic relations. This is due to the fact that mathematical thought is guided by the content of some concepts (such as the concept of the number) which does not seem to be exhausted by any formal system (Rav 1999, p. 29) (2007, p. 301)[77]. This is also due to the fact that the very notion of logical consequence is a semantical notion which is not reducible to syntactic derivation; even though what can be proved in a given formal system is strictly fixed within the system by a definite set of rules of inference, mathematics are made of several formal systems as well as of lots of unformalised reasonings whose criteria for acceptability cannot be unified into one single axiomatic system(Rav 1999, p. 11) (2007, pp. 311-312).

A second aspect in the claim that, even though *de facto* useful, diagrams can *de jure* be replaced by analytical proofs lies in the distinction between the “logic of discovery” – which would account for the practical (i.e. heuristic and time-saving) usefulness of diagrams – and the “logic of justification” – which would account for their theoretical dispensability –, as well as in the distinction between the mathematician’s conviction – for which diagrams can play important pedagogical and rhetorical roles – and the proof itself – which has nothing to do with these pedagogical and rhetorical aspects. Now both these distinctions have been seriously questioned by the work which has been done for the last sixty years in the epistemology of mathematical practice (starting from George Pólya and Imre Lakatos to the present work of Paolo Mancosu or, in Belgium, Jean-Paul Van Bendegem)[78]. This epistemological tradition has challenged the distinctions between discovery and justification on the basis that not all information and therefore not all justification is propositional. That there is visual and non-propositional information has been forcefully stressed by Jon Barwise and John Etchemendy (1991). More generally, twentieth century epistemology had to acknowledge that, if empirical evidence has to play a role in the justification of scientific statements, justification cannot just be a question of logical relations between propositions; since, even though it can be stated into propositions, experience is not itself propositional. Rather than just a *causal* source of mathematical statements, the vision of diagrams could thus be a *rational* – yet not merely logical – source of them[79].

Furthermore, informal logicians say, formal proofs need to be accepted as proofs. And such an acceptance itself rests on some informal reasoning (Krabbe 2008). Mathematical proofs very often involve informal considerations (Dove 2009) (Pease et al. 2009). And even the probative force of the formal part of the proof rests on some confidence in this kind of reasoning, which results itself from “intuitions” partly made of learned habits (Tappenden 2005)[80].

Now, if the probative force of formal proofs is not of a very different nature than the probative force of diagrams, the very question of whether proofs could or even should get rid of diagrams seems to loose part of their epistemological importance…

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[77] See also the comments of Smadja (2012, § 7) and Dove (2009, § 4).

[78] On this tradition, see Aberdein (2009) and Pease et al. (2009).

[79] This goes against the idea, which is still defended today by someone like Marcus Giaquinto, that experience plays the role of a “trigger” in the *genesis* of beliefs but does not *justify* them (Giaquinto M., “From symmetry perception to basic geometry”, in P. Mancosu et al. (eds.), *Visual Thinking in Mathematics*, op. cit., pp. 31, 47-48; “From symmetry perception to basis geometry”, p. 77).

[80] Judith Grabiner (1974) has shown that rigor, and hence the types of proof that are accepted by the mathematical community, is time-dependent.

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