A class of valid inequalities for multilinear 0-1 optimization problems

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Main contributions

Theoretical:

- Definition of a new class of valid inequalities, called **2-links**, for the standard linearization polytope.
- The 2-links (added to the standard linearization) provide a **complete description** for functions with **two monomials**.

Computational:

- Great improvements of the continuous **relaxation bounds**.
- Decrease of **computation times** (in many cases by factor 10).
- The number of 2-links is **only quadratic** in the number of terms.

Problem definition: multilinear 0-1 optimization

$$\min \sum_{S\in\mathcal{S}} a_S \prod_{i\in S} x_i + l(x)$$
$$\text{s. t. } x_i \in \{0,1\} \qquad i = 1,\dots,n$$

- $\mathcal{S}$: subsets of $\{1,\dots,n\}$ with $a_S \neq 0$ and $|S| \geq 2$,
- $l(x)$ linear part.

Standard linearization (SL) [2], [3]

$$\min \sum_{S\in\mathcal{S}} a_S y_S + l(x)$$
$$\text{s. t. } y_S \leq x_i \qquad \forall i \in S, \forall S \in \mathcal{S}$$
$$y_S \geq \sum_{i\in S} x_i - (|S|-1) \qquad \forall S \in \mathcal{S}$$

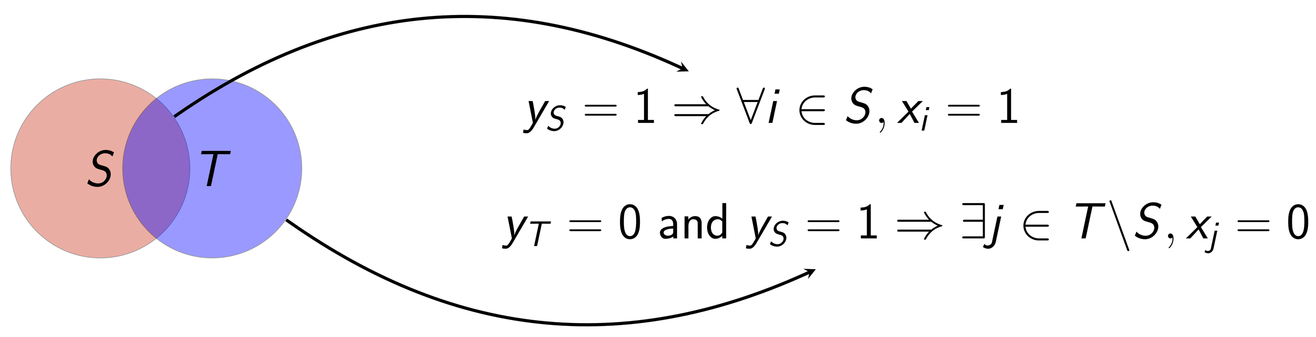
- for variables $x_i, y_S \in \{0,1\}$, the convex hull of feasible solutions is P^*_{SL},
- for continous variables $x_i, y_S \in [0,1]$, the set of feasible solutions is P_{SL}.

The 2-link inequalities [1]

Definition:
For $S, T \in \mathcal{S}$ and y_S, y_T such that $y_S = \prod_{i\in S} x_i$, $y_T = \prod_{i\in T} x_i$,

- the **2-link** associated with (S,T) is the linear inequality $y_S \leq y_T - \sum_{i\in T\setminus S} x_i + |T\setminus S|$,
- P^{2links}_{SL} is the polytope defined by the SL inequalities and the 2-links.

Interpretation:



Theorem 1: A complete description for the case of two monomials

For the case of two nonlinear monomials, $P^*_{SL} = P^{2links}_{SL}$, i.e., the **standard linearization** and the **2-links** provide a **complete description** of P^*_{SL}.

Theorem 2: Facet-defining inequalities for the case of two monomials

For the case of two nonlinear monomials defined by S, T with $|S \cap T| \geq 2$, the 2-links are **facet-defining** for P^*_{SL}.

Computational experiments: are the 2-links helpful for the general case?

Objectives:

- compare the **bounds** obtained when optimizing over P_{SL} and P^{2links}_{SL},
- compare the **computational performance** of **exact resolution methods**.

Software used: CPLEX 12.06.

Inequalities

- SL: standard linearization,
- cplex: CPLEX automatic cuts,
- 2L: 2-links.

Random instances: definition

Number of variables n, number of terms m. Monomials are uniformly distributed.

Fixed degree:

inst.	d	n	m	inst.	d	n	m
rf-a	3	400	800	rf-e	4	400	550
rf-b	3	400	900	rf-f	4	400	600
rf-c	3	600	1100	rf-g	4	600	750
rf-d	3	600	1200	rf-h	4	600	800

Random degree (proba. 2^{d-1}):

inst.	n	m	inst.	n	m
rr-a	200	600	rr-e	400	1000
rr-b	200	700	rr-f	600	1300
rr-c	200	800	rr-g	600	1400
rr-d	400	900	rr-h	600	1500

Instances inspired from image restoration problems (degree 4)

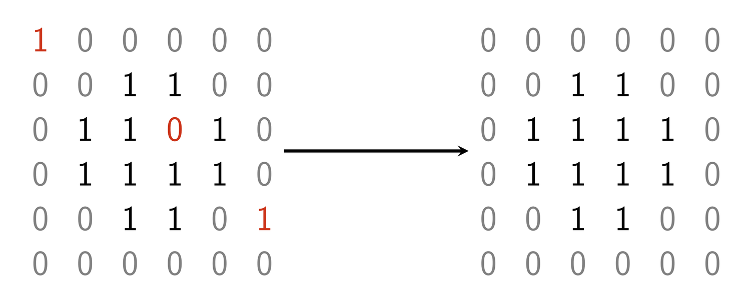
Base images:

- top left rect. (tl),
- centre rect. (cr),
- cross (cx).

Perturbations:

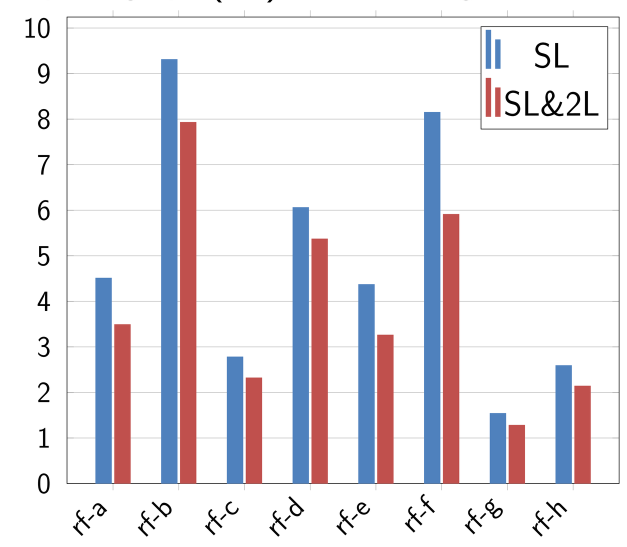
- none (n),
- low (l),
- high (h).

Image restoration

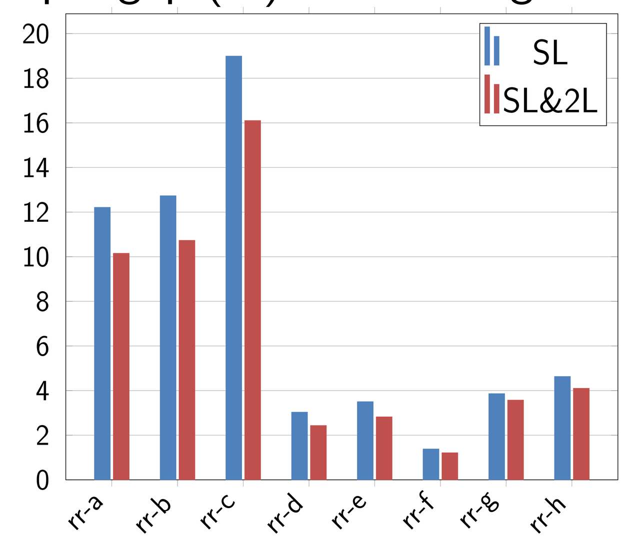


Results random instances

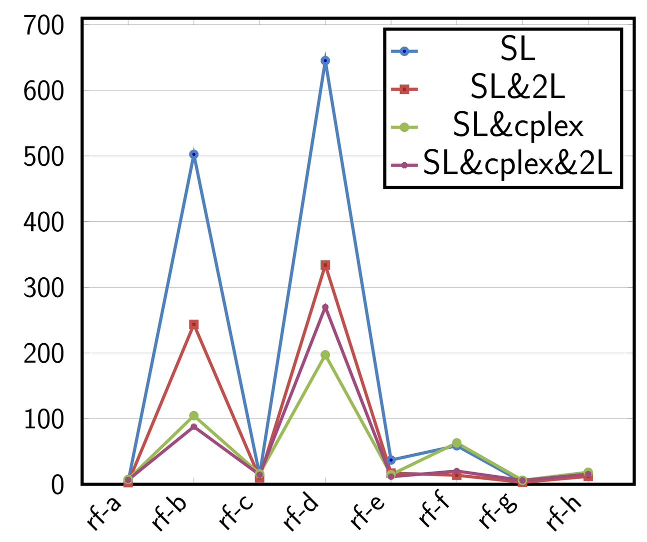
Opt. gap (%) fixed degree



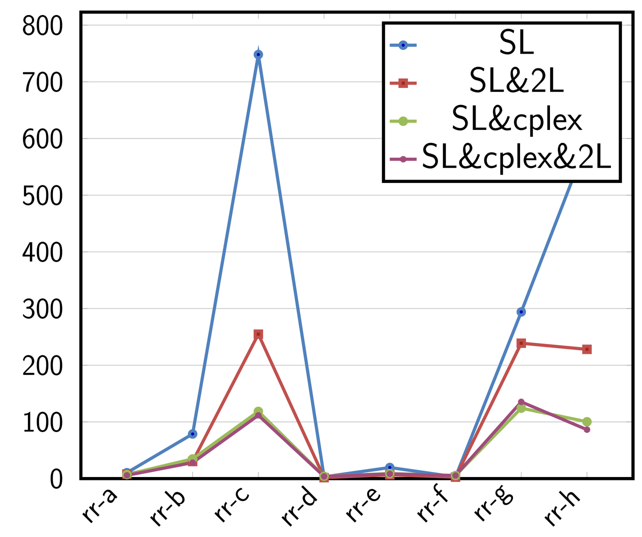
Opt. gap (%) random degree



Run times (in sec.) fixed degree

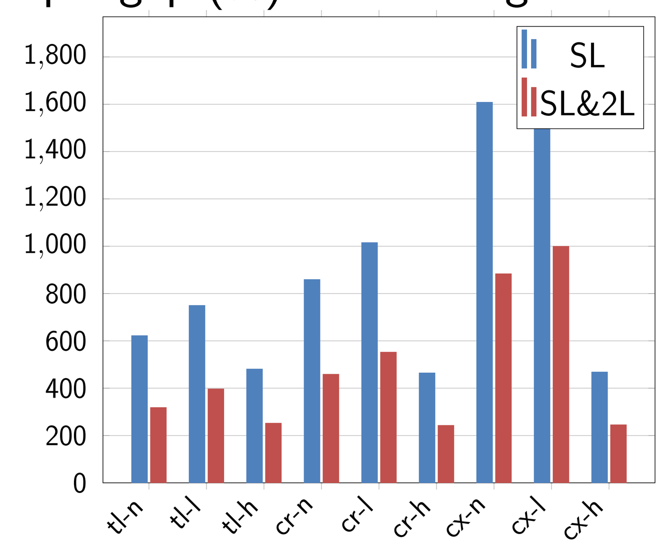


Run times (in sec.) random degree

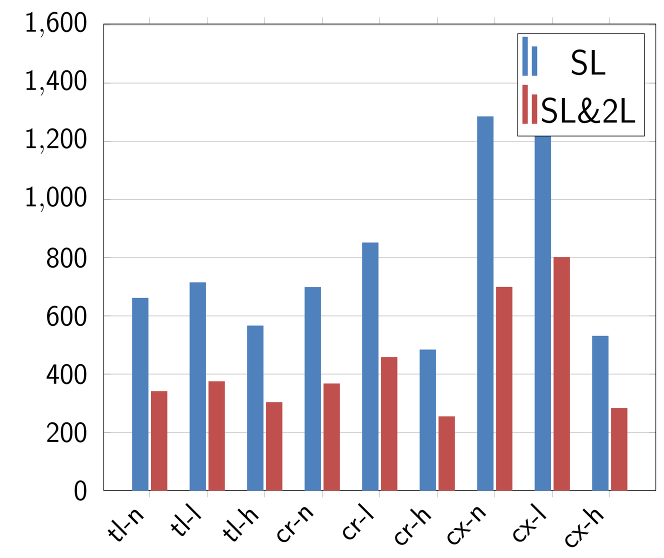


Results image restoration instances

Opt. gap (%) 10x15 images

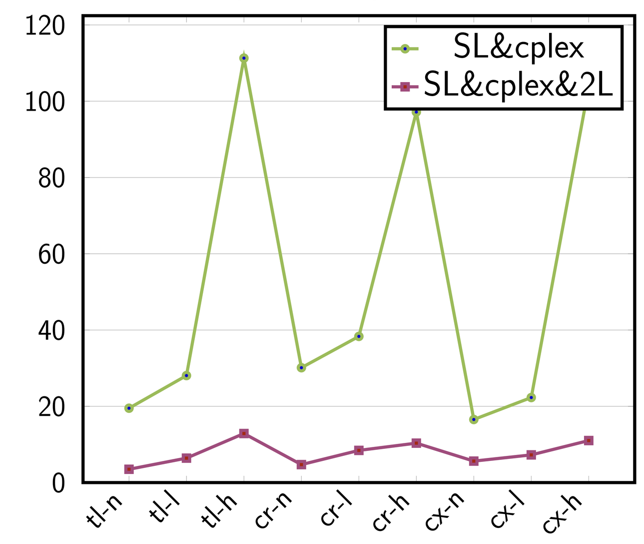


Opt. gap (%) 15x15 images



Run times (in sec.) 10x15 images

Run times (in sec.) 15x15 images



[1] Crama, Y., Rodríguez Heck, E. A class of valid inequalities for 0-1 optimization problems. E-print: http://orbi.ulg.ac.be/handle/2268/196789
[2] Fortet, R.: Applications de l'algèbre de Boole en recherche opérationelle. Revue Française d'Automatique, Informatique et Recherche Opérationelle 4(14), 17–26 (1960)
[3] Glover, F., Woolsey, E.: Further reduction of zero-one polynomial programming problems to zero-one linear programming problems. Operations Research 21(1), 156–161 (1973)