Generalized Pascal triangle for binomial coefficients of finite words

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Joint work with Julien LEROY and Michel RIGO

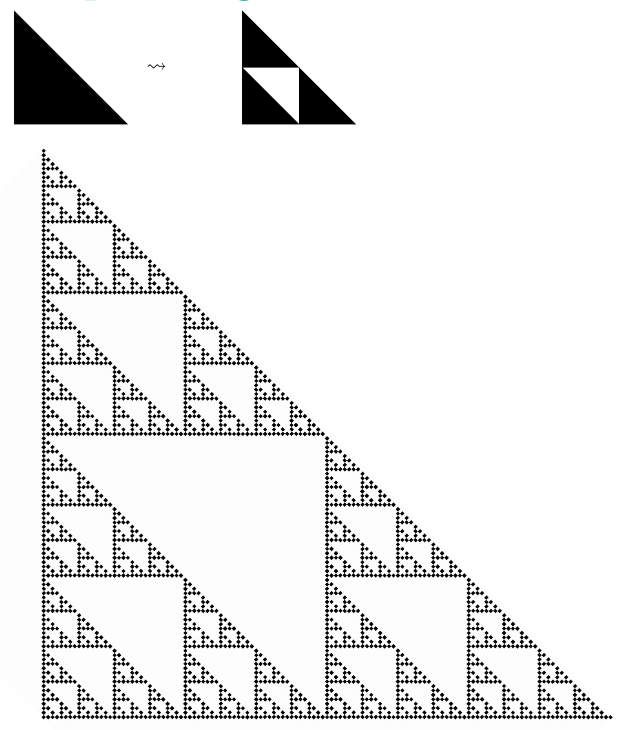
Pascal triangle and Sierpiński gasket

Pascal triangle

$\binom{m}{k}$ $m, k \in \mathbb{N}$

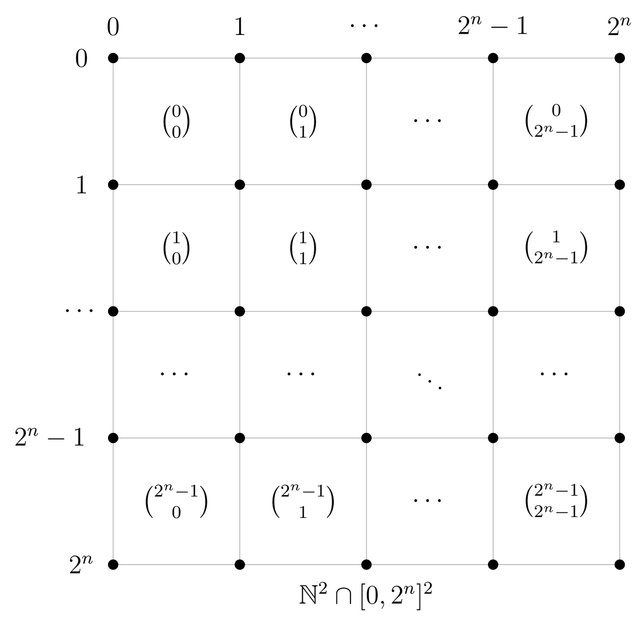
	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0
3	1	3	3	1	0	0	0	0
4	1	4	6	4	1	0	0	0
5	1	5	10	10	5	1	0	0
6	1	6	15	20	15	6	1	0
7	1	7	21	35	35	21	7	1

Sierpiński gasket



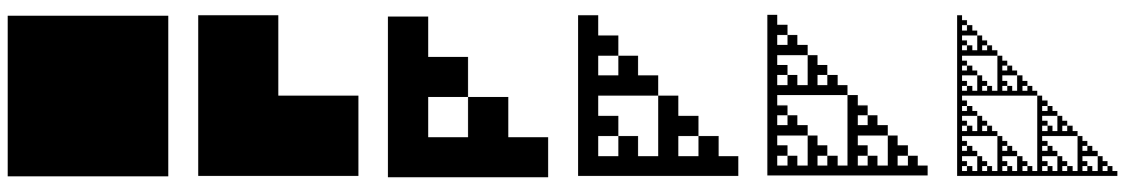
Link between these triangles?

For each $n \in \mathbb{N}$, consider the intersection of the lattice $\mathbb{N}^2$ with the region $[0, 2^n] \times [0, 2^n]$:



Color the unit square associated with the binomial coefficient $\binom{m}{k}$ in white if $\binom{m}{k} \equiv 0 \bmod 2$ and in black if $\binom{m}{k} \equiv 1 \bmod 2$.

If we normalize this region by a homothety of ratio $1/2^n$, we get a sequence of compacts in $[0, 1] \times [0, 1]$.



The elements of the latter sequence corresponding to $n \in \{0, \ldots, 5\}$.

In 1992, F. von Haeseler, H. O. Peitgen and G. Skordev showed that this sequence converges, for the Hausdorff distance, to the Sierpiński gasket when n tends to infinity.

Binomial coefficients of words

The binomial coefficient $\binom{u}{v}$ of two finite words u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword). This concept is a natural generalization of the binomial coefficients of integers. For a single letter alphabet $\{a\}$, we have

$$\binom{a^m}{a^k} = \binom{m}{k} \quad \forall m, k \in \mathbb{N}.$$

To define a new triangular array, we consider all the words over a finite alphabet and we order them by genealogical ordering (i.e. first by length, then by the classical lexicographic ordering for words of the same length assuming $0 < 1$). For the sake of simplicity, we mostly discuss the case of a 2-letter alphabet $\{0, 1\}$. We also consider the language of the base-2 expansions of integers, assuming without loss of generality that the non-empty words start with 1:

$$L = \mathrm{rep}_2(\mathbb{N}) = \{\varepsilon\} \cup 1\{0, 1\}^*.$$

The first few values of the generalized Pascal triangle are given in the following table.

	ε	1	10	11	100	101	110	111
ε	**1**	0	0	0	0	0	0	0
1	**1**	**1**	0	0	0	0	0	0
10	1	1	1	0	0	0	0	0
11	**1**	**2**	0	**1**	0	0	0	0
100	1	1	2	0	1	0	0	0
101	1	2	1	1	0	1	0	0
110	1	2	2	1	0	0	1	0
111	**1**	**3**	0	**3**	0	0	0	**1**

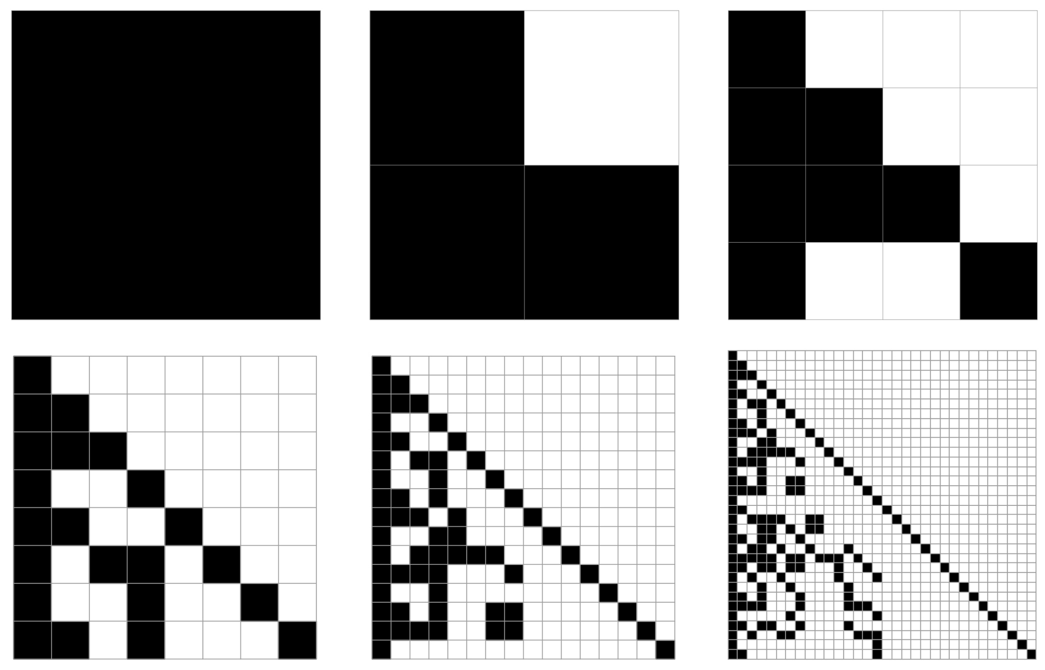
When only considering the words of the language $1^* \subset L$, we obtain the elements of the usual Pascal triangle (in **bold**).

Main results

Let $Q := [0, 1] \times [0, 1]$. Consider the sequence $(T_n)_{n \geq 0}$ of compact sets in $\mathbb{R}^2$ defined for all $n \geq 0$ by

$$T_n := \bigcup \left\{ (\mathrm{val}_2(v), \mathrm{val}_2(u)) + Q \mid u, v \in L_n, \binom{u}{v} \equiv 1 \bmod 2 \right\} \subset [0, 2^n] \times [0, 2^n].$$

Let $(U_n)_{n \geq 0}$ be the sequence of compact sets defined for all $n \geq 0$ by $U_n := \frac{T_n}{2^n} \subset [0, 1] \times [0, 1]$.



The sets $U_0, \ldots, U_5$.

Question: Does the sequence $(U_n)_{n \geq 0}$ converge to an analogue of the Sierpiński gasket and is it possible to describe the limit object?

The $(\star)$ condition: Let $(u, v) \in L \times L$. We say that (u, v) satisfies the $(\star)$ condition, if $(u, v) \neq (\varepsilon, \varepsilon)$, $\binom{u}{v} \equiv 1 \bmod 2$, $\binom{u}{v0} = 0$ and $\binom{u}{v1} = 0$.

Let (u, v) in $L \times L$ such that $|u| \geq |v| \geq 1$. We define a closed segment $S_{u,v}$ of slope 1 and length $\sqrt{2} \cdot 2^{-|u|}$ in $[0, 1] \times [1/2, 1]$. The endpoints of $S_{u,v}$ are given by $A_{u,v} := (0.0^{|u|-|v|}v, 0.u)$ and $B_{u,v} := A_{u,v} + (2^{-|u|}, 2^{-|u|})$.

Let $\mathcal{A}_0$ be the following compact set which is the closure of a countable union of segments:

$$\mathcal{A}_0 := \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}} \subset [0, 1] \times [1/2, 1].$$

Let c denote the homothety of center $(0, 0)$ and ratio $1/2$ and consider the map $h : (x, y) \mapsto (x, 2y)$. Consider the sequence $(\mathcal{A}_n)_{n \geq 0}$ of compact sets in $\mathbb{R}^2$ defined for all $n \geq 0$ by

$$\mathcal{A}_n := \bigcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq i}} h^j(c^i(\mathcal{A}_0)).$$

Lemma: The sequence $(\mathcal{A}_n)_{n \geq 0}$ is a Cauchy sequence.

Since we have a Cauchy sequence in the complete metric space $(\mathcal{H}(\mathbb{R}^2), d_h)$ (where d_h is the Hausdorff distance), the limit of $(\mathcal{A}_n)_{n \geq 0}$ is a well defined compact set denoted by $\mathcal{L}$.

Theorem: The sequence $(U_n)_{n \geq 0}$ converges to $\mathcal{L}$.

Extension to a more general context

For the sake of simplicity, we only considered odd binomial coefficients. It is straightforward to adapt our reasonings, constructions and results to a more general setting. Let p be a fixed prime and $r \in \{1, \ldots, p-1\}$. We can extend the definition of each compact set T_n to

$$T_{n,r} := \bigcup \left\{ (\mathrm{val}_2(v), \mathrm{val}_2(u)) + Q \mid u, v \in L_n, \binom{u}{v} \equiv r \bmod p \right\}$$

and introduce corresponding compact sets $U_{n,r}$.

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