Finite Element Analysis of the Electro-Mechanical Coupling in MEMS

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Abstract

This paper concerns the modelling of the strong electro-mechanical coupling appearing in micro-electro-mechanical systems (MEMS). These systems are very small devices (typical size of a few microns), in which electric phenomena as well as mechanical and dynamical phenomena exist. The coupling between the electric and mechanical fields induce non-linear terms in the dynamic equilibrium equations of these microscopic structures so that instability may occur. In this paper, the finite element method (FEM) is used to perform modal analysis around non-linear equilibrium positions, taking into account large displacements.

Keywords: Micro-Electro-Mechanical Systems, Finite Element Method, Non-linear Coupling

1 Introduction

To perform a correct modelling of micro-systems, all physical phenomena which can appear have to be considered. Because of the microscopic size of these structures, strong couplings may occur between the different physical fields, and some forces, which were negligible at macroscopic scales, play an important role. The main disciplines intervening in the study of micro-systems are : mechanics, electricity, thermics and fluidics. Surface forces such as Van der Waals forces may also be present. These forces tend to stick the surfaces together, what may cause manufacturing problems. The purpose of the present work is to model and to simulate, using a finite element approach, the coupling between the electric and mechanical fields in electroquasistatic micro-structures.

2 Definition of a reference problem

In order to validate the numerical results, the reference problem shown in figure 1 is considered. It consists in a capacitor made of two parallel discs with a voltage applied between them. The upper disc is moveable and is hold by a spring of stiffness k at its centre while the lower disc is fixed. This simplified problem is very representative of the mode of operation of micro-systems such as micro-actuators and micro-accelerometers.

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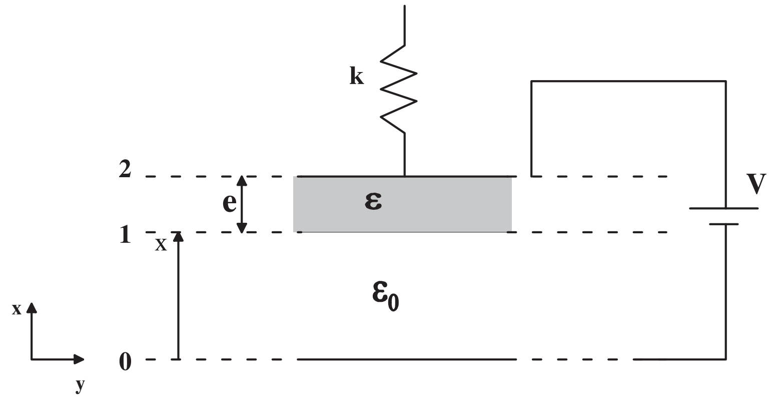

Figure 1: Definition of a reference problem

A preliminary non-dimensional analysis allows to define what are the assumptions of the problem to be considered. For this type of problem, the electrostatic assumption may be advocated [2]. For seek of simplicity, the electrodes of the capacitor are considered as infinite planes and the electric charges are supposed to be evenly distributed on the surfaces. This approximation allows to neglect fringing fields and to reduce the system to a one-dimension problem. The voltage is applied between planes '0' and '2' but because of the electrostatic assumption, the system behaves as if the voltage was applied between planes '0' and '1'. The capacitor is in vacuum and no damping is considered in the model. Only the strong coupling between the electric and the mechanical fields will be studied.

3 Mathematical modelling

3.1 Dynamic equilibrium equation

The dynamic equilibrium equation of the capacitor is written under the assumption of rigid plates. In this assembly, two mechanical forces occur. First, when the moveable plate moves down from its rest position x_0, an upward force is created :

$$\mathbf{f}_m = -k(x - x_0)\mathbf{e}_x \tag{3.1}$$

where k is the stiffness of the spring per unit area.
The second mechanical force to be considered is the force due to gravity :

$$\mathbf{f} = m\mathbf{g} \tag{3.2}$$

where m is the surface mass density of the plate.
Because of the small size of the structure, the gravitational force is very small compared to the electrical and the spring restoring forces, and may be neglected.

For the electrical part, the only known imposed value is the voltage V between the planes '0' and '2'. The electric potential on all the planes is:

$$V_0 = 0 \quad V_1 = V_2 = V \tag{3.3}$$

In this particular example, the magnitude of the electric field can be evaluated from the applied voltage by the relation :

$$E = \frac{V}{x} \tag{3.4}$$

The force experienced by a charge q in an electric field of intensity E is usually written $f = qE$. As the plates are perfect conductors, the electric charges concentrate on the surface. It results that the electric field rises instantaneously from zero inside the plates to its full value between the plates. For this reason, the average electric force is determined by [3] :

$$f_e = -\frac{1}{2}qE = -\frac{1}{2}q\frac{V}{x} = -\frac{1}{2}\epsilon_0\frac{V^2}{x^2} \tag{3.5}$$

where ϵ_0 is the permittivity of free space. This result can also be obtained by energy considerations in the capacitor.
Writing Newton's second law for the capacitor system, we obtain :

$$m\ddot{x} = -k(x - x_0) - \frac{1}{2}\epsilon_0\frac{V^2}{x^2} \tag{3.6}$$

3.2 Stability analysis

It can be noted from equation 3.6 that the dynamic behaviour of the structure depends on the applied voltage V, the spring stiffness k and its rest position x_0. A coupling term arises since the presence of the electric field adds a non-linear term in the equation of motion. The electric force applied to the surface induces an additional stiffness that modifies the natural frequency of the mechanical system.
In the state space, the non-linear equation of motion may be written using variables $y_1 = x$ and $y_2 = \dot{x}$. One obtains :

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\frac{k}{m}(y_1 - x_0) - \frac{1}{2m}\epsilon_0\frac{V^2}{y_1^2} \end{cases} \tag{3.7}$$

In the following, the influence of parameters V and k is discussed for a given value of x_0 equal to 1 μm. The solutions of the system corresponding to different initial conditions are shown in the phase diagrams of figure 2.
When no voltage is applied to the capacitor, the system is stable as shown in figure 2(a). When a voltage is applied, an instability zone occurs around the stability zone (figure 2(b),(c)). When the plates come closer, the electric force may become higher than the spring restoring force and the plates stick together. As voltage increases, the stable zone reduces and vanishes when the ratio $\frac{V^2}{k}$ exceeds a given threshold value equal to 0.33×10^{-7} in this example.

3.3 Natural frequency shifting

To calculate the natural frequency of the structure, the system must be linearised around its dynamic equilibrium position noted x_e. By defining the relative displacement as $x' = x - x_e$, the equation of motion becomes:

$$m\ddot{x}' + k(x' + x_e - x_0) + \frac{1}{2}\epsilon_0\frac{V^2}{(x' + x_e)^2} = 0 \tag{3.8}$$

With the assumption of small displacements around the dynamic equilibrium position, the equation of motion may be linearised in the form :

$$m\ddot{x}' + \left(k - \epsilon_0\frac{V^2}{x_e^3}\right)x' = -k(x_e - x_0) - \frac{1}{2}\epsilon_0\frac{V^2}{x_e^2} \tag{3.9}$$

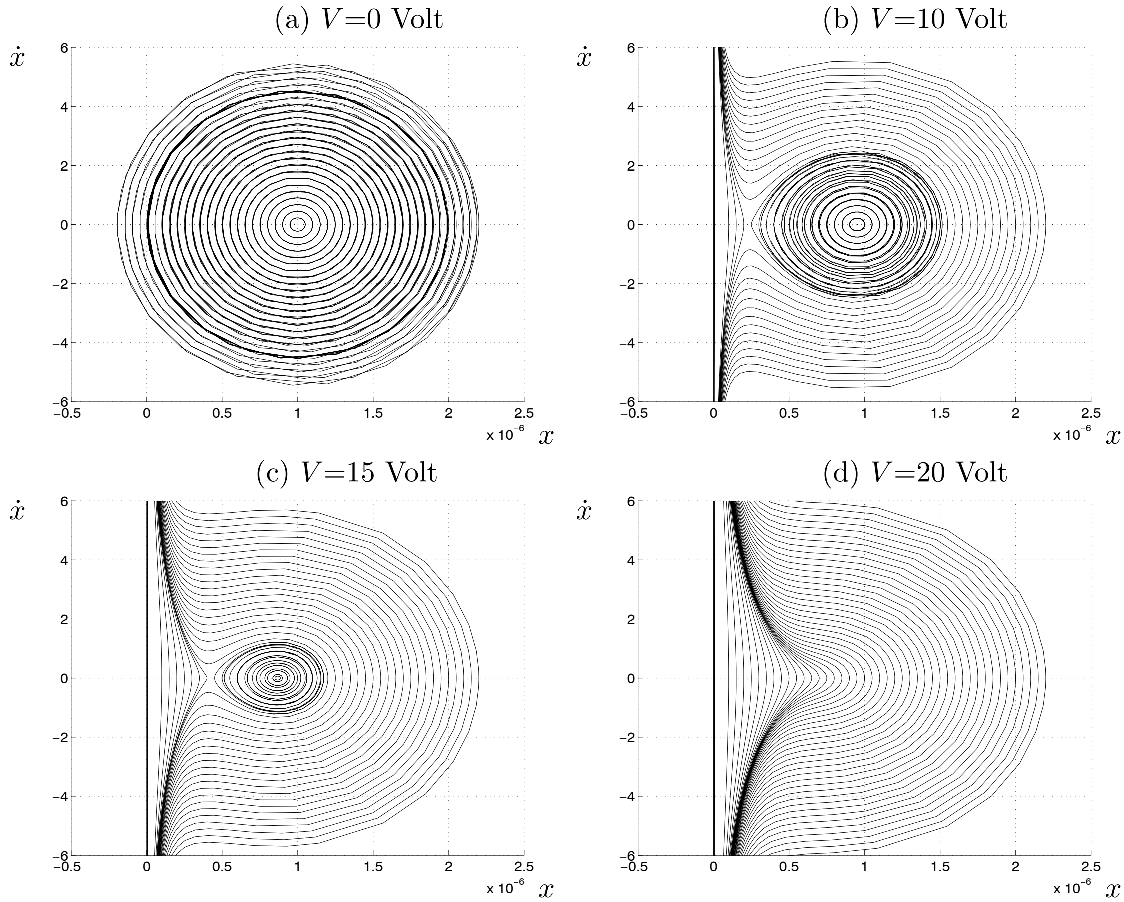

Figure 2: Phase diagrams for different values of voltage.

From equation (3.9), it results that the natural frequency ω_0 of the system is a function of the applied voltage :

$$\omega_0 = \sqrt{\frac{(k - \epsilon_0 \frac{V^2}{x_e^3})}{m}} \tag{3.10}$$

(Note that the natural frequency $\sqrt{k/m}$ of the pure mechanical system is retrieved if the electric field is equal to zero.)

Equation 3.10 shows that an increase of the voltage causes a decrease of the natural frequency of the system. Figure 3(a) shows the time evolution of the gap x between the plates for a stiffness value of 1×10^{10} N/m and for different voltage amplitudes. When the voltage increases, the dynamic equilibrium position is modified and the time period of the response increases. In this example, the initial position was taken to $x(0)$ =1.2 μm at time $t = 0$.

Because of the non-linearity induced by the electric force, the frequency also depends on the oscillation amplitude. This phenomenon does not appear in the previous expression of the natural frequency because of the linearisation of the equation of motion. This amplitude dependence is shown in figure 3(b) where it can be noticed that, for a constant voltage of 10 Volt, the oscillation frequency decreases as the amplitude increases.

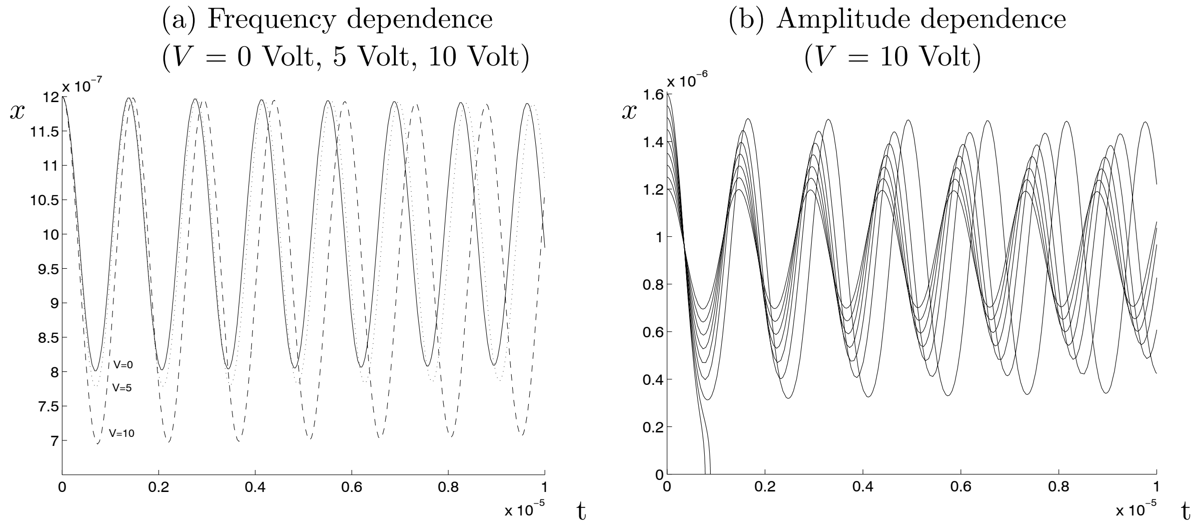

Figure 3: Non-linear dynamical behaviour of the capacitor ($V=$ 0 Volt (solid line), $V=$ 5 Volt (dotted line) and $V=$ 10 Volt (dashed line))

4 Finite element modelling

4.1 Static equilibrium equations

The governing equations associated to both the mechanical and the electric field are :

$$\begin{cases} \nabla.\mathbf{T} + \overline{\mathbf{f}} = 0 & \text{in } V_0 \\ \mathbf{T}.\mathbf{n} = \overline{\mathbf{t}} - \frac{1}{2}\rho\mathbf{E} & \text{on } \Gamma_T \end{cases} \quad \begin{cases} \nabla.\mathbf{D} - \overline{\rho} = 0 & \text{in } V_0 \\ \mathbf{D}.\mathbf{n} = \overline{d} & \text{on } \Gamma_d \end{cases} \tag{4.11}$$

where $\mathbf{T}$ is the stress tensor, $\overline{\mathbf{f}}$ is the applied body forces, $\overline{\mathbf{t}}$ the surface tractions imposed on Γ_T, $\mathbf{D}$ is the electric displacement tensor, $\overline{\rho}$ is the imposed charge density and $\overline{d}$ the electric displacement imposed on Γ_d. A force of electric origin appears in the mechanical equation to take into account the electric force induced by the charges on the surface. ρ is the charge density on the surface and $\mathbf{E}$ is the electric field. As previously stated, the factor $\frac{1}{2}$ is caused by the discontinuity of the electric field at the surface of the structure assumed as perfect conductor.

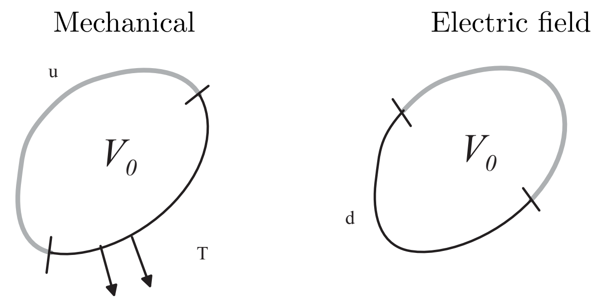

Figure 4: Mechanical and electric domains

The compatibility equations are:

$$\left\{\begin{array}{ll} S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) & \text{in } V_0 \\ \mathbf{u}=\overline{\mathbf{u}} & \text{on } \Gamma_u \end{array}\right. \quad \left\{\begin{array}{ll} \mathbf{E}=-\nabla\phi & \text{in } V_0 \\ \phi=\overline{\phi} & \text{on } \Gamma_\phi \end{array}\right. \tag{4.12}$$

where $\mathbf{u}$ et ϕ are respectively the mechanical displacement vector and the electric potential. These variables will be the unknowns of the problem in the following. S_{ij} is the strain tensor and $\mathbf{E}$ the electric field. $\overline{\mathbf{u}}$ and $\overline{\phi}$ are the imposed displacement and potential on the surfaces.

4.2 Weighted residual method

This method consists in the verification of the equilibrium equation by approximation means ([1],[4]). The equations are weighted by arbitrary variables which are the virtual displacement vector $\delta\mathbf{u}$ for the mechanical equation and the virtual potential $\delta\phi$ for the electric equation. The principle of the method is to integrate the weighted equations on the volume or on the surface on which they are defined.

$$\int_{V_0}(\nabla.\mathbf{T}+\overline{\mathbf{f}})\delta\mathbf{u}dV-\int_{\Gamma_T}(\mathbf{T}.\mathbf{n}+\frac{1}{2}\rho\mathbf{E}-\overline{\mathbf{t}})\delta\mathbf{u}dS+\int_{V_0}(\nabla.\mathbf{D}-\rho)\delta\Phi dV-\int_{\Gamma_d}(\mathbf{D}.\mathbf{n}-\overline{d})\delta\Phi dS=0 \tag{4.13}$$

Using the relation: $-a\,\nabla.\mathbf{b}=\nabla a.\mathbf{b}-\nabla.(a\mathbf{b})$, the weighted residual equation takes the form :

$$\begin{aligned}\int_{V_0}\nabla(\delta\mathbf{u})^T\mathbf{T}\,dV+\int_{V_0}\nabla(\delta\phi)^T\mathbf{D}\,dV = & \int_{V_0}(\delta\mathbf{u})^T(\overline{\mathbf{f}})\,dV+\int_{\Gamma_T}(\delta\mathbf{u})^T(-\frac{1}{2}\rho\mathbf{E}+\overline{\mathbf{t}})\,dS \\ & +\int_{V_0}(\delta\phi)^T(-\overline{\rho})\,dV+\int_{\Gamma_d}(\delta\phi)^T\overline{d}\,dS\end{aligned} \tag{4.14}$$

4.3 Finite Element discretisation

Each component of the displacement and of the potential fields may be approximated in terms of generalised co-ordinates and interpolation functions as follows :

$$\left\{\begin{array}{l} \widehat{\phi}(x,y)=\sum_i\beta_i(x,y)\Phi_i=\beta_\Phi\mathbf{\Phi} \\ \widehat{\mathbf{u}}(x,y)=\sum_i\beta_i(x,y)U_i=\beta_U\mathbf{U} \end{array}\right. \tag{4.15}$$

The electric field $\mathbf{E}$ and the strain components $\mathbf{S}$ are next evaluated by the relationship:

$$\left\{\begin{array}{l} \widehat{\mathbf{S}}(x,y)=\sum_i(\nabla\beta_i(x,y))U_i=\mathbf{B}_u\mathbf{U} \\ \widehat{\mathbf{E}}(x,y)=\sum_i-(\nabla\beta_i(x,y))\Phi_i=-\mathbf{B}_\Phi\mathbf{\Phi} \end{array}\right. \tag{4.16}$$

Substituting in equation (4.14), it follows :

$$\begin{aligned}&(\delta\mathbf{U})^T\underbrace{\left\{\int_{V_0}\mathbf{B}_u^T\mathbf{H}\mathbf{B}_u dV\right\}}_{\mathbf{K}_{uu}}\mathbf{U}-(\delta\mathbf{\Phi})^T\underbrace{\left\{\int_{V_0}\mathbf{B}_\Phi^T\epsilon\mathbf{B}_\Phi\delta\mathbf{\Phi}dV\right\}}_{\mathbf{K}_{\Phi\Phi}}\mathbf{\Phi}= \\ &(\delta\mathbf{U})^T\underbrace{\left\{\int_{V_0}\beta_U^T\overline{\mathbf{f}}dV+\int_S\beta_U^T(\overline{\mathbf{t}}-\frac{1}{2}\rho\mathbf{E})dS\right\}}_{\mathbf{F}}+(\delta\mathbf{\Phi})^T\underbrace{\left\{-\int_{V_0}\beta_\phi^T\overline{\rho}dV+\int_S\beta_\Phi^T\overline{d}dS\right\}}_{\mathbf{Q}}\end{aligned} \tag{4.17}$$

which leads to the equilibrium equation in the matrix form :

$$\begin{pmatrix}\mathbf{K}_{uu} & 0 \\ 0 & \mathbf{K}_{\Phi\Phi}\end{pmatrix}\begin{pmatrix}\mathbf{U} \\ \mathbf{\Phi}\end{pmatrix}=\begin{pmatrix}\mathbf{F} \\ \mathbf{Q}\end{pmatrix} \tag{4.18}$$

This last equation constitutes a non-linear coupled system since the electric force that appears in the right-hand side of the equation depends on the potential Φ and on the charge density ρ. Equation (4.18) may be rewritten in terms of the generalised coordinates $\mathbf{q}^T = \{\mathbf{U}, \mathbf{\Phi}\}$:

$$\mathbf{r}(\mathbf{q}) = \begin{pmatrix} \mathbf{K}_{uu} & 0 \\ 0 & \mathbf{K}_{\Phi\Phi} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{pmatrix} - \begin{pmatrix} \mathbf{F}(\mathbf{q}) \\ \mathbf{Q} \end{pmatrix} \tag{4.19}$$

where $\mathbf{r}$ is the residual vector.
The non-linear system (4.19) is then solved in an iterative manner by means of the Newton-Raphson method. Let us denote $\mathbf{q}^k$ an approximate value of $\mathbf{q}$ resulting from iteration k. The corrections $\mathbf{\Delta q}$ are computed by solving the linearised equation

$$\mathbf{r}(\mathbf{q}^{k+1}) = \mathbf{r}(\mathbf{q}^k) - \left[\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right]_{q^k} \mathbf{\Delta q} = 0 \tag{4.20}$$

where the Jacobian matrix $\mathbf{K}^t = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right]_{q^k}$ defines the tangent stiffness matrix.

5 Application to the reference problem

5.1 One-dimensional model

To illustrate the influence of the coupling between the mechanical and electric fields, we first consider the one-dimension model shown in figure 5 in which the capacitive system is made of three electric elements and three mechanical elements. The mechanical nodes are noted x_i and the electric nodes ϕ_i (i=1,...4). The boundary conditions are:

$$x_4 = 0, \quad \phi_1 = 0, \quad \phi_4 = V \tag{5.21}$$

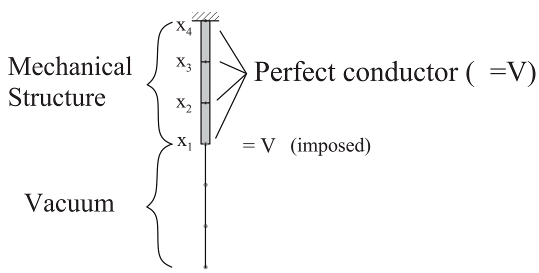

Figure 5: One-dimension problem

The tangent matrix corresponding to the free degrees-of-freedom of this problem takes the form :

$$\left(\begin{array}{cc|ccc} -5.3124\,10^{-5} & 2.6562\,10^{-5} & 0 & 0 & 0 \\ 2.6562\,10^{-5} & -5.3124\,10^{-5} & \mathbf{-265.62} & 0 & 0 \\ \hline 0 & \mathbf{-265.62} & \mathbf{1.29919\,10^{12}} & -1.30185\,10^{12} & 0 \\ 0 & 0 & -1.30185\,10^{12} & 2.6037\,10^{12} & -1.30185\,10^{12} \\ 0 & 0 & 0 & -1.30185\,10^{12} & 2.6037\,10^{12} \end{array}\right) \left(\begin{array}{c} \phi_2 \\ \phi_3 \\ \hline x_1 \\ x_2 \\ x_3 \end{array}\right) \tag{5.22}$$

The first two lines of this matrix correspond to the electric nodes and the other ones to the mechanical nodes. It can be observed that coupling terms appear between the mechanical and the electric degrees-of-freedom. It also follows that the stiffness of the structure decreases as voltage increases.

The equilibrium position of the system is then computed using the Newton-Raphson procedure described earlier. A modification of the static equilibrium position is observed when a voltage is applied as shown in figure 6(a). It is also found that the natural frequency of the structure decreases when voltage increases as shown in figure 6(b). These numerical results correspond to the analytical developments. In figures 6(a) and (b), the numerical results (black points) are compared to the analytical values (solid line) for both the equilibrium position and the natural frequency. The perfect concordance between these results allows to validate the numerical approach.

(a) Equilibrium position (b) Natural frequency shifting

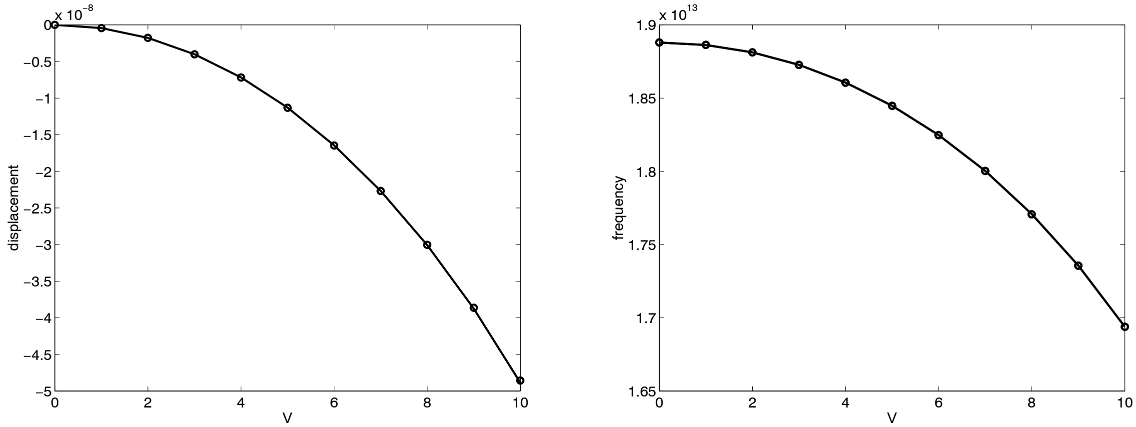

Figure 6: Equilibrium position and natural frequency according to the applied voltage for a stiffness of 1×10^{10} N/m

5.2 Axisymmetric model

The reference problem was also studied using an axisymmetric model in which a flexible disc (representing the upper capacitor plate) is fixed at its centre. Half of the structure with the corresponding finite element mesh is shown in figure 7. This figure shows the deformation of the plate submitted to the electric forces. In this example, the vacuum elements located under the plate are deformed and if the voltage becomes high, they may be distorted and even disappear. The stiffness matrices of very distorted element become ill-conditioned. In this case, one of the solution is to use adaptive meshing.

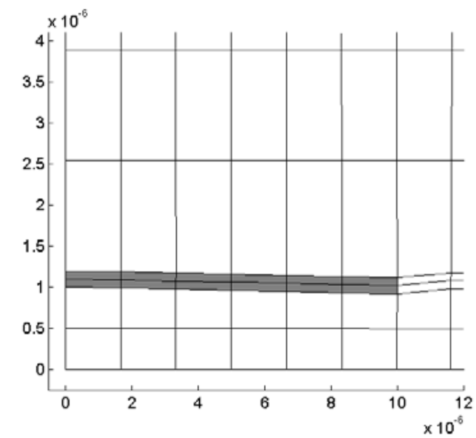

Figure 7: Fixed disc at its center

6 Conclusion

The finite element analysis of the electro-mechanical behaviour of MEMS requires the description of the strong coupling between the mechanical and the electric fields. It was shown that the electric forces introduce non-linear terms in the static equilibrium equations of capacitive structures. This causes a non-linear behaviour of the electrodes and consequently, a modification of the natural frequency around the equilibrium position. This behaviour was successfully reproduced by numerical simulations in one-dimension, three-dimensions and two-dimensions axisymmetric problems. Further improvement of the modelling could be performed by the use of adaptive meshing or by the application of the boundary element method.

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