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ABOUT THE MULTIFRACTAL NATURE OF CANTOR'S BIJECTION: BOUNDS FOR THE HÖLDER EXPONENT AT ALMOST EVERY IRRATIONAL POINT

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Abstract

In this note, we investigate the regularity of Cantor's one-to-one mapping between the irrational numbers of the unit interval and the irrational numbers of the unit square. In particular, we explore the fractal nature of this map by showing that its Hölder regularity lies between 0.35 and 0.72 almost everywhere (with respect to the Lebesgue measure).

Keywords: Cantor's Bijection; Hölder Exponent; Multifractal Analysis; Continued Fractions.

1. INTRODUCTION

In 1878,[1] Cantor proved that there exists a one-to-one correspondence between the points of the unit line segment $[0,1]$ and the points of the unit square $[0,1]^2$ (repeated application of this result gives a bijective correspondence between $[0,1]$ and $[0,1]^n$, where n is a natural number). About this discovery he wrote to Dedekind: "Je le vois, mais je ne le crois pas !" ("I see it, but I don't believe it!").[2,3]

The cardinals of $I = [0,1]\backslash([0,1] \cap \mathbb{Q})$ and $[0,1]$ being equal, Cantor simply constructed a one-to-one mapping between I and I^2 to show that the sets

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$[0,1]$ and $[0,1]^2$ are in bijection. Since this application (defined on I) is constructed via continued fractions, it is very hard to have any intuition about its regularity. When looking at its definition or at the graphical representation of each component (given for the first time here), it is not hard to convince oneself that the behavior of such a function is necessarily "erratic"; however, its (Hölder-)regularity has never been considered.

The set of the natural numbers is denoted by $\mathbb{N}$ (and does not contain 0). We set $E = [0,1]$, denote by D the set of the rational numbers of E and set $I = E \backslash D$. The set of the (infinite) sequences of natural numbers is denoted $\mathbb{N}^{\mathbb{N}}$; since this space is a countable product of metric spaces, if $\boldsymbol{a} = (a_j)_{j\in\mathbb{N}}$ and $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ are two elements of $\mathbb{N}^{\mathbb{N}}$, we define the usual distance

$$d(\boldsymbol{a},\boldsymbol{b}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}.$$

We will implicitly consider that $\mathbb{N}^{\mathbb{N}}$ is equipped with this distance, while E, D and I are endowed with the Euclidean distance.

Remark 1. Considering $\boldsymbol{a}$ and $\boldsymbol{b}$ as two infinite words on the alphabet $\mathbb{N}$,[4] one can also use the following ultrametric distance on $\mathbb{N}^{\mathbb{N}}$: if $\boldsymbol{a} = (a_j)_{j\in\mathbb{N}}$ and $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ both belong to $\mathbb{N}^{\mathbb{N}}$, let $\boldsymbol{a}\wedge\boldsymbol{b}$ denote the longest common prefix of $\boldsymbol{a}$ and $\boldsymbol{b}$, so that the length $|\boldsymbol{a}\wedge\boldsymbol{b}|$ of this prefix is equal to the lowest natural number j such that $a_j \neq b_j$ minus 1. A distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$d'(\boldsymbol{a},\boldsymbol{b}) = \begin{cases} 0 & \text{if } \boldsymbol{a} = \boldsymbol{b}, \\ 2^{-|\boldsymbol{a}\wedge\boldsymbol{b}|} & \text{if } \boldsymbol{a} \neq \boldsymbol{b}. \end{cases}$$

The following relations hold:

$$2^{-2} d \le d' < d.$$

For the sake of completeness, let us recall the following result.

Proposition 1. *The space $\mathbb{N}^{\mathbb{N}}$ (endowed with the distance defined above) is a separable complete metric space.*

Proof. If $\mathbb{N}^{\mathbb{N}}_n$ denotes the set $\{\boldsymbol{a} = (a_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} : a_j = 1 \forall j > n\}$ $(n \in \mathbb{N})$, one directly checks that $\cup_{n\in\mathbb{N}}\mathbb{N}^{\mathbb{N}}_n$ is dense in $\mathbb{N}^{\mathbb{N}}$. Moreover, if $\boldsymbol{a}_j$ is a Cauchy sequence of $\mathbb{N}^{\mathbb{N}}$, there exists a subsequence $\boldsymbol{b}_j$ such that $d(\boldsymbol{b}_j, \boldsymbol{b}_{j+1}) < 2^{-j}$ for any $j \in \mathbb{N}$. One easily checks that $\boldsymbol{b}_j$ converges to $\boldsymbol{a}_0 \in \mathbb{N}^{\mathbb{N}}$ as j tends to infinity, where $a_{0,k} = b_{k,k}$ $(k \in \mathbb{N})$. □

In this note, we first recall the construction of Cantor's bijection between I and I^2 based on continued fractions and give, as far as we know for the first time, a graphical representation of the two components of this map. We then construct a homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$ to show that Cantor's bijection between I and I^2 is a homeomorphism and that any extension of this mapping to E is necessarily discontinuous at every rational number. We also investigate the multifractal nature of this function. It is well known that most of the "historical" space filling functions are monoHölder with Hölder exponent equal to $1/2$[5,6]; here we show that for Cantor's bijection, almost every point of I (with respect to the Lebesgue measure) is associated to an Hölder exponent which belongs to an interval containing $1/2$ (more precisely, this interval is bounded by 0.35 and 0.72), while we can exhibit points with Hölder exponents lying outside this interval. All the obtained results strongly rely on the theory of the continued fractions (see e.g. Ref. 7).

The results obtained here show once more that questions easily formulated with continued fractions (e.g. what happens if "we split a continued fraction in two parts") often lead to difficult problems. This is maybe why continued fractions are closely connected with fractals and chaos.[8–12]

2. DEFINITIONS

2.1. Continued Fractions

Let us first recall the basic facts about continued fractions.[7] Here, we state the results for E, but they can be easily extended to the whole real line.

Let $\boldsymbol{a} = (a_j)_{j\in\{1,\ldots,n\}}$ be a finite sequence of positive real numbers $(n \in \mathbb{N})$; the expression $[a_1, \ldots, a_n]$ is recursively defined as follows:

$$[a_1] = \frac{1}{a_1} \quad \text{and} \quad [a_1, \ldots, a_m] = \frac{1}{a_1 + [a_2, \ldots, a_m]},$$

for any $m \in \{2, \ldots, n\}$. If $\boldsymbol{a} \in \mathbb{N}^n$, we say that $[a_1, \ldots, a_n]$ is a (simple) finite continued fraction.

Proposition 2. *For any $\boldsymbol{a} \in \mathbb{N}^n$ $(n \in \mathbb{N})$, $[a_1, \ldots, a_n]$ belongs to D. Conversely, for any $x \in D$, there exists a natural number n and a sequence $\boldsymbol{a} \in \mathbb{N}^n$ such that $x = [a_1, \ldots, a_n]$.*

The representation of a rational number as a continued fraction is not unique, as shown by the following remark; this will be used in the proof of Proposition 10.

Remark 2. If $\boldsymbol{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$) is such that $a_n > 1$, one has

$$[a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1].$$

Let us now define the notion of convergent. For $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ and for $j \in \mathbb{N} \cup \{-1, 0\}$, let us define recursively $p_j(\boldsymbol{a})$ and $q_j(\boldsymbol{a})$ by setting $p_{-1}(\boldsymbol{a}) = 1$, $q_{-1}(\boldsymbol{a}) = 0$, $p_0(\boldsymbol{a}) = 0$, $q_0(\boldsymbol{a}) = 1$ and

$$\begin{cases} p_j(\boldsymbol{a}) = a_j p_{j-1}(\boldsymbol{a}) + p_{j-2}(\boldsymbol{a}) \\ q_j(\boldsymbol{a}) = a_j q_{j-1}(\boldsymbol{a}) + q_{j-2}(\boldsymbol{a}) \end{cases}$$

for $j \in \mathbb{N}$. The quotient $p_j(\boldsymbol{a})/q_j(\boldsymbol{a})$ is called the convergent of order j of $\boldsymbol{a}$. They are intimately related to the continued fractions.

Proposition 3. *Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. For all $j \in \mathbb{N}$, we have*

$$\frac{p_j(\boldsymbol{a})}{q_j(\boldsymbol{a})} = [a_1, \ldots, a_j]$$

and the sequence $(p_j(\boldsymbol{a})/q_j(\boldsymbol{a}))_{j \in \mathbb{N}}$ converges. Furthermore, for all $j \in \mathbb{N}$, we have

$$q_j(\boldsymbol{a}) p_{j-1}(\boldsymbol{a}) - p_j(\boldsymbol{a}) q_{j-1}(\boldsymbol{a}) = (-1)^j.$$

The limit of the sequence $([a_1, \ldots, a_j])_{j \in \mathbb{N}}$ is called an infinite continued fraction and is denoted by $[a_1, \ldots]$. If the real number $x \in E$ is equal to $[a_1, \ldots]$, we say that $[a_1, \ldots]$ is a continued fraction corresponding to x. The following result states that the continued fraction is an instrument for representing the real numbers (of E).

Theorem 4. *We have $x \in I$ if and only if there exists an infinite continued fraction corresponding to x; moreover, this infinite continued fraction is unique.*

Proposition 5. *If $x \in E$ can be written as $x = [a_1, \ldots, a_n, r_{n+1}]$, with $a_1, \ldots, a_n \in \mathbb{N}$ and $r_{n+1} \in [1, \infty)$, the following relation holds*:

$$x = \frac{p_n(\boldsymbol{a}) r_{n+1} + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) r_{n+1} + q_{n-1}(\boldsymbol{a})},$$

with $\boldsymbol{a} = (a_j)_{j \in \{1, \ldots, n\}}$.

A sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ is ultimately periodic of period $k \in \mathbb{N}$ if there exists J such that $a_{j+k} = a_j$ for any $j \geq J$. In this case, the corresponding continued fraction $[a_1, \ldots]$ is also called ultimately periodic of period k. The quadratic numbers (of E) are characterized by their corresponding continued fractions.

Theorem 6. *An element of I is a quadratic number if and only if the corresponding continued fraction is ultimately periodic.*

If $\boldsymbol{a}$ is an element of $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^n$ ($n \in \mathbb{N}$), we will sometimes simply write $[\boldsymbol{a}]$ instead of $[a_1, \ldots]$ or $[a_1, \ldots, a_n]$ respectively.

Let us now give a brief introduction to the notion of the metric theory of continued fractions. Since, for any $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}, [\boldsymbol{a}]$ corresponds to an irrational number $x \in I$, one can consider, for each $j \in \mathbb{N}$, the term a_j as a function of x: $a_j = a_j(x)$. Let us fix $j \in \mathbb{N}$ and write $x = [a_1, \ldots, a_{j-1}, r_j]$, with $r_j \in [1, \infty)$. It is easy to check that for any $k \in \mathbb{N}$, we have, if j is odd,

$$a_j = k \quad \text{if and only if} \quad \frac{1}{k+1} < r_j \leq \frac{1}{k}$$

and, if j is even,

$$a_j = k \quad \text{if and only if} \quad k \leq r_j < k + 1.$$

For any $j \in \mathbb{N}$, a_j is thus a piecewise constant function. Moreover, a_j is non-increasing if j is odd and non-decreasing if j is even. The functions a_1 and a_2 are represented in Fig. 1. Let $x = [\boldsymbol{a}]$ be an irrational number; for $n \in \mathbb{N}$, we set

$$I_n(x) = \{y = [\boldsymbol{b}] \in I : b_j = a_j \text{ if } j \in \{1, \ldots, n\}\}.$$

We will say that $I_n(x)$ is an interval of rank n. For any $n \in \mathbb{N}$, $I_n(x) \subset I_{n+1}(x) \subset I$ and $\lim_n I_n(x) = \{x\}$. Indeed, using Proposition 5 with $r_{n+1} = 1$ and $r_{n+1} \to \infty$, one gets

$$I_n(x) = \left(\frac{p_n(\boldsymbol{a})}{q_n(\boldsymbol{a})}, \frac{p_n(\boldsymbol{a}) + p_{n-1}(\boldsymbol{a})}{q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a})} \right) \cap I,$$

if n is even (if n is odd, the endpoints of the interval are reversed). Every interval of rank n is partitioned into a denumerably infinite number of intervals of rank $n + 1$. We will denote by $|I_n(x)|$ the Lebesgue measure of $I_n(x)$. One has, using Proposition 3,

$$|I_n(x)| = \frac{1}{q_n(\boldsymbol{a})(q_n(\boldsymbol{a}) + q_{n-1}(\boldsymbol{a}))}.$$

2.2. Cantor's Bijection

Cantor's bijection (see Ref. 1) is a one-to-one mapping between I and I^2. If $x \in I$, let $[a_1, \ldots]$ be the corresponding continued fraction and define the applications f_1 and f_2 as follows:

$$f_1(x) = [a_1, a_3, \ldots, a_{2j+1}, \ldots] \quad \text{and}$$

$$f_2(x) = [a_2, a_4, \ldots, a_{2j}, \ldots].$$

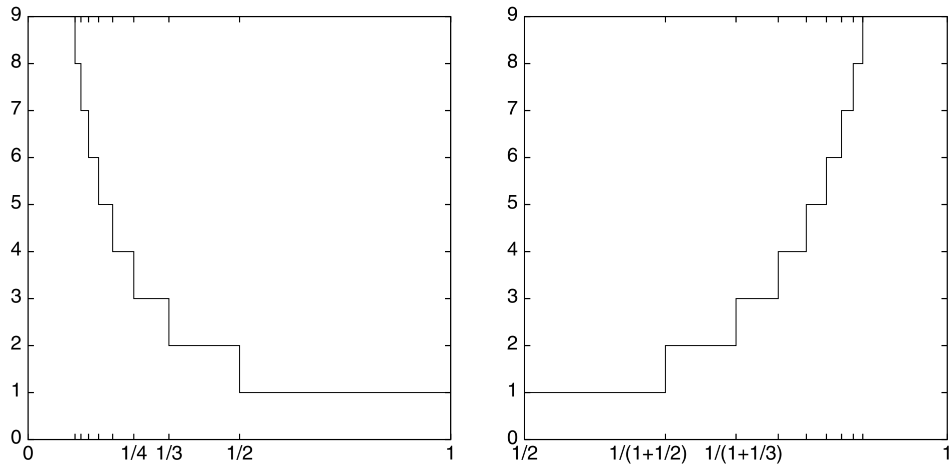

Fig. 1 The functions $x \mapsto a_1(x)$ (left panel) and $x \mapsto a_2(x)$ if $a_1(x) = 1$ (right panel). This illustrates the fact that $I_1(x)$ is partitioned into a denumerably infinite number of intervals of rank 2; in this case, $I_2(x) \subset [1/2, 1] \cap I$, since $a_1(x) = 1$ if and only if $x \in [1/2, 1] \cap I$.

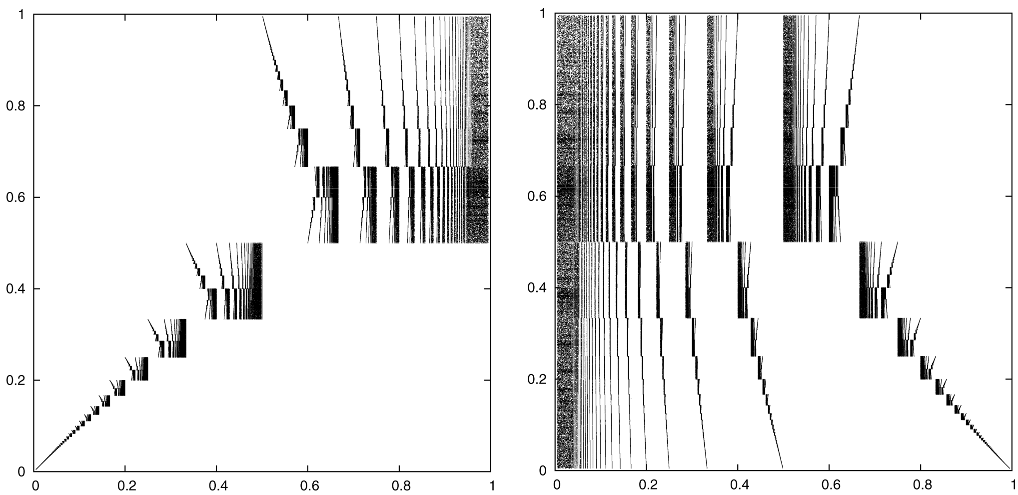

Fig. 2 The functions f_1 (left panel) and f_2 (right panel).

These applications are represented in Fig. 2. Theorem 4 implies that the application

$$f : I \to I^2 \quad x \mapsto (f_1(x), f_2(x))$$

is a one-to-one mapping. If Q denotes the set of the quadratic numbers of I, f is a one-to-one mapping between Q to Q^2. Since the cardinals of E and I are equal, f can be extended to a one-to-one mapping from E to E^2.

One can already show that Cantor's bijection is continuous (on I). However, we will be more precise in the next section, using simpler arguments.

Remark 3. For any $n \in \mathbb{N}$ and any $x \in I$, f_1 maps the interval $I_n(x)$ to $I_m(f_1(x))$, where $m = n/2$ if n is even and $m = (n + 1)/2$ if n is odd. This indeed shows that f_1 is a continuous function; obviously, the same argument can be applied to f_2.

3. CONTINUITY OF CANTOR'S BIJECTION ON I

Let $x \in I$; we write $\varphi(x) = \boldsymbol{a}$ if $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies $x = [\boldsymbol{a}]$.

Proposition 7. *The application φ is an homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let x_j be a sequence on I that converges to $x_0 \in I$. The fact that $\varphi(x_j)$ converges to $\varphi(x_0)$ is a direct consequence of Euclid's algorithm, but it is even simpler when one has the metric theory of continued fractions at one's disposal. For any $n \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $j \geq J$ implies $x_j \in I_n(x_0)$, which is sufficient.

Now let $\boldsymbol{a}_j$ be a sequence on $\mathbb{N}^{\mathbb{N}}$ that converges to $\boldsymbol{a}_0 \in \mathbb{N}^{\mathbb{N}}$ and set $x_j = [a_{j,1}, \ldots]$, $x_0 = [a_{0,1}, \ldots]$. For $\varepsilon > 0$, let $n \in \mathbb{N}$ such that

$$q_n(\boldsymbol{a}_0)(q_n(\boldsymbol{a}_0) + q_{n-1}(\boldsymbol{a}_0)) > \frac{1}{\varepsilon}.$$

Since there exists $J \in \mathbb{N}$ such that $x_j \in I_n(x_0)$ whenever $j \geq J$, one has

$$|x_0 - x_j| \leq |I_n(x_0)| < \varepsilon$$

for such indexes. □

We can thus define the application $[\cdot]$ as follows:

$$[\cdot] : \mathbb{N}^{\mathbb{N}} \to I \quad \boldsymbol{a} \mapsto \varphi^{-1}(\boldsymbol{a}).$$

Since $(\mathbb{N}^{\mathbb{N}}, d)$ is a separable complete metric space, we have reobtained the following well-known result.

Corollary 8. *The space I is a Polish space.*

Proposition 9. *Cantor's bijection f is an homeomorphism between I and I^2.*

Proof. This is trivial since the application

$$\psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \quad (\boldsymbol{a}, \boldsymbol{b}) \mapsto \boldsymbol{c},$$

where

$$c_j = \begin{cases} a_{(j+1)/2} & \text{if } j \text{ is odd} \\ b_{j/2} & \text{if } j \text{ is even} \end{cases}$$

is an homeomorphism. □

Netto's theorem[13] guarantees that such a function f cannot be extended to a continuous function from E to E^2. The following result gives additional information.

Proposition 10. *Any extension of Cantor's bijection to E is discontinuous at any rational number.*

Proof. Let $x \in D$; there exists $k \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{N}^k$ with $a_k > 1$ such that

$$x = [a_1, \ldots, a_k] = [a_1, \ldots, a_k - 1, 1].$$

Let $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$ and set $x_j = [a_1, \ldots, a_k, r_j]$, $y_j = [a_1, \ldots, a_k - 1, 1, r_j]$ with $r_j = j + [\boldsymbol{b}]$. Both the sequences x_j and y_j converge to x and it is easy to check that $\lim_j f(x_j) \neq \lim_j f(y_j)$. □

The previous proposition shows that the Hölder regularity at rational points of any extension of f to E is equal to zero. In what follows, we will therefore limit ourselves on the Hölder regularity of f defined on I.

4. HÖLDER REGULARITY OF CANTOR'S BIJECTION ON *I*

In this section, we give some preliminary results about the Hölder regularity (see e.g. Ref. 14 and references therein) of Cantor's bijection on I. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the floor function and $\lceil x \rceil$ the ceil function: $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$, $\lceil x \rceil = \inf\{k \in \mathbb{Z} : x \leq k\}$.

Let $\alpha \in [0, 1]$; a continuous and bounded real function g defined on $A \subset \mathbb{R}$ belongs to the Hölder space $\Lambda^\alpha(x)$ with $x \in A$ if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x - y|^\alpha,$$

for any $y \in A$. The Hölder exponent $h_g(x)$ of g at x is defined as follows:

$$h_g(x) = \sup\{\alpha \in [0, 1] : g \in \Lambda^\alpha(x)\}.$$

Following Ref. 14, the function g is said to be multifractal if there exist $x, x' \in A$ such that $h_g(x) \neq h_g(x')$. Let us mention that, if $h_g(x) < 1$, then g is not differentiable at x.

Let us now state our main result.

Theorem 11. *Let $x = [\boldsymbol{a}]$ be an element of I and $y \in I_n(x) \backslash I_{n+1}(x)$. One has*

$$\frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2 \rceil} \log a_{2j-1}}{\frac{1}{n}\sum_{j=1}^{n+3} \log(a_j + 1) + \frac{1}{n} C_1(n)} \leq \frac{\log|f_1(x) - f_1(y)|}{\log|x - y|}$$

and

$$\frac{\log|f_1(x) - f_1(y)|}{\log|x - y|} \leq \frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2 \rceil + 3} \log(a_{2j-1} + 1) + \frac{1}{2n} C_2(n)}{\frac{1}{n}\sum_{j=1}^{n} \log a_j},$$

where

$$C_1(n) = \frac{\log 2}{2} + \log \max\left(\frac{a_{n+2} + 2}{a_{n+2} + 1}, \frac{a_{n+3} + 2}{a_{n+3} + 1}\right)$$

and

$$C_2(n) = \frac{\log 2}{2} + \log \max\left(\frac{a_{2\lceil n/2 \rceil + 3} + 2}{a_{2\lceil n/2 \rceil + 3} + 1}, \frac{a_{2\lceil n/2 \rceil + 5} + 2}{a_{2\lceil n/2 \rceil + 5} + 1}\right).$$

Proof. Let $x = [\boldsymbol{a}] = [a_1, \ldots]$ be an element of I and consider

$$y = [a_1, \ldots, a_n, b_{n+1}, b_{n+2}, \ldots],$$

with $b_{n+1} \neq a_{n+1}$; for the sake of simplicity, one can suppose that n is even. We will bound $|x - y|$

and $|f_1(x) - f_1(y)|$ with terms depending on $\boldsymbol{a}$ and n only.

Since $I_n(x) = I_n(y)$, one has $|x-y| \leq |I_n(x)|$ and

$$|I_n(x)| = \frac{1}{q_n^2(\boldsymbol{a})} \frac{1}{1 + q_{n-1}(\boldsymbol{a})/q_n(\boldsymbol{a})} \leq \frac{1}{q_n^2(\boldsymbol{a})}.$$

Moreover, since

$$\begin{aligned} q_n(\boldsymbol{a}) &= a_n q_{n-1}(\boldsymbol{a}) + q_{n-2}(\boldsymbol{a}) \geq a_n q_{n-1}(\boldsymbol{a}) \\ &\geq a_n(a_{n-1} q_{n-2}(\boldsymbol{a}) + q_{n-3}(\boldsymbol{a})) \\ &\geq a_n \cdots a_3(a_2 q_1(\boldsymbol{a}) + q_0(\boldsymbol{a})) \\ &\geq a_n \cdots a_1, \end{aligned}$$

one gets

$$|x - y| \leq \frac{1}{a_1^2 \cdots a_n^2}.$$

The same reasoning can be applied to

$$f_1(x) = [a_1, a_3, \ldots, a_{n-1}, a_{n+1}, \ldots]$$

and

$$f_1(y) = [a_1, a_3, \ldots, a_{n-1}, b_{n+1}, b_{n+3}, \ldots]$$

to obtain

$$|f_1(x) - f_1(y)| \leq |I_{n/2}(f_1(x))| \leq \frac{1}{a_1^2 a_3^2 \cdots a_{n-1}^2}.$$

For the lower bound of $|x-y|$, let us remark that $I_{n+1}(x) \cap I_{n+1}(y) = \emptyset$, but the distance between $I_{n+1}(x)$ and $I_{n+1}(y)$ can be zero. However, for any fixed $j \in \mathbb{N}$, there exists a denumerably infinite number of intervals of rank $n+1+j$ in between $I_{n+1+j}(x)$ and $I_{n+1+j}(y)$, i.e. there exists a denumerably infinite number of $z \in I$ such that $z' \in I_{n+1+j}(z)$ implies $x < z' < y$ or $y < z' < x$. If $z = [\boldsymbol{c}]$ is such an element, one has

$$\begin{aligned} |x - y| \geq |I_{n+3}(z)| &\geq \frac{1}{q_{n+3}(\boldsymbol{c})(q_{n+3}(\boldsymbol{c}) + q_{n+2}(\boldsymbol{c}))} \\ &\geq \frac{1}{2q_{n+3}^2(\boldsymbol{c})}. \end{aligned}$$

The relations

$$\begin{aligned} q_{n+3}(\boldsymbol{c}) &= c_{n+3} q_{n+2}(\boldsymbol{c}) + q_{n+1}(\boldsymbol{c}) \leq (c_{n+3} + 1) q_{n+2}(\boldsymbol{c}) \\ &\leq (c_{n+3} + 1)(c_{n+2} q_{n+1}(\boldsymbol{c}) + q_n(\boldsymbol{c})) \\ &\leq (c_{n+3} + 1) \cdots (c_1 + 1) \end{aligned}$$

lead to

$$|I_{n+3}(z)| \geq \frac{1}{2(c_1 + 1)^2 \cdots (c_{n+3} + 1)^2}.$$

Now let

$$j_0 = \begin{cases} n+2 & \text{if } x < y, \\ n+3 & \text{if } y < x, \end{cases}$$

one can choose z such that $c_j = a_j$ for any $j \in \mathbb{N}$ except for the index j_0 for which $c_{j_0} = a_{j_0} + 1$, so that $z > x$ in the case $x < y$ and $z < x$ in the case $y < x$. Moreover, $I_{n+1}(z) = I_{n+1}(x) \neq I_{n+1}(y)$, so that $x < z < y$ in the case $x < y$ and $y < z < x$ in the case $y < x$. One therefore has

$$\begin{aligned} |x - y| &\geq |I_{n+3}(z)| \\ &\geq \frac{1}{2(a_1 + 1)^2 \cdots (a_{n+2} + 1)^2 (a_{n+3} + 2)^2}, \end{aligned}$$

or

$$\begin{aligned} |x - y| &\geq |I_{n+3}(z)| \\ &\geq \frac{1}{2(a_1 + 1)^2 \cdots (a_{n+2} + 2)^2 (a_{n+3} + 1)^2}, \end{aligned}$$

depending on the value of j_0. Without loss of generality, one can assume that j_0 corresponds to the largest integer in such inequalities.

Now there also exists $w = [d_1, \ldots]$ such that $I_{n/2+3}(w)$ lies between $I_{n/2+3}(f_1(x))$ and $I_{n/2+3}(f_1(y))$; moreover one can choose w such that $d_j = a_{2j-1}$ for any j except for one index $j_0 \in \{n/2 + 2, n/2+3\}$, for which $d_{j_0} = a_{2j_0-1} + 1$. One thus has

$$\begin{aligned} &|f_1(x) - f_1(y)| \\ &\quad \geq |I_{n/2+3}(w)| \\ &\quad \geq \frac{1}{2(a_1+1)^2 (a_3+1)^2 \cdots (a_{n+3}+1)^2 (a_{n+5}+2)^2}. \end{aligned}$$

Putting all these inequalities together and taking the logarithm, one gets

$$\begin{aligned} &\frac{-2\sum_{j=1}^{n/2} \log a_{2j-1}}{-\log 2 - 2\sum_{j=1}^{n+3} \log(a_j + 1) - 2\log\left(\frac{a_{n+3}+2}{a_{n+3}+1}\right)} \\ &\quad \leq \frac{\log|f_1(x) - f_1(y)|}{\log|x - y|} \end{aligned}$$

and

$$\begin{aligned} &\frac{\log|f_1(x) - f_1(y)|}{\log|x - y|} \\ &\quad \leq \frac{-\log 2 - 2\sum_{j=1}^{n/2+3} \log(a_{2j-1} + 1) - 2\log\left(\frac{a_{n+5}+2}{a_{n+5}+1}\right)}{-2\sum_{j=1}^{n} \log a_j}. \qquad \square \end{aligned}$$

Of course, the same reasoning can be applied to f_2, leading to the same result.

Theorem 12. *Let $x = [\boldsymbol{a}]$ be an element of I and $y \in I_n(x) \backslash I_{n+1}(x)$. One has*

$$\frac{\frac{1}{n}\sum_{j=1}^{\lfloor n/2\rfloor} \log a_{2j}}{\frac{1}{n}\sum_{j=1}^{n+3} \log(a_j+1) + \frac{1}{n}C_1(n)} \le \frac{\log|f_2(x)-f_2(y)|}{\log|x-y|}$$

and

$$\frac{\log|f_2(x)-f_2(y)|}{\log|x-y|} \le \frac{\frac{1}{n}\sum_{j=1}^{\lfloor n/2\rfloor+3} \log(a_{2j}+1) + \frac{1}{n}C_2(n)}{\frac{1}{n}\sum_{j=1}^{n} \log a_j},$$

where C_1 is defined as in Theorem 11 *and*

$$C_2(n) = \frac{\log 2}{2} + \log\max\left(\frac{a_{2\lfloor n/2\rfloor+4}+2}{a_{2\lfloor n/2\rfloor+4}+1}, \frac{a_{2\lfloor n/2\rfloor+6}+2}{a_{2\lfloor n/2\rfloor+6}+1}\right).$$

Let x_a be any quadratic number whose continued fraction expansion is ultimately periodic of period one with number a: $x_a = [\ldots, a, a, \ldots, a, \ldots]$. The previous results imply

$$\lim_{a\to\infty} h_{f_1}(x_a) = \lim_{a\to\infty} h_{f_2}(x_a) = \frac{1}{2}.$$

To obtain a generic result about the regularity of Cantor's bijection, we need a direct consequence of the ergodic theorem on continued fractions.[15] We say that a property P concerning sequences of $\mathbb{N}^{\mathbb{N}}$ holds almost everywhere if for almost every $x \in I$ (with respect to the Lebesgue measure), the sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ such that $x = [\boldsymbol{a}]$ satisfies P. The following result can be obtained from the main theorem of Ref. 16.

Theorem 13. *For any $k \in \mathbb{N} \cup \{0\}$, almost every sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies*

$$\frac{1}{n}\sum_{j=1}^{n}\log(a_j+k), \frac{1}{n}\sum_{j=1}^{n}\log(a_{2j}+k), \times \frac{1}{n}\sum_{j=1}^{n}\log(a_{2j-1}+k) \to \log K_k,$$

as n goes to infinity, where K_k is defined by:

$$K_k = \prod_{j=1}^{\infty}\left(1+\frac{1}{j(j+2)}\right)^{\log(j+k)/\log 2}.$$

The seminal result $\frac{1}{n}\sum_{j=1}^{n}\log a_j \to \log K_0$ was proven in Ref. 7; K_0 is called the Khintchine's constant. Here, we will be interested in the values

$$\log K_0 \approx 0.987849056\cdots \quad \text{and}$$
$$\log K_1 \approx 1.409785988\cdots.$$

Using Theorems 11, 12 and 13 as n goes to infinity (or equivalently as y tends to x), we get the following result.

Corollary 14. *For almost every $x \in I$, one has*

$$h_{f_1}(x), h_{f_2}(x) \in \left[\frac{\log K_0}{2\log K_1}, \frac{\log K_1}{2\log K_0}\right].$$

Remark 4. The insiders of ergodic theory will certainly recognize the Birkhoff theorem (with the Gauss transformation, which preserves the Gauss measure and which is ergodic for this measure) behind some arguments to prove Corollary 14.

Thanks to Theorem 11 (and Theorem 12), we can exactly determine the Hölder exponent of f_1 (and of f_2) at some points of I. For example, let $\boldsymbol{a^{(1)}}, \boldsymbol{a^{(2)}}, \boldsymbol{a^{(3)}} \in \mathbb{N}^{\mathbb{N}}$ be the sequences defined by

$$a_j^{(1)} = \begin{cases} k^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases}, \quad a_j^{(2)} = k^j \quad \text{and}$$

$$a_j^{(3)} = \begin{cases} 1 & \text{if } j \text{ is even} \\ k^j & \text{if } j \text{ is odd} \end{cases}$$

for any $j \in \mathbb{N}$, with $k > 1$. Using Theorem 11, it is easy to check that

$$h_{f_1}([\boldsymbol{a^{(1)}}]) = 0, \quad h_{f_1}([\boldsymbol{a^{(2)}}]) = \frac{1}{2} \quad \text{and}$$

$$h_{f_1}([\boldsymbol{a^{(3)}}]) = 1.$$

We then obtain the following corollary.

Corollary 15. *The functions f_1 and f_2 are multifractal.*

Under some conditions, it is possible to improve Corollary 14 thanks to a refinement of the bounds of Theorems 11 and 12. Taking the notations and conventions of the proof of Theorem 11, we have

$$\frac{2\log(q_{n/2}(\boldsymbol{a}'))}{\log(2)+2\log(q_{n+3}(\boldsymbol{c}))} \le \frac{\log|f_1(x)-f_1(y)|}{\log|x-y|} \le \frac{\log(2)+2\log(q_{n/2+3}(\boldsymbol{d}))}{2\log(q_n(\boldsymbol{a}))}, \tag{1}$$

where $\boldsymbol{a}' = (a_{2j-1})_{j\in\mathbb{N}}$. In order to estimate the Hölder exponent of f_1 at x, it only remains to take the limit as n tends to infinity in the previous inequalities.

On one hand, we have the following result.[17,18]

Theorem 16. *For almost every sequence $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{b})) = \frac{\pi^2}{12\log(2)}.$$

The real number $\pi^2/(12\log(2))$ is known as the Lévy's constant. On the other hand, we have

$$\lim_{n\to+\infty} \frac{1}{n+3} \log(q_{n+3}(\boldsymbol{c})) = \lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{a})) \tag{2}$$

and similarly, we also have

$$\lim_{n\to+\infty} \frac{1}{\frac{n}{2}+3} \log(q_{n/2+3}(\boldsymbol{d})) = \lim_{n\to+\infty} \frac{2}{n} \log(q_{n/2}(\boldsymbol{a}')) \tag{3}$$

(if all these limits exist). It only remains to compare Expressions (2) and (3), which is not evident. In any case, from Inequality (1) and from the above, we have the following proposition.

Proposition 17. *Let $x = [\boldsymbol{a}]$ be an element of I and let $\boldsymbol{a}' := (a_{2j-1})_{j\in\mathbb{N}}$. If we assume that*

$$\lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{a})) = \lim_{n\to+\infty} \frac{1}{n} \log(q_n(\boldsymbol{a}')) = \frac{\pi^2}{12\log(2)}, \tag{4}$$

then we have

$$h_{f_1}(x) = \frac{1}{2}.$$

There is of course a similar result for f_2. With Theorem 16, we can hope for equality (4) to be satisfied for almost every sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. The following conjecture is thus natural.

Conjecture 18. *For almost every $x \in [0,1]$, we have*

$$h_{f_1}(x) = h_{f_2}(x) = \frac{1}{2}.$$

Finally, the question of what is yielded by the multifractal formalism is also natural and could help to conjecture about the true multifractal spectrum. Let us first recall some definitions; for more details, the reader is referred to e.g. Refs. 5 and 14. For functions such as the one studied above, considering global information about the regularity can be enlightening. To do so, one introduces the multifractal spectrum d of a function f as follows:

$$d : h \mapsto \dim_{\mathcal{H}}\{x : h_f(x) = h\},$$

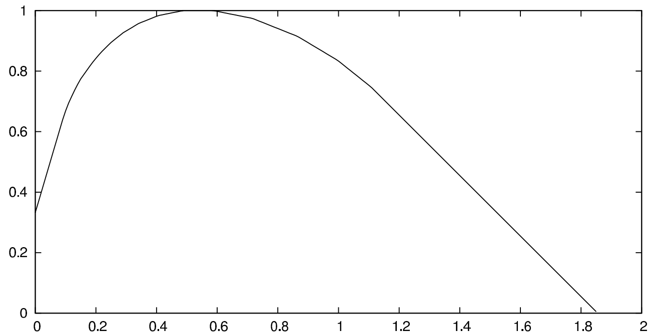

Fig. 3 A computation of the multifractal spectrum associated to f_1 (the spectrum associated to f_2 is identical).

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.[19] In some way, $d(h)$ gives the size of the set of points sharing the same Hölder exponent h (the Hölder space $\Lambda^\alpha(x)$ can be defined for $\alpha \geq 1$). However, it is often difficult to directly determine or compute such a spectrum starting from a general function f. This is why several methods, called multifractal formalisms, have been proposed. The so numerically obtained function does not necessarily correspond to the original spectrum; nevertheless, it can be shown that it leads to the expected result in numerous cases (e.g. for self-similar functions). Here we have applied two multifractal formalisms to Cantor's bijection: the wavelet leaders method and the leaders profile method (this last one is described in Ref. 20). Both lead to what is shown in Fig. 3, the result being the same whether f_1 or f_2 is considered. As expected, the maximum is reached for $h = 1/2$. One can see that for the largest values of h, the spectrum seems to be linear. Values greater than 1 reveal the existence of points at which the function is differentiable (on I). However, the numerical results obtained here only indicate that there could exist one small set of points (associated to a Hölder exponent close to 1.85) at which the function is differentiable. In such a case, the multifractal formalisms will display a linear behavior.

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